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<th>Title</th>
<th>Decentralized H ∞-controller design for nonlinear systems</th>
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<td>Author(s)</td>
<td>Yang, GH; Wang, J; Soh, CB; Lam, J</td>
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the PPC policy. In this case, one approach is to identify a stable service period and use it as a supervisor for another, more intuitive (e.g., PCLB) policy. As a supervisor, the purpose of the PPC policy is simply to guarantee stability of the system. In practice, the supervising PPC policy is invoked when some measure of system performance (e.g., the sum of the buffer levels) exceeds some preset threshold (i.e., once the sum of FMS buffer levels grows beyond some preset limit, a predefined, stable periodic service sequence is implemented).

Next, note that in simulation studies for the policies introduced here we have uncovered some interesting points [5]. First, in attempting to formulate a general rule of thumb for determining the suitability of PC control for a given FMS, we make the following observation: The less variance there is among processing rates along individual paths, the better PC policies will perform (with respect to distributed policies). The reason for this is that because PC policies mandate that all buffers on a given path be processed at a single rate (the minimum processing rate of all buffers on the path), any buffers on the path that are able to be processed at a faster rate than the minimum processing rate are constrained to be processed at a lower rate than they would be processed at in a distributed control scheme. In general, then, for systems with very high processing rate “skew” along individual paths, we may be wiser to choose a distributed policy. However, it may be possible to choose paths intelligently so as to minimize the adverse affects of processing rate skew.

Finally, we would like to emphasize that PC policies will not yield stability for all FMS that would be stable under a distributed scheduling approach where the FMS satisfies a capacity constraint. For example, if there is a high amount of “processing rate skew” along the paths PC policies may not be stable and a distributed policy may be.

REFERENCES


Decentralized $H_{\infty}$-Controller Design for Nonlinear Systems

Guang-Hong Yang, Jianliang Wang, C. B. Soh, and James Lam

Abstract—This paper considers the decentralized $H_{\infty}$-controller design problem for nonlinear systems. Sufficient conditions for the solution of the problem are presented in terms of solutions of Hamilton–Jacobi inequalities. The resulting design guarantees local asymptotic stability and ensures a predetermined $L_{\infty}$-gain bound on the closed-loop system.

Index Terms—Decentralized control, Hamilton–Jacobi inequality, nonlinear $H_{\infty}$ control, nonlinear system.

I. INTRODUCTION

In the area of the decentralized control of large scale systems, numerous important advances have been accomplished in the past two decades [9], [11]. Recently, the decentralized $H_{\infty}$-control problem for linear systems has been considered in [7], [8], [10], and [13]. In particular, Veillette et al. [13] presented a decentralized $H_{\infty}$-controller design procedure in terms of solutions of the modified algebraic Riccati equations, and the result has also been extended to discrete-time linear systems [7]. In [8], another sufficient condition for the decentralized $H_{\infty}$-control problem is derived, under which the decentralized solution can be constructed from the central controller solution from the standard $H_{\infty}$-control theory in [2].

In recent years, the problem of central controller design to solve the $H_{\infty}$-control problem (or in short, the central $H_{\infty}$-control problem) for nonlinear systems has been extensively investigated by several authors [1], [3]–[6], [12]. In particular, Van der Schaft [12] has shown that the solution of the $H_{\infty}$-control problem via state feedback can be determined from the solution of a Hamilton–Jacobi equation (or inequality), which is the nonlinear version of the Riccati equation for the corresponding linear $H_{\infty}$-control problem. In the case of measurement feedback, a set of sufficient conditions has also been given in [1], [4], and [6] in terms of the solutions of a pair of Hamilton-Jacobi inequalities, and the necessity of these sufficient conditions has been discussed in [1] and [5].

