

the PPC policy. In this case, one approach is to identify a stable service period and use it as a supervisor for another, more intuitive (e.g., PCLB) policy. As a supervisor, the purpose of the PPC policy is simply to guarantee stability of the system. In practice, the supervising PPC policy is invoked when some measure of system performance (e.g., the sum of the buffer levels) exceeds some preset threshold (i.e., once the sum of FMS buffer levels grows beyond some preset limit, a predefined, stable periodic service sequence is implemented).

Next, note that in simulation studies for the policies introduced here we have uncovered some interesting points [5]. First, in attempting to formulate a general rule of thumb for determining the suitability of PC control for a given FMS, we make the following observation: The less variance there is among processing rates along individual paths, the better PC policies will perform (with respect to distributed policies). The reason for this is that because PC policies mandate that all buffers on a given path be processed at a single rate (the minimum processing rate of all buffers on the path), any buffers on the path that are able to be processed at a faster rate than the minimum processing rate are constrained to be processed at a lower rate than they would be processed at in a distributed control scheme. In general, then, for systems with very high processing rate "skew" along individual paths, we may be wiser to choose a distributed policy. However, it may be possible to choose paths intelligently so as to minimize the adverse affects of processing rate skew.

Finally, we would like to emphasize that PC policies will not yield stability for all FMS that would be stable under a distributed scheduling approach where the FMS satisfies a capacity constraint. For example, if there is a high amount of "processing rate skew" along the paths PC policies may not be stable and a distributed policy may be.

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Decentralized H_∞ -Controller Design for Nonlinear Systems

Guang-Hong Yang, Jianliang Wang, C. B. Soh, and James Lam

Abstract—This paper considers the decentralized H_∞ -controller design problem for nonlinear systems. Sufficient conditions for the solution of the problem are presented in terms of solutions of Hamilton–Jacobi inequalities. The resulting design guarantees local asymptotic stability and ensures a predetermined L_2 -gain bound on the closed-loop system.

Index Terms—Decentralized control, Hamilton–Jacobi inequality, nonlinear H_∞ control, nonlinear system.

I. INTRODUCTION

In the area of the decentralized control of large scale systems, numerous important advances have been accomplished in the past two decades [9], [11]. Recently, the decentralized H_∞ -control problem for linear systems has been considered in [7], [8], [10], and [13]. In particular, Veillette *et al.* [13] presented a decentralized H_∞ -controller design procedure in terms of solutions of the modified algebraic Riccati equations, and the result has also been extended to discrete-time linear systems [7]. In [8], another sufficient condition for the decentralized H_∞ -control problem is derived, under which the decentralized solution can be constructed from the central controller solution from the standard H_∞ -control theory in [2].

In recent years, the problem of central controller design to solve the H_∞ -control problem (or in short, the central H_∞ -control problem) for nonlinear systems has been extensively investigated by several authors [1], [3]–[6], [12]. In particular, Van der Schaft [12] has shown that the solution of the H_∞ -control problem via state feedback can be determined from the solution of a Hamilton–Jacobi equation (or inequality), which is the nonlinear version of the Riccati equation for the corresponding linear H_∞ -control problem. In the case of measurement feedback, a set of sufficient conditions has also been given in [1], [4], and [6] in terms of the solutions of a pair of Hamilton–Jacobi inequalities, and the necessity of these sufficient conditions has been discussed in [1] and [5].

In this paper, we consider the decentralized H_∞ -control problem for nonlinear systems by using the Hamilton–Jacobi inequality approach. The results given in this paper are extensions of existing results on the linear decentralized H_∞ -control problem [13], [8] and nonlinear central H_∞ -control problem [4]. The paper is organized as follows. The system description and problem statement are given in Section II. The main results are given in Section III, followed by a numerical example in Section IV to illustrate the design procedure and the effectiveness of the proposed method. Finally, some concluding remarks are given in Section V.

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II. PROBLEM STATEMENT

Consider a nonlinear system Σ described by equations of the form

$$\Sigma: \dot{x} = f(x) + g_1(x)w_0 + \sum_{i=1}^q g_{2i}(x)u_i \quad (1)$$

$$z = [h_1(x) \quad u_1 \quad \cdots \quad u_q]^T \quad (2)$$

$$y_i = h_{2i}(x) + w_i, \quad i = 1, \dots, q \quad (3)$$

where x is a state vector defined in a neighborhood X of the origin in R^n ; the u_i 's are the local control inputs with $u_i \in R^{m_i}$; the y_i 's are the local measurement with $y_i \in R^{p_i}$; w_0 and the w_i 's are square-integrable disturbances; and z is an output to be regulated. The functions $f(x)$, $g_1(x)$, $h_1(x)$, $g_{2i}(x)$, and $h_{2i}(x)$ ($i = 1, \dots, q$) are all known smooth mappings of appropriate dimensions, defined in X with $f(0) = 0$, $h_1(0) = 0$, and $h_{2i}(0) = 0$ ($i = 1, \dots, q$). For convenience, we denote

$$\begin{aligned} u &= [u_1^T \quad u_2^T \quad \cdots \quad u_q^T]^T, \quad y = [y_1^T \quad y_2^T \quad \cdots \quad y_q^T]^T \\ w_e &= [w_0^T \quad w_1^T \quad \cdots \quad w_q^T]^T \\ g_2(x) &= [g_{21}(x) \quad g_{22}(x) \quad \cdots \quad g_{2q}(x)] \\ h_2(x) &= [h_{21}^T(x) \quad h_{22}^T(x) \quad \cdots \quad h_{2q}^T(x)]^T. \end{aligned}$$

The decentralized control structure constrains each control input u_i to be generated by an independent controller which uses only the corresponding measurement y_i .

