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Fault Tolerant Decentralized $H_{\infty}$ Control for Symmetric Composite Systems

Shoudong Huang, James Lam, Guang-Hong Yang, and Siying Zhang

Abstract—This note discusses a class of large-scale systems composed of symmetrically interconnected identical subsystems. We consider the decentralized $H_{\infty}$ control design problem and study the fault tolerance of the resulting system by exploiting the special structure of the systems, a sufficient condition for the existence of a decentralized $H_{\infty}$ controller is derived. Moreover, for the nominal case as well as for contingent situations characterized by control channel failures, the poles and the $H_{\infty}$-norm of the closed-loop system can be calculated easily based on certain systems of reduced dimensions. Consequently, the tolerance to actuator failure can be easily tested.

Index Terms—Decentralized control, fault tolerance, $H_{\infty}$ control, large-scale systems.

I. INTRODUCTION

In the last decade, a great deal of attention has been paid to the $H_{\infty}$ control of dynamic systems, and some important design procedures have been established (e.g., [1]–[3]). Unfortunately, these control designs may result in unsatisfactory performance or even unexpected instability in the event of control component failures (e.g., actuator failures and sensor failures). Since failures of control components do occur in real world applications, they should be taken into account when a practical control system is designed.

Recently, Veillette et al. [4] studied the design of reliable control systems. The resulting control systems provide guaranteed stability and satisfy an $H_{\infty}$-norm disturbance attenuation bound only when all control components are operational, but also in case of actuator or sensor outages in the systems. The reliable control using redundant controllers was studied in [5].

This note considers a special kind of large-scale system—symmetric composite systems. Symmetric composite systems are composed of identical subsystems which are symmetrically interconnected. These systems are encountered in electric power systems, industrial manipulators, computer networks, etc. (see [6]–[8] for other examples and references). Many analyses and design problems for symmetric composite systems can be simplified because of the special structure of the system. For example, Lunze [6] discussed the stability, controllability, and observability for such systems. The output regulation problem is investigated in [9]. Hovd and Skogestad [7] studied the observability for such systems. The output regulation problem is can be simplified because of the special structure of the system.

II. SYSTEM DESCRIPTION AND PROBLEM STATEMENT

Consider a large-scale system composed of $N$ subsystems, the $i$th subsystem is given by

$$\dot{x}_i = A_{i}x_i + \sum_{k=1}^{N}A_{x_{ik}}x_k + B_{1i}u_i + G_{1i}w_i + \sum_{k=1}^{N}G_{2i}w_k$$

$$z_i = C_{i}x_i + D_{1i}u_i$$

where $i = 1, \ldots, N$ and $x_i \in \mathbb{R}^{n_i}$, $u_i \in \mathbb{R}^{m_i}$, $w_i \in \mathbb{R}^{r_i}$, $z_i \in \mathbb{R}^{s_i}$ ($i = 1, \ldots, N$) are the $n_i$-, $m_i$-, $r_i$-, and $s_i$-dimensional state, control input, exogenous input, and penalty, respectively. $A_{i}, A_{x_{ik}} \in \mathbb{R}^{n_i \times n_i}$, $B_{1i} \in \mathbb{R}^{n_i \times m_i}$, $G_{1i} \in \mathbb{R}^{s_i \times r_i}$, $C_{i} \in \mathbb{R}^{s_i \times n_i}$, $D_{1i} \in \mathbb{R}^{s_i \times m_i}$. Then the state-space model of the overall system is

$$\dot{x} = Ax + Bu + Gw$$

$$z = Cx + Du$$

where $x = [x_1^T, \ldots, x_N^T]^T$, $u = [u_1^T, \ldots, u_N^T]^T$, $w = [w_1^T, \ldots, w_N^T]^T$, $z = [z_1^T, \ldots, z_N^T]^T$, and $A \in \mathbb{R}^{Nn \times Nn}$, $B \in \mathbb{R}^{Nn \timesNm}$, $G \in \mathbb{R}^{Ns \times Nr}$, $C \in \mathbb{R}^{Ns \times Nn}$, $D \in \mathbb{R}^{Ns \timesNm}$ have the structure

$$A = \begin{bmatrix} A_1 & A_2 & \ldots & A_N \\ A_2 & A_1 & \ldots & A_N \\ \vdots & \vdots & \ddots & \vdots \\ A_N & A_{N-1} & \ldots & A_1 \end{bmatrix}, \quad G = \begin{bmatrix} G_1 & G_2 & \ldots & G_N \\ G_2 & G_1 & \ldots & G_N \\ \vdots & \vdots & \ddots & \vdots \\ G_N & G_{N-1} & \ldots & G_1 \end{bmatrix}$$

$$B = \text{diag}[B_1, \ldots, B_N], \quad C = \text{diag}[C_1, \ldots, C_N], \quad D = \text{diag}[D_1, \ldots, D_N].$$