In this paper, we consider the decentralized $H_{\infty}$-control problem for nonlinear systems by using the Hamilton–Jacobi inequality approach. The results given in this paper are extensions of existing results on the linear decentralized $H_{\infty}$-control problem [13], [8] and nonlinear central $H_{\infty}$-control problem [4]. The paper is organized as follows. The system description and problem statement are given in Section II. The main results are given in Section III, followed by a numerical example in Section IV to illustrate the design procedure and the effectiveness of the proposed method. Finally, some concluding remarks are given in Section V.
II. PROBLEM STATEMENT

Consider a nonlinear system $\Sigma$ described by equations of the form

$$\dot{x} = f(x) + g_1(x)w_0 + \sum_{i=1}^{n_q} g_{2i}(x)u_i,$$  
(1)

$$z = [h_1(x) \quad u_1 \quad \cdots \quad u_q]^T,$$  
(2)

$$y_i = h_{2i}(x) + w_{ei}, \quad i = 1, \ldots, q,$$  
(3)

where $x$ is a state vector defined in a neighborhood $X$ of the origin in $\mathbb{R}^n$; the $u_i$'s are the local control inputs with $u_i \in \mathbb{R}^{n_i}$; the $y_i$'s are the local measurement with $y_i \in \mathbb{R}^{n_i}$; $w_0$ and the $w_i$'s are square-integrable disturbances; and $z$ is an output to be regulated. The functions $f(x)$, $g_1(x)$, $h_1(x)$, $g_{2i}(x)$, and $h_{2i}(x)$ ($i = 1, \ldots, q$) are all known smooth mappings of appropriate dimensions, defined in $X$ with $f(0) = 0$, $h_1(0) = 0$, and $h_{2i}(0) = 0$ ($i = 1, \ldots, q$). For convenience, we denote

$$u = [u_1^T \quad u_2^T \quad \cdots \quad u_q^T]^T,$$  

$$w_e = [w_0^T \quad w_1^T \quad \cdots \quad w_q^T]^T,$$  

$$g_{2i}(x) = [g_{21}(x) \quad g_{22}(x) \quad \cdots \quad g_{2q}(x)]^T,$$  

$$h_{2i}(x) = [h_{21}(x) \quad h_{22}(x) \quad \cdots \quad h_{2q}(x)]^T.$$  

The decentralized control structure constrains each control input $u_i$ to be generated by an independent controller which uses only the corresponding measurement $y_i$.

Decentralized $H_{\infty}$-Controller Design (DHCD) Problem:

Given the system $\Sigma$ described by (1)–(3) and a positive constant $\gamma$, find controllers of the following form:

$$\dot{\xi}_i = a_i(\xi_i) + b_i(\xi_i)y_i, \quad \xi_i \in \mathbb{R}^{n_i},$$  
(4)

$$u_i = c_i(\xi_i), \quad i = 1, \ldots, q,$$  
(5)

such that the resulting closed-loop system is locally asymptotically stable and has a local $L_2$ gain less than or equal to $\gamma$.

The following notion of detectability from [3] will be used in the sequel.

Definition 2.1: Suppose $f(0) = 0$ and $h(0) = 0$. The pair $\{f, h\}$ is said to be locally detectable if there exists a neighborhood $U$ of the point $x = 0$ such that, if $x(t)$ is any integral curve of $\dot{x} = f(x)$ satisfying $x(0) \in U$, then $h(x(t))$ is defined for all $t \geq 0$ and $h(x(t)) \neq 0$ for all $t \geq 0$ implies $\lim_{t \to \infty} x(t) = 0$.

For some fundamental notions and results of nonlinear $H_{\infty}$-control theory, the reader is referred to [4] or [12].

III. MAIN RESULTS

The main results are presented in the following two subsections. Section III-A gives a solution to the DHCD problem, assuming that an observer gain is given. Then, in Section III-B, two methods are given for the observer gain design.