Decentralized H_∞ -Controller Design (DHCD) Problem: Given the system Σ described by (1)–(3) and a positive constant γ , find controllers of the following form:

$$\dot{\xi}_i = a_i(\xi_i) + b_i(\xi_i)y_i, \quad \xi_i \in R^{v_i} \quad (4)$$

$$u_i = c_i(\xi_i), \quad i = 1, \dots, q \quad (5)$$

such that the resulting closed-loop system is locally asymptotically stable and has a local L_2 gain less than or equal to γ .

The following notion of detectability from [3] will be used in the sequel.

Definition 2.1: Suppose $f(0) = 0$ and $h(0) = 0$. The pair $\{f, h\}$ is said to be *locally detectable* if there exists a neighborhood U of the point $x = 0$ such that, if $x(t)$ is any integral curve of $\dot{x} = f(x)$ satisfying $x(0) \in U$, then $h(x(t))$ is defined for all $t \geq 0$ and $h(x(t)) = 0$ for all $t \geq 0$ implies $\lim_{t \rightarrow \infty} x(t) = 0$.

For some fundamental notions and results of nonlinear H_∞ -control theory, the reader is referred to [4] or [12].

III. MAIN RESULTS

The main results are presented in the following two subsections. Section III-A gives a solution to the DHCD problem, assuming that an observer gain is given. Then, in Section III-B, two methods are given for the observer gain design.

A. A Solution to the DHCD Problem

For the nonlinear system Σ described by (1)–(3) and a smooth positive definite function $V : R^n \rightarrow R_+$ [with its Jacobian matrix being $V_x(x)$], denote

$$\alpha_1(x) = \frac{1}{2\gamma^2} g_1^T(x) V_x^T(x) \quad (6)$$

$$\alpha_2(x) = -\frac{1}{2} g_2^T(x) V_x^T(x) = [\alpha_{21}^T(x) \quad \cdots \quad \alpha_{2q}^T(x)]^T \quad (7)$$

$$\bar{\alpha}_2(\xi) = [\alpha_{21}^T(\xi_1) \quad \alpha_{22}^T(\xi_2) \quad \cdots \quad \alpha_{2q}^T(\xi_q)]^T \in R^q \quad (8)$$

$$\bar{g}_1(\xi) = \text{diag}\{g_1(\xi_1), g_1(\xi_2), \dots, g_1(\xi_q)\} \quad (9)$$

$$\bar{h}_2(\xi) = \text{diag}\{h_{21}(\xi_1), h_{22}(\xi_2), \dots, h_{2q}(\xi_q)\} \quad (10)$$

$$\bar{f}(\xi) = \begin{bmatrix} f(\xi_1) + g_1(\xi_1)\alpha_1(\xi_1) + g_2(\xi_1)\alpha_2(\xi_1) - g_2(\xi_1)\bar{\alpha}_2(\xi) \\ f(\xi_2) + g_1(\xi_2)\alpha_1(\xi_2) + g_2(\xi_2)\alpha_2(\xi_2) - g_2(\xi_2)\bar{\alpha}_2(\xi) \\ \vdots \\ f(\xi_q) + g_1(\xi_q)\alpha_1(\xi_q) + g_2(\xi_q)\alpha_2(\xi_q) - g_2(\xi_q)\bar{\alpha}_2(\xi) \end{bmatrix} \quad (11)$$

where $\xi = [\xi_1^T \quad \xi_2^T \quad \cdots \quad \xi_q^T]^T$ is an nq dimensional vector with $\xi_i \in R^{n_i}$, and define the following matrix which is to be determined later:

$$\begin{aligned} \bar{L}(\xi) &= \text{diag}\{L_1(\xi_1), L_2(\xi_2), \dots, L_q(\xi_q)\}, \\ L_i(\xi_i) &\in R^{n_i \times p_i}, \quad \xi_i \in R^{n_i}. \end{aligned} \quad (12)$$

Then the following theorem presents a sufficient condition for the solution of the DHCD problem and gives a controller of order qn .

Theorem 3.1: Consider the system Σ described by (1)–(3) and a positive constant γ . Suppose that the following conditions hold.

- 1) The pair $\{f, h_1\}$ is locally detectable.
- 2) There exists a C^2 positive definite function $V(x)$, locally defined in a neighborhood of $x = 0$ and vanishing at $x = 0$, which satisfies the Hamilton–Jacobi inequality

$$\begin{aligned} H_s(x, V_x^T) &\triangleq V_x f(x) + h_1^T(x) h_1(x) + \gamma^2 \alpha_1^T(x) \alpha_1(x) \\ &\quad - \alpha_2^T(x) \alpha_2(x) \leq 0 \end{aligned} \quad (13)$$

where V_x is the Jacobian matrix of $V(x)$.

- 3) There exist $n \times p_i$ matrix-valued functions $L_i(\xi_i)$ ($i = 1, \dots, q$) such that the following Hamilton–Jacobi inequality admits a C^2 positive definite solution $Q(\xi)$ that is locally defined in a neighborhood of $\xi = 0$ and vanishing at $\xi = 0$

$$\begin{aligned} H_{\text{do}}(\xi, Q_\xi^T) &\triangleq Q_\xi [\bar{f}(\xi) - \bar{L}(\xi) \bar{h}_2(\xi)] + \bar{\alpha}_2^T(\xi) \bar{\alpha}_2(\xi) \\ &\quad + \frac{1}{4\gamma^2} Q_\xi \bar{g}_1(\xi) \bar{g}_1^T(\xi) Q_\xi^T + \frac{1}{4\gamma^2} Q_\xi \bar{L}(\xi) \bar{L}^T(\xi) Q_\xi^T \leq 0. \end{aligned} \quad (14)$$

Furthermore, the Hessian matrix of $H_{\text{do}}(\xi, Q_\xi^T)$ is nonsingular at $\xi = 0$.