Remark 1: Just as in [6] and [9], we shall hereafter refer system (1) to a symmetric composite system. In [7], Hovd and Skogestad called a system with this structure a parallel system, whereas in Sundareshan and Elbanna [8], it was a symmetrically interconnected system.
For a symmetric composite system (1), the decentralized $H_\infty$ control problem under consideration is to design a decentralized state feedback control law

$$ u_i = K_i x_i, \quad i = 1, \ldots, N $$

such that

S1) $\text{spec}(A + BK) \subset C^-$, $(K = \text{diag}[K_1, \ldots, K_l])$, where $C^-$ denotes the open left-half plane.

S2) The transfer matrix $T(s)$ of the closed-loop system

$$
\dot{x} = (A + BK)x + Gw \\
z = (C + DK)x
$$

satisfies $\|T\|_\infty \leq \alpha$ for some prescribed $\alpha > 0$.

Remark 2: Since all the subsystems in system (1) are identical, it is an intuitive idea to use decentralized controller of the form (2). Although it had been pointed out in [7], [12], and [13] that decentralized control with identical local controllers is not optimal for all cases, we may still prefer the decentralized controller of the form (2) because of practical reasons, such as easier maintenance and tuning [12].

III. DECENTRALIZED $H_\infty$ CONTROL

In the rest of this note, we denote

$$
A_x = A_1 - A_2, \quad A_s = A_1 + (N - 1)A_2 \\
G_x = G_1 - G_2, \quad G_s = G_1 + (N - 1)G_2.
$$

The following theorem gives a sufficient condition for the existence of a decentralized $H_\infty$ controller of the form (2).

Theorem 1: Suppose that system (1) satisfies the following two assumptions:

H1) $R_1 = D_1^T D_1$ is nonsingular;

H2) for every real number $\omega$,

$$
\begin{align*}
\text{rank} \begin{bmatrix} A_x - j\omega I & B_1 \\ C_1 & D_1 \end{bmatrix} &= n + m \\
\text{rank} \begin{bmatrix} A_s - j\omega I & B_1 \\ C_1 & D_1 \end{bmatrix} &= n + m \quad (j = \sqrt{-1}).
\end{align*}
$$

Let $\alpha$ be a positive constant. Suppose that there exists a symmetric definite positive matrix $P_1$ such that the following two Riccati algebraic inequalities hold:

$$
\begin{align*}
& (A_x - B_1 R_1^{-1} D_1^T C_1) P_1 + P_1 (A_x - B_1 R_1^{-1} D_1^T C_1)^T + P_1 \frac{1}{\alpha^2} G_s G_x^T - B_1 R_1^{-1} B_1^T P_1 = 0 \\
& + C_1^T (I - D_1 R_1^{-1} D_1^T) C_1 < 0 \\
& (A_s - B_1 R_1^{-1} D_1^T C_1) P_1 + P_1 (A_s - B_1 R_1^{-1} D_1^T C_1)^T + P_1 \frac{1}{\alpha^2} G_s G_x^T - B_1 R_1^{-1} B_1^T P_1 = 0 \\
& + C_1^T (I - D_1 R_1^{-1} D_1^T) C_1 < 0
\end{align*}
$$

then the decentralized state feedback control law

$$
u_i = K_i x_i = -R_1^{-1} \begin{bmatrix} B_1^T P_1 + D_1^T C_1 \\ \end{bmatrix} x_i, \quad i = 1, \ldots, N
$$

stabilizes system (1), and the closed-loop transfer matrix satisfies $\|T\|_\infty \leq \alpha$.

The proof of Theorem 1 is based on the following notations and results.