A. A Solution to the DHCD Problem

For the nonlinear system $\Sigma$ described by (1)–(3) and a smooth positive definite function $V : \mathbb{R}^n \to \mathbb{R}_+$ [with its Jacobian matrix being $V_x(x)$], denote

$$\alpha_1(x) = \frac{1}{2}g_{21}^T(x)V_x(x)$$  

$$\alpha_2(x) = \frac{1}{2}g_{22}^T(x)V_x(x) = [\alpha_{21}(x) \cdots \alpha_{2q}(x)]^T,$$  

$$\gamma_1(\xi) = \left[\begin{array}{c} g_{21}(\xi_1) \\ g_{22}(\xi_2) \\ \vdots \\ g_{2q}(\xi_q) \end{array}\right],$$  

$$\gamma_2(\xi) = \left[\begin{array}{c} \gamma_1(\xi_1) \\ \gamma_2(\xi_2) \\ \vdots \\ \gamma_q(\xi_q) \end{array}\right],$$  

$$\gamma(\xi) = \left[\begin{array}{c} \gamma_1(\xi) \\ \gamma_2(\xi) \end{array}\right] \in \mathbb{R}^{n_q}.$$  

The following theorem presents a sufficient condition for the solution of the DHCD problem and gives a controller of order $\gamma n_q$.

Theorem 3.1: Consider the system $\Sigma$ described by (1)–(3) and a positive constant $\gamma$. Suppose that the following conditions hold.

1) The pair $\{f, h_1\}$ is locally detectable.

2) There exists a $C^2$ positive definite function $V(x)$, locally defined in a neighborhood of $x = 0$ and vanishing at $x = 0$, which satisfies the Hamilton–Jacobi inequality

$$H_\gamma(x, V_x) \triangleq V_x f(x) + h_1^T(x)h_1(x) + \gamma^2\alpha_1^T(x)\alpha_1(x) - \alpha_2^T(x)\alpha_2(x) \leq 0$$  

where $V_x$ is the Jacobian of $V(x)$.

3) There exist $n \times p_i$ matrix-valued functions $L_i(\xi_i)$ ($i = 1, \ldots, q$) such that the following Hamilton–Jacobi inequality admits a $C^2$ positive definite solution $Q(\xi)$ which is locally defined in a neighborhood of $\xi = 0$ and vanishing at $\xi = 0$

$$H_{\gamma,0}(\xi, Q(\xi)) \triangleq Q_{\xi} f(\xi) - \bar{L}(\xi)h_2(\xi) + \bar{a}_2^T(\xi)\gamma_2(\xi)$$  

$$+ \frac{1}{4\gamma^2} Q_{\xi} \bar{L}(\xi)Q_{\xi}^T + \frac{1}{4\gamma^2} \gamma_2^T(\xi)\bar{L}(\xi)Q_{\xi}^T \leq 0.$$  

Furthermore, the Hessian matrix of $H_{\gamma,0}(\xi, Q(\xi))$ is nonsingular at $\xi = 0$.

Then there exist controllers of the form (4) and (5) that solves the DHCD problem for the system $\Sigma$. Furthermore, a particular such a controller of order $\gamma n_q$ is given by

$$\dot{\xi}_i = f(\xi_i) + g_1(\xi_i)\alpha_1(\xi_i) + g_2(\xi_i)\alpha_2(\xi_i) + L_i(\xi_i)(y_i - h_2(\xi_i)),$$  

$$u_i = \alpha_2(\xi_i), \quad i = 1, \ldots, q.$$  

The following preliminaries are required in the proof of Theorem 3.1.

By applying the controllers given by (15) and (16) to the system $\Sigma$ of (1)–(3), the resulting closed-loop system $\Sigma_c$ is described by

$$\dot{x} = f_c(x) + g_1(x)w_e,$$  

$$z = [h_{21}^T(x) \quad \alpha_{21}^T(\xi_1) \quad \alpha_{22}^T(\xi_2) \quad \cdots \quad \alpha_{2q}^T(\xi_q)]^T$$  

where $x = [x^T \quad \xi]^T$ is of dimension $n + n_q$, and we have the equations shown at the bottom of the next page. Denote (19) and (20), as shown at the bottom of the next page. Then we have the following lemma.