Then there exist controllers of the form (4) and (5) that solves the DHCD problem for the system Σ . Furthermore, a particular such a controller of order nq is given by

$$\dot{\xi}_i = f(\xi_i) + g_1(\xi_i)\alpha_1(\xi_i) + g_2(\xi_i)\alpha_2(\xi_i) + L_i(\xi_i)(y_i - h_{2i}(\xi_i)) \quad (15)$$

$$u_i = \alpha_{2i}(\xi_i), \quad i = 1, \dots, q. \quad (16)$$

The following preliminaries are required in the proof of Theorem 3.1.

By applying the controllers given by (15) and (16) to the system Σ of (1)–(3), the resulting closed-loop system Σ_c is described by

$$\Sigma_c: \dot{x}_e = f_e(x_e) + g_e(x_e)w_e \quad (17)$$

$$z = [h_1^T(x) \quad \alpha_{21}^T(\xi_1) \quad \alpha_{22}^T(\xi_2) \quad \cdots \quad \alpha_{2q}^T(\xi_q)]^T \quad (18)$$

where $x_e = [x^T \quad \xi^T]^T$ is of dimension $n + nq$, and we have the equations shown at the bottom of the next page. Denote (19) and (20), as shown at the bottom of the next page. Then we have the following lemma.

Lemma 3.2: Let $W(x_e) = Q(\bar{x} - \xi)$, where $\bar{x} = [x^T \quad \cdots \quad x^T]^T \in R^{nq}$ with $x \in R^n$. Then under the assumptions of Theorem 3.1, there exists a neighborhood of $x_e = 0$ in which the following inequality holds:

$$W_{x_e} \bar{f}_e(x_e) + h_e^T(x_e) h_e(x_e) + \frac{1}{4\gamma^2} W_{x_e} g_e(x_e) g_e^T(x_e) W_{x_e}^T \leq 0. \quad (21)$$

Proof: Let $e = \bar{x} - \xi$ and $I_c = [I, \dots, I]^T \in R^{nq \times n}$ where $I \in R^{n \times n}$ is the identity matrix. Then $W_{x_e} = (Q_e(e)I_c, -Q_e(e))$. Note that

$$\begin{aligned} g_2(x)\bar{\alpha}_2(\xi) &= \sum_{i=1}^q g_{2i}(x)\alpha_{2i}(\xi_i) \\ &= (g_2(x) - g_2(\xi_j))\bar{\alpha}_2(\xi) \\ &\quad + g_2(\xi_j)\bar{\alpha}_2(\xi), \quad j = 1, \dots, q. \end{aligned}$$

By (17), (19) and (20), we have

$$\begin{aligned} H(e, x) &= W_{x_e}\bar{f}_e(x_e) + h_e^T(x_e)h_e(x_e) + \frac{1}{4\gamma^2}W_{x_e}g_e(x_e)g_e^T(x_e)W_{x_e}^T \\ &= Q_e(e) \begin{bmatrix} f(x) + g_1(x)\alpha_1(x) - f(\xi_1) - g_1(\xi_1)\alpha_1(\xi_1) \\ \vdots \\ f(x) + g_1(x)\alpha_1(x) - f(\xi_q) - g_1(\xi_q)\alpha_1(\xi_q) \end{bmatrix} \\ &\quad + Q_e(e) \begin{bmatrix} (g_2(x) - g_2(\xi_1))\bar{\alpha}_2(\xi) + g_2(\xi_1)\bar{\alpha}_2(\xi) - g_2(\xi_1)\alpha_2(\xi_1) \\ \vdots \\ (g_2(x) - g_2(\xi_q))\bar{\alpha}_2(\xi) + g_2(\xi_q)\bar{\alpha}_2(\xi) - g_2(\xi_q)\alpha_2(\xi_q) \end{bmatrix} \\ &\quad - Q_e(e)\bar{L}(\xi) \begin{bmatrix} h_{21}(x) - h_{21}(\xi_1) \\ \vdots \\ h_{2q}(x) - h_{2q}(\xi_q) \end{bmatrix} + [\bar{\alpha}_2(\bar{x}) - \bar{\alpha}_2(\xi)]^T \\ &\quad \times [\bar{\alpha}_2(\bar{x}) - \bar{\alpha}_2(\xi)] + \frac{1}{4\gamma^2}Q_e(e)I_c g_1(x)g_1^T(x)I_c^T Q_e^T(e) \\ &\quad + \frac{1}{4\gamma^2}Q_e(e)\bar{L}(\xi)\bar{L}^T(\xi)Q_e^T(e) \end{aligned} \quad (22)$$

where $\bar{L}(\cdot)$ is as defined in (12). From the above equality, it is easy to see that $H(0, x) = 0$, $[\frac{\partial H(e, x)}{\partial e}]_{e=0} = 0$. Thus, $H(e, x)$ can be expressed as $H(e, x) = e^T R(e, x)e$ for some continuous matrix-valued function $R(e, x)$. By (14) and (22), we have $R(0, 0) = [\frac{\partial^2 H_{do}(\xi, Q_\xi^T)}{\partial \xi^2}]_{\xi=0}$. Since the Hessian matrix of $H_{do}(\xi, Q_\xi^T)$ is negative definite, the function $H(e, x)$ is nonpositive in a neighborhood of $(e, x) = (0, 0)$, which further implies that the inequality (21) holds. \square

Lemma 3.3: Under the assumption 3) of Theorem 3.1, the equilibrium $\xi = 0$ of the system

$$\dot{\xi} = \bar{f}(\xi) - \bar{L}(\xi)\bar{h}_2(\xi) \quad (23)$$

is locally asymptotically stable.