For a positive integer $p$, we denote

$$
m_k = \begin{bmatrix} 1 & v_k & v_k^2 & \cdots & v_k^{p-1} \end{bmatrix}^T, \quad k = 1, 2, \ldots, p
$$

where $v_k = \exp(2\pi i (k-1)/p)$, $k = 1, 2, \ldots, p$, i.e., $v_k$ is a root of the equation $v^p = 1$.

Let $t = (p+1)/2$ if $p$ is odd, $t = p/2$ if $p$ is even. Denote $r_1 = m_1 = [1 \ 1 \ \cdots \ 1]^T$, $r_{(p/2)+1} = m_{(p/2)+1}$ if $p$ is an even number, $r_{j} = (1/\sqrt{2}) (m_j + m_{p+2-j})$, $r_{p+2-j} = (j/\sqrt{2}) (m_j - m_{p+2-j})$ $(i = 2, 3, \ldots, t)$. Define

$$
R_p = \frac{1}{\sqrt{p}} \begin{bmatrix} r_1 & r_2 & \cdots & r_p \end{bmatrix}.
$$

Then from the results in [7], $R_p$ is a real orthogonal matrix, and the following lemma holds.

Lemma 1 [7]: For a positive integer $p \geq 2$, let

$$
D_p = \begin{bmatrix} a & b & \cdots & b \\ b & a & \cdots & b \\ \vdots & \vdots & \ddots & \vdots \\ b & b & \cdots & a \end{bmatrix} \in \mathbb{R}^{p \times p}
$$

where $a, b$ are two arbitrarily given numbers. Then we have

$$
R_p^{-1} D_p R_p = \text{diag} [a + (p-1)b, a - b, \ldots, a - b] \in \mathbb{R}^{p \times p}.
$$

In this note, we further denote

$$
T_{p,i} = R_p \otimes I_i
$$

where $I_i$ is the $i \times i$ identity matrix and $\otimes$ denotes the Kronecker product. Then from Lemma 1 we have

$$
\begin{align*}
T_{Nn}^{-1} A T_{Nn} &= \text{diag} [A_n, A_0, \ldots, A_s] \\
T_{Nn}^{-1} G T_{Nn} &= \text{diag} [G_n, G_0, \ldots, G_s] \\
T_{Nn}^{-1} B T_{Nn} &= \text{diag} [B_1, \ldots, B_1] \\
T_{Nn}^{-1} C T_{Nn} &= \text{diag} [C_1, \ldots, C_1] \\
T_{Nn}^{-1} D T_{Nn} &= \text{diag} [D_1, \ldots, D_1].
\end{align*}
$$

From [2, Theorem 2.4.1], the following lemma holds.

Lemma 2 [2]: Consider system (1); suppose

i) $R = D^T D$ is nonsingular;

ii) for every real number $\omega$:

$$
\begin{align*}
& \text{rank} \begin{bmatrix} A - j\omega I & B \\ C & D \end{bmatrix} = N_n + N_m.
\end{align*}
$$

Let $\alpha$ be a given positive constant. If there exists a symmetric definite positive matrix $P$ such that the following Riccati algebraic inequality holds:

$$
\begin{align*}
& (A - B R_1^{-1} D^T C) P + P (A - B R_1^{-1} D^T C)^T + P \frac{1}{\alpha^2} G G^T - B R_1^{-1} B^T P + C^T (I - D R_1^{-1} D^T) C < 0
\end{align*}
$$

then the state feedback control law $u = Kx = -R_1^{-1} (B^T P + D^T C)x$ stabilizes the system (1) and the closed-loop transfer matrix satisfies $\|T\|_\infty \leq \alpha$. 
Proof of Theorem 1: From H1), $R = D^TD$ is nonsingular. From H2) and (8), we have

$$\text{rank} \begin{bmatrix} A - j\omega I & B \\ C & D \end{bmatrix} = \text{rank} \begin{bmatrix} T_{N_n}^{-1} & 0 \\ 0 & T_{N_m}^{-1} \end{bmatrix} \begin{bmatrix} A - j\omega I & B \\ C & D \end{bmatrix} \begin{bmatrix} T_{N_n} & 0 \\ 0 & T_{N_m} \end{bmatrix}$$

$$= \text{rank} \begin{bmatrix} T_{N_n}^{-1}(A - j\omega I)T_{N_n} & T_{N_n}^{-1}BT_{N_m} \\ T_{N_m}^{-1}CT_{N_n} & T_{N_m}^{-1}DT_{N_m} \end{bmatrix}$$

$$= \text{rank} \begin{bmatrix} A - j\omega I & B_1 \\ C_1 & D_1 \end{bmatrix} = (n + m) + (N - 1)(n + m)$$

Thus, i) and ii) in Lemma 2 hold. Suppose (4) and (5) hold, let

$$P = \text{diag}[P_1, \ldots, P_1].$$

Noting that $T_{N_n}^{-1} = T_{N_n}^T$, from (8), we have

$$T_{N_n}^T \left[ (A - BR^{-1}D^T)P + P(A - BR^{-1}D^T)C \right] + P \left( \frac{1}{\alpha^2} GG^T - BR^{-1}B^T \right)$$

$$= C^T(I - DR^{-1}D^T)CT_{N_n}$$

$$= \text{diag} \left[ \begin{bmatrix} A_1 - B_1R_{11}^{-1}D_1^T C_1 \\ \cdots \\ A_n - B_1R_{11}^{-1}D_1^T C_1 \end{bmatrix}^T, \begin{bmatrix} A_1 - B_1R_{11}^{-1}D_1^T C_1 \\ \cdots \\ A_n - B_1R_{11}^{-1}D_1^T C_1 \end{bmatrix}^T \right]$$

$$+ \text{diag}[P_1, P_1, \ldots, P_1]$$

$$+ \text{diag}[P_1, P_1, \ldots, P_1]$$

$$\text{From (4) and (5),}

$$T_{N_n}^T \left[ (A - BR^{-1}D^T)P + P(A - BR^{-1}D^T)C \right] + P \left( \frac{1}{\alpha^2} GG^T - BR^{-1}B^T \right)$$

$$= C^T(I - DR^{-1}D^T)CT_{N_n} < 0.$$

Thus (9) holds.
IV. TOLERANCE TO ACTUATOR FAILURE

This section studies the tolerance to actuator failure of the decentralized controller (2). For a given $\gamma > 0$, we want to find the integer $l_0$ which corresponds to the smallest number of failures that make the closed-loop system unstable or cause the closed-loop system to violate the disturbance attenuation bound $\gamma$. It will be shown that $l_0$ can be obtained easily as a result of the special structure of system (1). The main results of this section are given by the following theorems.

**Theorem 3:** Consider the closed-loop system (3), when only one of the subsystem controllers fail, the set of poles of the resulting closed-loop system is

$$\text{spec}(A_s) = \text{spec}(A_s + B_1 K_1)$$

Moreover, in this case, the $H_\infty$-norm of the resulting closed-loop transfer matrix is

$$\|T_l\|_\infty = \max\{\|T_{l1}\|_\infty, \|T_{l2}\|_\infty\}$$

where

$$T_{l1}(s) = \frac{C_1}{0} \begin{bmatrix} C_1 + D_1 K_1 \\ 0 \end{bmatrix}$$

$$T_{l2}(s) = \frac{G_1}{\sqrt{N - T} G_2} \begin{bmatrix} A_1 + (N - 2) A_2 + B_1 K_1 \\ 0 \end{bmatrix}^{-1}$$

and $T_{l2}(s)$ is defined in (12).

**Theorem 4:** Consider the closed-loop system (3), for positive integer $l$ ($2 \leq l \leq N - 2$), when $l$ of the subsystem controllers fail, the set of poles of the resulting closed-loop system is

$$\text{spec}(A_s) = \text{spec}(A_s) \cup \text{spec}(A_s + B_1 K_1)$$

Moreover, in this case, the $H_\infty$-norm of the resulting closed-loop transfer matrix is

$$\|T_l\|_\infty = \max\{\|T_{l1}\|_\infty, \|T_{l2}\|_\infty, \|T_{l3}\|_\infty\}$$

where

$$T_{l1}(s) = \frac{C_1}{0} \begin{bmatrix} C_1 + D_1 K_1 \\ 0 \end{bmatrix}$$

$$T_{l2}(s) = \frac{G_1}{\sqrt{N - T} G_2} \begin{bmatrix} A_1 + (N - 2) A_2 + B_1 K_1 \\ 0 \end{bmatrix}^{-1}$$

and $T_{l2}(s)$ is defined in (12).