The proof is straightforward and omitted.

Proof of Theorem 3.1: A straightforward calculation using (13), (16), (6), and (8) yields the following inequality:

$$\begin{aligned} &V_x[f(x) + g_1(x)w_0 + g_2(x)u] + \|h_1(x)\|^2 + \|u\|^2 \\ &\quad - \gamma^2\|w_0\|^2 - \gamma^2\|w\|^2 \\ &\leq \sum_{i=1}^q \|\alpha_{2i}(\xi_i) - \alpha_{2i}(x)\|^2 - \gamma^2\|w_0 - \alpha_1(x)\|^2 - \gamma^2\|w\|^2. \end{aligned} \quad (24)$$

Let

$$\begin{aligned} U(x_e) &= V(x) + W(x_e), \\ \bar{\alpha}_1(x) &= [[\alpha_1(x)]^T \quad 0 \quad \dots \quad 0]^T. \end{aligned}$$

Then, by using (17)–(19), (24), and Lemma 3.2, it follows that

$$\begin{aligned} &U_{x_e}(f_e(x_e) + g_e(x_e)w_e) + \|z\|^2 - \gamma^2\|w_e\|^2 \\ &= V_x[f(x) + g_1(x)w_0 + g_2(x)\bar{\alpha}_2(\xi)] + W_{x_e}[f_e(x_e) \\ &\quad + g_e(x_e)w_{x_e}] + \|z\|^2 - \gamma^2\|w_e\|^2 \\ &\leq \sum_{i=1}^q \|\alpha_{2i}(\xi_i) - \alpha_{2i}(x)\|^2 - \gamma^2\|w_0 - \alpha_1(x)\|^2 \\ &\quad - \gamma^2\|w\|^2 + W_{x_e}[f_e(x_e) + g_e(x_e)w_{x_e}] \\ &= -\gamma^2\|w_0 - \alpha_1(x)\|^2 - \gamma^2\|w\|^2 + h_e^T(x_e)h_e(x_e) \\ &\quad + W_{x_e}[\bar{f}_e(x_e) + g_e(x_e)(w_e - \bar{\alpha}_1(x))] \\ &\leq -\gamma^2\|w_0 - \alpha_1(x)\|^2 - \gamma^2\|w\|^2 + W_{x_e}\bar{f}_e(x_e) \\ &\quad + h_e^T(x_e)h_e(x_e) + \frac{1}{4\gamma^2}W_{x_e}g_e(x_e)g_e^T(x_e)W_{x_e}^T \\ &\quad + \gamma^2\|w_e - \bar{\alpha}_1(x)\|^2 \\ &= W_{x_e}\bar{f}_e(x_e) + h_e^T(x_e)h_e(x_e) \\ &\quad + \frac{1}{4\gamma^2}W_{x_e}g_e(x_e)g_e^T(x_e)W_{x_e}^T \leq 0. \end{aligned} \quad (25)$$

This shows that $U(x_e)$ is a storage function for the closed-loop system Σ_c , with respect to the supply rate $s = \gamma^2\|w_e\|^2 - \|z\|^2$. This further implies that the system Σ_c has an L_2 gain less than or equal to γ . Let $w_e = 0$. Then from inequality (25), we have

$$\frac{dU(x_e(t))}{dt} = U_{x_e}f_e(x_e) \leq -\|h_1(x(t))\|^2 - \|\bar{\alpha}_2(\xi(t))\|^2. \quad (26)$$

Since $U(x_e)$ is positive definite, it follows that the equilibrium $(x, \xi) = (0, 0)$ of the closed-loop system Σ_c is stable. To prove asymptotic stability, note that any trajectory satisfying $\frac{dU(x_e)}{dt} = 0$ is necessarily a trajectory of $\dot{x} = f(x) + g_2(x)\bar{\alpha}_2(\xi)$ such that $x(t)$ is

$$\begin{aligned} f_e(x_e) &= \begin{bmatrix} f(x) + \sum_{i=1}^q g_{2i}(x)\alpha_{2i}(\xi_i) \\ f(\xi_1) + g_1(\xi_1)\alpha_1(\xi_1) + g_2(\xi_1)\alpha_2(\xi_1) - L_1(\xi_1)h_{21}(\xi_1) + L_1(\xi_1)h_{21}(x) \\ \vdots \\ f(\xi_q) + g_1(\xi_q)\alpha_1(\xi_q) + g_2(\xi_q)\alpha_2(\xi_q) - L_q(\xi_q)h_{2q}(\xi_q) + L_q(\xi_q)h_{2q}(x) \end{bmatrix} \\ g_e(x_e) &= \text{diag}\{g_1(x), L_1(\xi_1), \dots, L_q(\xi_q)\} \end{aligned}$$

$$\begin{aligned} \bar{f}_e(x_e) &= f_e(x_e) + [[g_1(x)\alpha_1(x)]^T \quad 0 \quad \dots \quad 0]^T \\ &= \begin{bmatrix} f(x) + g_1(x)\alpha_1(x) + \sum_{i=1}^q g_{2i}(x)\alpha_{2i}(\xi_i) \\ f(\xi_1) + g_1(\xi_1)\alpha_1(\xi_1) + g_2(\xi_1)\alpha_2(\xi_1) - L_1(\xi_1)h_{21}(\xi_1) + L_1(\xi_1)h_{21}(x) \\ \vdots \\ f(\xi_q) + g_1(\xi_q)\alpha_1(\xi_q) + g_2(\xi_q)\alpha_2(\xi_q) - L_q(\xi_q)h_{2q}(\xi_q) + L_q(\xi_q)h_{2q}(x) \end{bmatrix} \end{aligned} \quad (19)$$