**Remark 5:** When $l$ ($2 \leq l \leq N - 2$) of the subsystem controllers fail, the resulting closed-loop system can be regarded as composed of two symmetric composite systems: one is an $n$-dimensional “open-loop system” (with no $K_1$ in it), another is an $(N - l)$-dimensional “closed-loop system” (with $K_1$ in every subsystem). In (15), spec($A_s$) is part of the poles of the “open-loop system,” spec($A_s + B_1 K_1$) is part of the poles of the “closed-loop system,” and

$$\text{spec}\left[\begin{bmatrix} A_1 + (l - 1) A_2 \\ \sqrt{N - l} A_2 \end{bmatrix} \begin{bmatrix} A_1 + (N - l - 1) A_2 + B_1 K_1 \\ \sqrt{N - l} A_2 \end{bmatrix}\right]$$

is the rest of the poles. The $H_\infty$-norm result can be explained similarly.

**Theorem 5:** Consider the closed-loop system (3), when $N - 1$ of the subsystem controllers fail, the set of poles of the resulting closed-loop system is

$$\text{spec}(A_{s(N - 1)}) = \text{spec}(A_s) \cup \text{spec}\left[\begin{bmatrix} A_1 + (N - 2) A_2 \\ \sqrt{N - 2} A_2 \end{bmatrix} \begin{bmatrix} A_1 + B_1 K_1 \\ 0 \end{bmatrix}\right]$$

Moreover, in this case, the $H_\infty$-norm of the resulting closed-loop transfer matrix is

$$\|T_{N - 1}\|_\infty = \max\{\|T_{N - 1}\|_\infty, \|T_s\|_\infty\}$$

where

$$T_{N - 1}(s) = \frac{C_1}{0} \begin{bmatrix} C_1 + D_1 K_1 \\ 0 \end{bmatrix}$$

$$T_s(s) = \frac{G_1}{\sqrt{N - 2} G_2} \begin{bmatrix} A_1 + (N - 2) A_2 + B_1 K_1 \\ 0 \end{bmatrix}^{-1}$$

and $T_{N - 1}(s)$ is defined in (18).

**Remark 6:** Theorems 4 and 5 show that the part of the poles of the open-loop system (1), given by spec($A_s$), cannot be changed when more than two controllers failures occur. Hence, spec($A_s$) $\subset C^\infty$ is a necessary condition for the closed-loop system to tolerate more than two controllers failure.

The proofs of Theorems 3–5 require the following lemma.

**Lemma 3:** For positive integers $p \geq 2$ and $q \geq 2$, let

$$E_{p,q} = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{bmatrix} \in \mathbb{R}^{p \times q}.$$
Then the following equality holds:

$$R_p^{-1} E_{pe} R_q = \begin{bmatrix} \sqrt{p/q} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \in \mathbb{R}^{p \times q}$$

where $R_p$ and $R_q$ are defined by (6).

**Proof:** The lemma can be established through straightforward algebraic manipulations. \(\square\)

For Theorems 3–5, we only prove Theorem 4. The proofs of Theorems 3 and 5 are similar and thus omitted.

**Proof of Theorem 4:** Consider the closed-loop system (3), since the subsystems of system (1) are symmetrically interconnected, without loss of generality, we can assume that the first $l$ of the subsystem controllers fail. In this case, the decentralized controller becomes

$$u_i = 0, \quad i = 1, \ldots, l$$
$$u_i = K_i x_i, \quad i = l + 1, \ldots, N.$$

Thus, the resulting closed-loop system matrix becomes as shown in (18a) at the bottom of the page.

Denote $W_1 = A_1 + (l-1)A_2$, $W_2 = \sqrt{l(N-l)}A_2$, $W_3 = A_1 + (N-l-1)A_2 + B_1K_1$. Then from Lemmas 1 and 3, we have

$$\text{spec}(A_{cl}) = \text{spec}\left( \begin{bmatrix} T_{in}^{-1} & 0 \\ 0 & T_{in}^{-1} \end{bmatrix} A_{cl} \begin{bmatrix} T_{in}^{-1} & 0 \\ 0 & T_{in}^{-1} \end{bmatrix} \right)$$

where $T_{in}$ and $T_{in}^{-1}$ are defined in (7) as shown in (18b) at the bottom of the page. Thus (15) holds.

Moreover, the resulting closed-loop transfer matrix becomes

$$T_i(s) = \text{diag}[C_1, \cdots, C_1 + D_1K_1, \cdots, C_1 + D_1K_1]$$
$$\times (sI - A_{cl})^{-1}G.$$

Since premultiplication or postmultiplication of $T_i(s)$ by orthogonal matrices will leave the $\mathcal{H}_\infty$-norm unchanged, hence we have

$$\|T\|_{\infty} = \left\| \begin{bmatrix} T_{ir}^{-1} \\ 0 \end{bmatrix} T_i(s) \begin{bmatrix} T_{ir} & 0 \\ 0 & T_{ir} \end{bmatrix} \right\|_{\infty}$$
$$= \text{diag}[C_1, C_1 + D_1K_1, C_1, \cdots, C_1 + D_1K_1, C_1 + D_1K_1]$$
$$\times \text{diag}\left( \begin{bmatrix} sI - W_1 & W_2 \\ W_2 & W_3 \end{bmatrix} \right)^{-1},$$
$$\times \text{diag}\left( \begin{bmatrix} \sqrt{l(N-l)}G_1 \right.$$
$$\sqrt{l(N-l)}G_2 \\ \sqrt{l(N-l)}G_2 \cdot G_1 + (N-l-1)G_2 \cdot G_1$$
$$\right)$$
$$= \max\{||T_{ir}||_{\infty}, ||T_{ir}||_{\infty}, ||T_{ir}||_{\infty}\}.$$

Thus (16) holds.

From Theorems 3–5, the poles and the $\mathcal{H}_\infty$-norm of the resulting closed-loop system can be easily computed when arbitrary controller failures occur. Thus, after decentralized controller (2) (the gain matrix $K_1$) is obtained, the fault tolerance of the controller ($h_0$) can be

\begin{align}
A_{cl} &= \begin{bmatrix}
A_1 & A_2 & \cdots & A_2 & A_2 & \cdots & A_2 \\
A_2 & A_1 & \cdots & A_2 & A_2 & \cdots & A_2 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
A_2 & A_2 & \cdots & A_1 & A_2 & \cdots & A_2 \\
A_2 & A_2 & \cdots & A_2 & A_1 + B_1K_1 & \cdots & A_2 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
A_2 & A_2 & \cdots & A_2 & \cdots & \cdots & A_1 + B_1K_1 \\
\end{bmatrix} \\
W_1 &= \begin{bmatrix} A_1 & \cdots & \cdots \\ \cdots & A_n \\ \cdots & \cdots \end{bmatrix} \\
W_2 &= \begin{bmatrix} A_1 & \cdots & \cdots \\ \cdots & A_n \end{bmatrix} \\
W_3 &= \begin{bmatrix} A_1 + B_1K_1 & \cdots & \cdots \\ \cdots & A_n + B_1K_1 \end{bmatrix} \\
\text{spec}(A_{cl}) &= \text{spec}\left( \begin{bmatrix} A_1 & \cdots & \cdots \\ \cdots & A_n \end{bmatrix} \right) \\
\text{spec}(A_{cl}) &= \text{spec}\left( \begin{bmatrix} W_1 & W_2 \\ W_2 & W_3 \end{bmatrix} \right) \\
\text{spec}(A_{cl}) &= \text{spec}\left( \begin{bmatrix} W_1 & W_2 \\ W_2 & W_3 \end{bmatrix} \right),
\end{align}
assessed by computing the poles and the $H_\infty$-norm of different actuator failure cases. In next section, we shall provide a possible design procedure and an example to illustrate the details.

V. A POSSIBLE DESIGN PROCEDURE AND EXAMPLE

Using Theorems 1–5, for a given $\gamma > 0$, a design scheme for a decentralized $H_\infty$ controller is suggested and its fault tolerance properties are tested for the symmetric composite system (1) as follows.

A. Possible Design Procedure

Step 1) Select $\alpha$ and $\epsilon$, ($0 < \alpha < \gamma$, $\epsilon > 0$, for example, $\alpha = \gamma/2$) solve Riccati equations

$$
\begin{align*}
(A_x - B_1 R_1^{-1} D_1^T C_1)^T P_x + P_x \left( A_x - B_1 R_1^{-1} D_1^T C_1 \right) \\
+ P_x \left( \frac{1}{\alpha^2} G_x G_x^T - B_1 R_1^{-1} B_1^T \right) P_x \\
+ C_1^T \left( I - D_1 R_1^{-1} D_1^T \right) C_1 + \epsilon I = 0
\end{align*}
$$

(19)

and

$$
\begin{align*}
(A_x - B_1 R_1^{-1} D_1^T C_1)^T P_x + P_x \left( A_x - B_1 R_1^{-1} D_1^T C_1 \right) \\
+ P_x \left( \frac{1}{\alpha^2} G_x G_x^T - B_1 R_1^{-1} B_1^T \right) P_x \\
+ C_1^T \left( I - D_1 R_1^{-1} D_1^T \right) C_1 + \epsilon I = 0
\end{align*}
$$

(20)

to obtain $P_x$ and $P_o$.