$$h_e(x_e) = [\alpha_{21}(x) - \alpha_{21}(\xi_1) \quad \dots \quad \alpha_{2q}(x) - \alpha_{2q}(\xi_q)]^T \quad (20)$$

bounded, and $h_1(x(t)) = 0$ and $\bar{\alpha}_2(\xi) = 0$ for all $t \geq 0$. From the detectability of the pair $\{f, h_1\}$, it follows that $\lim_{t \rightarrow \infty} x(t) = 0$. Thus, the ω -limit set of such a trajectory is a subset of

$$M = \{(x, \xi) : x = 0, \bar{\alpha}_2(\xi) = 0\}.$$

Any initial condition on this ω -limit set yields a trajectory in which $x(t) = 0$ for all $t \geq 0$, while from (17) and $\bar{\alpha}_2(\xi) = 0$, $\xi(t)$ is a trajectory of

$$\dot{\xi} = \bar{f}_0(\xi) - \bar{L}(\xi)\bar{h}_2(\xi) = \bar{f}(\xi) - \bar{L}(\xi)\bar{h}_2(\xi)$$

where

$$\bar{f}_0(\xi) = \begin{bmatrix} f(\xi_1) + g_1(\xi_1)\alpha_1(\xi_1) + g_2(\xi_1)\alpha_2(\xi_1) \\ f(\xi_2) + g_1(\xi_2)\alpha_1(\xi_2) + g_2(\xi_2)\alpha_2(\xi_2) \\ \vdots \\ f(\xi_q) + g_1(\xi_q)\alpha_1(\xi_q) + g_2(\xi_q)\alpha_2(\xi_q) \end{bmatrix}.$$

From Lemma 3.3, it follows that $\lim_{t \rightarrow \infty} \xi(t) = 0$ and by the invariance principle, the closed-loop system Σ_c is locally asymptotically stable. \square

Remark 3.4: Theorem 3.1 presents an approach to solve the DHCD problem for nonlinear systems. If $q = 1$, then Theorem 3.1 becomes [4, Lemma 3.2], which is a solution to the central H_∞ control problem for the system Σ . It should be noted that the inequality (14) in Theorem 3.1 still contains undetermined observer gains $L_i(\xi_i)$'s. The problem of how to design these observer gains will be treated in the next subsection.

B. Observer Gain Design

Now we look at how to design these observer gains $L_i(\xi_i)$, $i = 1, 2, \dots, q$. Two methods will be presented. The first method makes use of the centralized observer design result in [4], and the idea is similar to that of Paz [8] for linear local observers. But our result here is for nonlinear systems. The second method gives a decentralized observer design in terms of solutions to a matrix inequality that implies the Hamilton–Jacobi inequality (14) in condition 3) of Theorem 3.1.

B.1 The First Method: First, we review the result for nonlinear centralized observer design of [4]. Suppose that the assumptions 1) and 2) of Theorem 3.1 hold. Suppose that there exists a C^2 positive definite function $S(x)$, locally defined in a neighborhood of $x = 0$ and vanishing at $x = 0$, which satisfies the Hamilton–Jacobi inequality

$$H_o(x, S_x^T) \triangleq S_x[f(x) + g_1(x)\alpha_1(x)] + \alpha_2^T(x)\alpha_2(x) - \gamma^2 h_2^T(x)h_2(x) + \frac{1}{4\gamma^2} S_{xx}g_1(x)g_1^T(x)S_x^T \leq 0$$

and the Hessian matrix of $H_o(x, S_x^T)$ is nonsingular at $x = 0$. Suppose also that there exists a C^2 matrix-valued function $L(x) \in R^{n \times p}$ with $p = \sum_{i=1}^q p_i$ such that

$$S_x L(x) = 2\gamma^2 h_2^T(x). \quad (27)$$

Then, from in [4, Th. 3.1], the centralized controller

$$\begin{aligned} \dot{\eta} &= f(\eta) + g_1(\eta)\alpha_1(\eta) + g_2(\eta)\alpha_2(\eta) + L(\eta)(y - h_2(\eta)) \\ u &= \alpha_2(\eta) \end{aligned}$$

locally stabilizes the system Σ and guarantees that the resulting closed-loop system has an L_2 gain less than or equal to γ .

For decentralized nonlinear control, we have the following result. Note that, in the decentralized case, $L(x) = [L_1(x) \ L_2(x) \ \dots \ L_q(x)]$ with $L_i(x) \in R^{n \times p_i}$, $i = 1, \dots, q$, and condition (27) becomes

$$S_x L_i(x) = 2\gamma^2 h_{2i}^T(x), \quad i = 1, \dots, q. \quad (28)$$

Then by using Theorem 3.1, we have the following corollary which is an extension of Theorem 2.1 in [8] for the linear DHCD problem.

Corollary 3.5: Suppose that assumptions 1) and 2) of Theorem 3.1 hold, and that the local observer gains $L_i(\xi_i)$, $i = 1, 2, \dots, q$ given in (28) are such that the Hamilton–Jacobi inequality (14) has a C^2 local positive definite solution $Q(\xi)$ in a neighborhood of $\xi = 0$, vanishing at $\xi = 0$, and the Hessian matrix of $H_{do}(\xi, Q_\xi^T)$ is nonsingular at $\xi = 0$. Then the controller given by (15) and (16) with the above observer gains solves the DHCD problem for the nonlinear system Σ .

B.2 The Second Method: Next, we present the second approach to the observer gain design. Since $f(x)$ and $h_{2i}(x)$ ($i = 1, \dots, q$) are smooth functions with $f(0) = 0$ and $h_{2i}(0) = 0$, there exist smooth matrix-valued functions $A(x)$ and $C_{2i}(x)$ ($i = 1, \dots, q$) such that

$$f(x) = A(x)x, \quad h_{2i}(x) = C_{2i}(x)x. \quad (29)$$

Theorem 3.6: Under conditions 1) and 2) of Theorem 3.1, we assume that $V_x = 2x^T P(x)$ with $P(x)$ being a C^2 matrix-valued function. Furthermore, we assume that [in place of 3) of Theorem 3.1] there exists a C^2 matrix-valued functions $T(\xi)$, locally defined and nonsingular in a neighborhood of $\xi = 0$, of the form¹

$$T(\xi) = \begin{bmatrix} T_{11}(\xi_1) & T_{12}(\xi) & \cdots & T_{1q}(\xi) \\ T_{21}(\xi) & T_{22}(\xi_2) & \cdots & T_{2q}(\xi) \\ \vdots & \vdots & \ddots & \vdots \\ T_{q1}(\xi) & T_{q2}(\xi) & \cdots & T_{qq}(\xi_q) \end{bmatrix}, \quad T_{ii}(\xi_i) \in R^{n \times n} \quad (30)$$

that satisfies the matrix inequality

$$\begin{aligned} &T(\xi)\bar{A}_c^T(\xi) + \bar{A}_c(\xi)T^T(\xi) + T(\xi)K_c^T(\xi)K_c(\xi)T^T(\xi) \\ &\quad - \gamma^2 T(\xi)\bar{C}^T(\xi)\bar{C}(\xi)T^T(\xi) + \gamma^2 [T(\xi) \\ &\quad - T_D(\xi)]\bar{C}^T(\xi)\bar{C}(\xi)[T(\xi) - T_D(\xi)]^T + \frac{1}{\gamma^2} \bar{g}_1(\xi)\bar{g}_1^T(\xi) < 0 \end{aligned} \quad (31)$$

where $\bar{g}_1(\xi)$ is as defined in (9), and there exists a positive definite function $Q(\xi)$ with $Q(0) = 0$ such that $Q_\xi = 2\xi^T T^{-1}(\xi)$ where

$$\begin{aligned} T_D(\xi) &= \text{diag}\{T_{11}(\xi_1), T_{22}(\xi_2), \dots, T_{qq}(\xi_q)\} \\ \bar{A}_c(\xi) &= \text{diag}\{\bar{A}(\xi_1), \bar{A}(\xi_2), \dots, \bar{A}(\xi_q)\} \end{aligned} \quad (32)$$

$$- \begin{bmatrix} g_2(\xi_1) \\ \vdots \\ g_2(\xi_q) \end{bmatrix} \text{diag}\{g_{21}^T(\xi_1)P^T(\xi_1), g_{22}^T(\xi_2)P^T(\xi_2), \dots, g_{2q}^T(\xi_q)P^T(\xi_q)\} \quad (33)$$

$$\bar{A}(x) = A(x) + \frac{1}{\gamma^2} g_1(x)g_1^T(x)P^T(x) - g_2(x)g_2^T(x)P^T(x) \quad (34)$$

$$\begin{aligned} K_c(\xi) &= \text{diag}\{-g_{21}^T(\xi_1)P^T(\xi_1), -g_{22}^T(\xi_2)P^T(\xi_2), \dots, \\ &\quad - g_{2q}^T(\xi_q)P^T(\xi_q)\} \end{aligned} \quad (35)$$

$$\bar{C}(\xi) = \text{diag}\{C_{21}(\xi_1), C_{22}(\xi_2), \dots, C_{2q}(\xi_q)\}. \quad (36)$$

Denote

$$\bar{L}(\xi) = \gamma^2 T_D(\xi)\bar{C}^T(\xi). \quad (37)$$

Then the controller given by (15) and (16) with the observer gain as specified by (37) and (12) solves the DHCD problem for the system Σ .

¹Note the special structure in $T(\xi)$ that the diagonal entries $T_{ii}(\xi_i)$ is a function of ξ_i only, not ξ !

Proof: From inequality (32), and definitions (32) and (37), it follows that

$$\begin{aligned} & T(\xi)[\bar{A}_c(\xi) - \bar{L}(\xi)\bar{C}(\xi)]^T + [\bar{A}_c(\xi) - \bar{L}(\xi)\bar{C}(\xi)]T^T(\xi) \\ & + T(\xi)K_c^T(\xi)K_c(\xi)T^T(\xi) + \frac{1}{\gamma^2}\bar{g}_1(\xi)\bar{g}_1^T(\xi) \\ & + \frac{1}{\gamma^2}\bar{L}(\xi)\bar{L}^T(\xi) < 0 \end{aligned}$$

which further implies that

$$\begin{aligned} & [\bar{A}_c(\xi) - \bar{L}(\xi)\bar{C}(\xi)]^T[T^{-1}(\xi)]^T + T^{-1}(\xi)[\bar{A}_c(\xi) - \bar{L}(\xi)\bar{C}(\xi)] \\ & + K_c^T(\xi)K_c(\xi) + \frac{1}{\gamma^2}T^{-1}(\xi)\bar{g}_1(\xi)\bar{g}_1^T(\xi)[T^{-1}(\xi)]^T \\ & + \frac{1}{\gamma^2}T^{-1}(\xi)\bar{L}(\xi)\bar{L}^T(\xi)[T^{-1}(\xi)]^T < 0. \end{aligned}$$