Step 2) Test Riccati inequality

$$
\begin{align*}
(A_x - B_1 R_1^{-1} D_1^T C_1)^T P_x + P_x \left( A_x - B_1 R_1^{-1} D_1^T C_1 \right) \\
+ P_x \left( \frac{1}{\alpha^2} G_x G_x^T - B_1 R_1^{-1} B_1^T \right) P_x \\
+ C_1^T \left( I - D_1 R_1^{-1} D_1^T \right) C_1 < 0.
\end{align*}
$$

(21)

If (21) holds, then let $P_1 = P_x$, go to Step 7.

Step 3) Test Riccati inequality

$$
\begin{align*}
(A_x - B_1 R_1^{-1} D_1^T C_1)^T P_x + P_x \left( A_x - B_1 R_1^{-1} D_1^T C_1 \right) \\
+ P_x \left( \frac{1}{\alpha^2} G_x G_x^T - B_1 R_1^{-1} B_1^T \right) P_x \\
+ C_1^T \left( I - D_1 R_1^{-1} D_1^T \right) C_1 < 0.
\end{align*}
$$

(22)

If (22) holds, then let $P_1 = P_x$, go to Step 7.

Step 4) Let $K_1 = - R_1^{-1} \left( B_1^T P_x + D_1^T C_1 \right)$ (or let $K_1 = - R_1^{-1} \left( B_1^T P_x + D_1^T C_1 \right)$).

Step 5) Compute spec($A_x$) and $\|T\|_\infty$ (using Theorem 2). If spec($A_x$) $\subset C^-$ and $\|T\|_\infty \leq \gamma$, then go to Step 8.

Step 6) Go back to Step 1, select $\alpha$ and $\epsilon$ again (decrease $\epsilon$ and/or increase $\alpha$).

Step 7) Let $K_1 = - R_1^{-1} \left( B_1^T P_x + D_1^T C_1 \right)$.

Step 8) The decentralized $H_\infty$ control law can be chosen as $u_i = K_1 x_i$, $i = 1, \ldots, N$.

Step 9) Let $l = 1$.

Step 10) Compute spec($A_{xi}$) and $\|T\|_\infty$ (using Theorems 3–5).

Step 11) If spec($A_{xi}$) $\subset C^-$ and $\|T\|_\infty \leq \gamma$, then let $l = l + 1$, go back to Step 10.

Step 12) Let $l_0 = l$, and one can conclude that the closed-loop system will maintain its stability with $\|T\|_\infty \leq \gamma$ when less than $l_0$ of the subsystem controllers fail.

Remark 7: If for some $\alpha$ and $\epsilon$, (21) or (22) holds, then the above algorithm will converge, and we can obtain both the decentralized $H_\infty$ controller and its tolerance level to actuator failure. If (21) and (22) do not hold, we suggest choosing $K_1$ as in Step 4 and using Step 5 to test its stabilization and disturbance attenuation properties. This choice very often works in our numerical examples. Up till now, a systematic method for choosing $K_1$ to ensure spec($A_x$) $\subset C^-$ and $\|T\|_\infty \leq \gamma$ is not available.

Remark 8: Before starting the design procedure, we should first compute spec($A$) = spec($A_x$) $\cup$ spec($A_o$) and the $H_\infty$-norm of the open-loop transfer matrix

$$
\|T\|_\infty = \max \{ \|T_{11}\|_\infty, \|T_{12}\|_\infty \}
$$

where $T_{11}(s) = C_1(sI - A_x)^{-1}G_x$, and $T_{12}(s)$ is defined in (18). If spec($A$) $\subset C^-$ and $\|T\|_\infty \leq \gamma$, then we do not need to design the controller. On the other hand, if we need to design the controller, this computation will also simplify the computation of spec($A_{xi}$) and $\|T\|_\infty$ in Step 10.