By using definitions (29) and (33)–(36), and by comparing with $H_{do}(\xi, Q_\xi^T)$ in (14), we can see that the above strict inequality implies that condition 3) of Theorem 3.1 is satisfied. Also, the nonsingularity of the Hessian matrix of $H_{do}(\xi, Q_\xi^T)$ is implied by the above strict inequality. Thus, the proof is completed from Theorem 3.1. \square

It should be noted that, apart from the special structure as mentioned in the footnote to Theorem 3.6, $T(\xi)$ must satisfy the matrix inequality (31) and must also be related to a gradient function [i.e., $Q_\xi = 2\xi^T T^{-1}(\xi)$]. In general, $T(\xi)$ may not be symmetric. Therefore, the task of solving for such a $T(\xi)$ is indeed a nontrivial one. But a constant solution (which renders a linear observer gain) can be obtained easily as follows.

Corollary 3.7: Under conditions 1) and 2) of Theorem 3.1, we assume that $V_x = 2x^T P(x)$ with $P(x)$ being a C^2 matrix-valued function. Furthermore, we assume that [in place of 3) of Theorem 3.1] there exists a symmetric positive definite matrix \bar{T} that satisfies the matrix inequality

$$\begin{aligned} & \bar{T}\bar{A}_c^T(0) + \bar{A}_c(0)\bar{T} + \bar{T}K_c^T(0)K_c(0)\bar{T} - \gamma^2\bar{T}\bar{C}^T(0)\bar{C}(0)\bar{T} \\ & + [\bar{T} - \bar{T}_D]\bar{C}^T(0)\bar{C}(0)[\bar{T} - \bar{T}_D] + \frac{1}{\gamma^2}\bar{g}_1(0)\bar{g}_1^T(0) < 0 \end{aligned} \quad (38)$$

where

$$\begin{aligned} \bar{T} &= \begin{bmatrix} \bar{T}_{11} & \bar{T}_{12} & \cdots & \bar{T}_{1q} \\ \bar{T}_{12} & \bar{T}_{22} & \cdots & \bar{T}_{2q} \\ \vdots & \vdots & \ddots & \vdots \\ \bar{T}_{1q} & \bar{T}_{2q} & \cdots & \bar{T}_{qq} \end{bmatrix}, \quad \bar{T}_D = \text{diag}\{\bar{T}_{11}, \bar{T}_{22}, \dots, \bar{T}_{qq}\} \\ T_{ii} &= T_{ii}^T \in R^{n \times n}. \end{aligned} \quad (39)$$

Denote

$$\bar{L} = \gamma^2 \bar{T}_D \bar{C}^T(0). \quad (40)$$

Then the controller given by (15) and (16) with the above observer gain in (40) solves the DHCD problem for the system Σ .

Remark 3.8: Theorem 3.6 presents a design method for local observer gains $L_i(\xi_i)$ ($i = 1, \dots, q$), which is based on the existence of solutions of the form (30) to the nonlinear matrix inequality (31). From Corollary 3.7, the linear local observer gains can be designed by solving a linear matrix inequality. These results generalize to nonlinear decentralized control systems the results given in [13] for linear decentralized control systems. The results in [13] are given in terms of solutions of modified algebraic Riccati equations.

IV. AN EXAMPLE

Consider the following nonlinear system:

$$\dot{x} = \begin{bmatrix} -2x_1 + x_1x_2^2 \\ x_2^3 \end{bmatrix} x + \begin{bmatrix} 1 \\ x_1 \end{bmatrix} w_0 + \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

TABLE I

HERE “—” INDICATES THAT THE SYSTEM BECOMES UNSTABLE

γ_{\max}	$A = 0.1$	$A = 0.17$	$A = 0.18$	$A = 0.23$
Linear	0.1726	0.1732	—	—
Nonlinear	0.1726	0.1728	0.1728	0.1767

$$\begin{aligned} z &= [x_1 \quad x_2^4 \quad u_1 \quad u_2]^T \\ y &= \begin{bmatrix} 2x_1 + 2x_2 \\ 2x_1 \end{bmatrix} + \begin{bmatrix} w_{11} \\ w_{12} \end{bmatrix} \end{aligned}$$

which is of the form Σ in (1)–(3). Hence, $q = 2$. We will design a decentralized controller for the above system by using Theorem 3.1 and Corollary 3.7.

For this example, it is easy to check that $\{f, h_1\}$ is locally detectable. By solving (13), the following positive definite solutions are obtained with a minimum γ value of $\gamma_d = 0.48$:

$$\begin{aligned} V(x) &= 0.3642x_1^2 - 0.03706x_1x_2 + 0.04331x_2^2 \\ &\quad - 0.04442x_1^3 + 0.1324x_1^2x_2 - 0.08808x_1x_2^2 \\ &\quad + 0.04536x_2^3 + 0.1612x_1^4 - 0.7842x_1^3x_2 \\ &\quad + 2.1051x_1^2x_2^2 - 2.1938x_1x_2^3 + 1.1772x_2^4. \end{aligned}$$

The observer gain is taken as linear by solving (38) of Corollary 3.7

$$\bar{L} = \begin{bmatrix} L_1 & 0 \\ 0 & L_2 \end{bmatrix} = \begin{bmatrix} 0.3645 & 0.4029 & 0 & 0 \\ 0 & 0 & 0.6231 & -0.2519 \end{bmatrix}^T.$$

Finally the controller is given by (15) and (16) with $\alpha_1(\xi_i)$ and $\alpha_2(\xi_i)$ as defined in (6) and (7).