In the following, we use an example to illustrate the design procedure stated above. All $H_\infty$-computations in the example are performed with the $\mu$-Analysis and Synthesis Toolbox for MATLAB.

Example: Consider the voltage/reactive power behavior of a multimachine power system, the overall system consists of several synchronous machines including their PI-voltage controller, which feed the load through a distribution net [6]. The system can be modeled by

$$
\begin{align*}
\dot{x} &= \begin{bmatrix} 2.51 & -0.16 \\ 2.55 & 0 \end{bmatrix} x + \sum_{k=1, k \neq i}^{N} \begin{bmatrix} -0.065 & 0 \\ -0.0027 & 0 \end{bmatrix} z_k \\
&+ \begin{bmatrix} 0.9 \\ -1 \end{bmatrix} u_i + \begin{bmatrix} 0.2 \\ 0 \end{bmatrix} w_i + \sum_{k=1, k \neq i}^{N} \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix} w_k \\
z_i &= \begin{bmatrix} 2.54 & 0 \end{bmatrix} x_i + u_i, \quad i = 1, 2, \ldots, N.
\end{align*}
$$

Suppose $N = 20$, computing directly, we have

$$
\begin{align*}
A_x &= \begin{bmatrix} -2.445 & -0.16 \\ 2.5527 & 0 \end{bmatrix} \\
A_o &= \begin{bmatrix} -3.745 & -0.16 \\ 2.4987 & 0 \end{bmatrix} \\
G_x &= \begin{bmatrix} 0.1 \\ 0 \end{bmatrix} \\
G_o &= \begin{bmatrix} 2.1 \\ 2 \end{bmatrix}
\end{align*}
$$

Suppose $\gamma = 0.8$, we choose $\alpha = 0.4$, $\epsilon = 0.0002$, solving the Riccati equations (19) and (20), we have

$$
\begin{align*}
P_x &= \begin{bmatrix} 0.001 & 0.0011 \\ 0.0011 & 0.0014 \end{bmatrix} \\
P_o &= \begin{bmatrix} 0.000694 & 0.000625 \\ 0.000625 & 0.0006025 \end{bmatrix}
\end{align*}
$$
By testing, we know that inequalities (21) and (22) do not hold, we try by choosing

$$K_1 = -R_1^{-1} \left( B_1^T P_0 + D_1^T C_1 \right)$$

and obtain $K_1 = [-2.5397, 0.0003]$. From Theorem 2, we get $\text{spec}(A_1) = \{-4.552, -0.179, -0.8517, -2.3089, -0.1802, -0.1352\} \subseteq C^-$ and $||T||_{\infty} = 0.0083 < \gamma$. Thus the decentralized $H_\infty$ control law can be chosen as

$$u_i = K_1 x_i = [-2.5397, 0.0003] x_i, \quad i = 1, \ldots, N.$$ 

For $l = 1, 2, 3, 4$, Theorems 3 and 4 are used to compute $\text{spec}(A_i)$ and $||T||_{\infty}$. The results are summarized in Table I.

Since for $l = 1, 2, 3$, $\text{spec}(A_i) \subseteq C^-$ and $||T||_{\infty} < \gamma$, but $||T||_{\infty} > \gamma$, hence $l_0 = 4$. As a result, the closed-loop system will maintain its stability and the transfer matrix will satisfy $||T||_{\infty} \leq \gamma$ when less than four subsystem controllers fail.

VI. CONCLUSION

In this note, we studied the state feedback decentralized $H_\infty$ control for symmetric composite systems. First, we gave a sufficient condition for the existence of a decentralized $H_\infty$ controller. Second, we proved that the poles and the $H_\infty$-norm of the closed-loop system can be computed easily, even when some actuator faults eliminate the state feedback in some of the subsystems. Using these results, we then know the tolerance to actuator failure as soon as the decentralized state feedback controller is designed.

Since only a sufficient condition for the existence of a state feedback decentralized $H_\infty$ controller is obtained, further work is still needed before a complete design framework can be established. Moreover, the fault tolerant decentralized $H_\infty$ control for symmetric composite systems via output feedback is also a further research problem.

It should be noted that the special structure of symmetric composite systems allows us to use the methodology presented in this note. The methodology is not suitable for general large-scale systems, since the computation of the poles and the $H_\infty$-norm is computationally more demanding.

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