As there is no viable method for computing the H_∞ -norm of an affine nonlinear system, the following approximate L_2 -gain γ_{\max} is computed:

$$\gamma_{\max} = \max_{t \geq 0} \sqrt{\frac{\int_0^t [z_1^2(\tau) + z_2^2(\tau) + z_3^2(\tau) + z_4^2(\tau)] d\tau}{\int_0^t [w_0^2(\tau) + w_{11}^2(\tau) + w_{12}^2(\tau)] d\tau}}.$$

If the H_∞ -norm of the closed-loop system is γ_r , then obviously $0 \leq \gamma_{\max} \leq \gamma_r \leq \gamma_d$.

In computer simulations, the plant disturbance $w_0(t)$ is taken as $w_0(t) = A \cdot \sin(4\pi t)$, the measurement noise $w_{11}(t)$ is taken as a square wave of an amplitude $\pm A$ and a frequency 10 Hz, and $w_{12}(t)$ is taken as a saw-tooth wave of an amplitude A and a frequency of 7 Hz. Here the amplitude A is variable. The initial conditions of the system and the controller are all set to zero. The simulation results are given in Table I. Here, “Nonlinear” refers to the controller that we designed just now, and “Linear” refers to the controller after dropping all nonlinear terms in our nonlinear controller (namely, the controller by using the linear methods of [13]).

It can be seen from Table I that, for small disturbances/noises ($A = 0.1$), the linear and the nonlinear controllers give the same closed-loop disturbance attenuation γ_{\max} . Also the L_2 -attenuation γ_{\max} is less than the designed value of $\gamma_d = 0.48$. When the disturbances/noises are increased to $A = 0.18$, the linear controller fails to stabilize the system, but the nonlinear controller not only stabilizes the system but also provides adequate disturbance attenuation. In fact, the nonlinear controller can meet the design specification for a disturbance level of up to $A = 0.23$.

Of course, the nonlinear method presented in this paper is a local one. This is confirmed by the simulation result that if the disturbance/noise level is increased beyond $A = 0.23$, the nonlinear controller fails to stabilize the system.

V. CONCLUSION

This paper presents a sufficient condition for the *DHCD* problem for nonlinear systems. The resulting controller guarantees local asymptotic stability and provides a predetermined L_2 -gain bound on the closed-loop system. Two design methods of the local observers are given: one is based on the centralized observer gain and another one is related to the solution of the matrix inequalities. The results are extensions of those in [8] and [13] for the case of linear systems.

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A Methodology for the Design of Optimal Traffic Shapers in Communication Networks

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Abstract—The authors consider the problem of optimally regulating the source traffic in a communication network to simultaneously satisfy a finite number of affine burstiness constraints. They prove that an optimal solution is a series connection of correspondingly dimensioned "leaky buckets." They propose a simple "fork-join" implementation of the optimal solution and study extensions to the problem of optimally shaping the traffic flow to meet a burstiness constraint specified by a concave increasing function. A consequence of their optimality results is that permutations of leaky buckets in a series connection are input–output equivalent.

Index Terms—Communication networks, flow control, Skorokhod reflection mapping.

I. PRELIMINARIES

In this paper we consider the problem of designing flow control schemes in a communication network. Flow control is necessary for the regulation and shaping of a source traffic stream, which must interact and share network resources with other traffic streams after it is admitted. Therefore, one normally requires the admitted flow to satisfy certain "burstiness" or "shaping" constraints. It is also desirable that the controller be optimal, in that the offered traffic is transmitted as quickly as possible.

A general model for a traffic process is a nonnegative sigma-finite Borel measure A on the time axis \mathbb{R}_+ . This is represented by an increasing right-continuous process $\{A_t, t \geq 0\}$; the interpretation is that for $0 \leq s \leq t$, $A_t - A_s$ gives the volume of traffic (in cells) on the time interval $(s, t]$. Write \mathcal{M} for the collection of such processes. We write A^S for the restriction of A on $S \subseteq \mathbb{R}_+$, defined by $A_t^S := \int_{S \cap [0, t]} dA_s$. We also define a partial ordering on \mathcal{M} by $A \leq B \iff A_t \leq B_t$, for all $t \geq 0$. We say that $A \in \mathcal{M}$ is $(\sigma_{i0}, \sigma_i, \rho_i)_{i=1, \dots, n}$ constrained iff, for all $0 \leq s \leq t$

$$A_t \leq \min_{1 \leq i \leq n} \{\sigma_{i0} + \rho_i t\}, \quad A_t - A_s \leq \min_{1 \leq i \leq n} \{\sigma_i + \rho_i(t - s)\}. \quad (1)$$

Here, $\sigma_i \geq \sigma_{i0} \geq 0$, $\rho_i \geq 0$, for all i . For $n = 1$ we simply say that A is (σ_0, σ, ρ) constrained. The above definitions are discussed in Anantharam [1] and Cruz [4], [5], and they also closely match the standard shaping descriptors that have been adopted in practice for high-speed networks. More generally, for f_0, f arbitrary concave increasing functions from \mathbb{R}_+ into \mathbb{R}_+ , we say that A is (f_0, f) constrained iff

$$A_t \leq f_0(t), \quad A_t - A_s \leq f(t - s). \quad (2)$$

Of course, (1) is a convenient special case of (2). A *traffic regulator*, or *flow controller* is simply a map $\varphi: \mathcal{M} \rightarrow \mathcal{M}$. Some properties that such a map may possess are as follows.

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