Lemma 3.2: Let $W(x, \epsilon) = Q(\bar{x} - \xi)$, where $\bar{x} = [x^T \quad \xi]^T \in \mathbb{R}^{n+n_q}$ with $x \in \mathbb{R}^n$. Then under the assumptions of Theorem 3.1, there exists a neighborhood of $x = 0$ in which the following inequality holds:

$$W_{x_\epsilon}f_c(x_\epsilon) + h_{21}^T(x_\epsilon)h_2(x_\epsilon) + \frac{1}{4\gamma^2} W_{x_\epsilon}g_1(x_\epsilon)g_{11}^T(x_\epsilon)W_{x_\epsilon}^T \leq 0.$$  

(21)
Proof: Let \( e = \bar{x} - \xi \) and \( I_t = [I \cdots I]^T \in \mathbb{R}^{n \times n} \) where \( I \in \mathbb{R}^{n \times n} \) is the identity matrix. Then \( W_{ee} = (Q_t(e)I_t - Q_t(e)) \).

Note that
\[
g_2(x)\bar{\alpha}_2(\xi) = \sum_{i=1}^{q} g_2(x)\alpha_2(\xi_i) = (g_2(x) - g_2(\xi))\bar{\alpha}_2(\xi) + g_2(\xi)\bar{\alpha}_2(\xi), \quad j = 1, \ldots, q.
\]

By (17), (19) and (20), we have
\[
H(e, x) = W_{ee}\bar{f}_t(x) + h_t(e, x)g_t(x) + \frac{1}{4\gamma^2} W_{ee}\theta(x, y)^T(x)W_{ee}^T
\]
\[
= Q_t(e)\left[ f(x) + g_1(x)\alpha_1(x) - f(\xi) - g_1(\xi)\alpha_1(\xi) \right] + \cdots + Q_t(e)\left[ (g_2(x) - g_2(\xi))\bar{\alpha}_2(\xi) + g_2(\xi)\bar{\alpha}_2(\xi) - g_2(\xi)\alpha_2(\xi) \right] - Q_t(e)\left[ h_{21}(x) - h_{21}(\xi) \right]
\]
\[
= Q_t(e)\left[ \bar{L}(\xi) \right] + [\bar{\alpha}_2(\bar{x}) - \bar{\alpha}_2(\xi)]^T + \frac{1}{4\gamma^2} Q_t(e)I_t h(x)g_t(x)^TQ_t(e)
\]
\[
\frac{1}{4\gamma^2} Q_t(e)\bar{L}(T)Q_t(e)
\] (22)

where \( \bar{L}(\xi) \) is as defined in (12). From the above equality, it is easy to see that \( H(0, x) = 0 \) and \( \frac{\partial H(e, x)}{\partial e} = 0 \). Thus, \( H(e, x) \) can be expressed as \( H(e, x) = e^T R(e, x) e \) for some continuous matrix-valued function \( R(e, x) \). By (14) and (22), we have \( R(0, 0) = 0 \). Since the Hessian matrix of \( H_{ee}(\xi, Q_t^T) \) is negative definite, the function \( H(e, x) \) is nonpositive in a neighborhood of \( (e, x) = (0, 0) \), which further implies that the inequality (21) holds.

Lemma 3.3: Under the assumption 3) of Theorem 3.1, the equilibrium \( \xi = 0 \) of the system
\[
\dot{x} = \bar{f}(\xi) - \bar{L}(\xi)\bar{\alpha}_2(\xi)
\]
(23)
is locally asymptotically stable. The proof is straightforward and omitted.

Proof of Theorem 3.1: A straightforward calculation using (13), (16), (6), and (8) yields the following inequality:
\[
V_t[f(x) + g_1(x)\alpha_1(x) + g_2(x)\alpha_2(x) + ||h_1(x)||^2 + ||\bar{\alpha}_2(x)||^2 - \gamma^2||\alpha_2(x)||^2 - \gamma^2||\bar{\alpha}_2(x)||^2] 
\]
\[
\leq \sum_{i=1}^{q} ||\alpha_2(x_i) - \alpha_2(x)||^2 - \gamma^2||\alpha_2(x)||^2 - \gamma^2||\bar{\alpha}_2(x)||^2.
\]
(24)

Let
\[
U(x_e) = V(x) + W(x_e),
\]
\[
\bar{\alpha}_1(x) = ||\alpha_1(x)||^T 0 \cdots 0^T
\]

Then, by using (17)–(19), (24), and Lemma 3.2, it follows that
\[
U_{ee}(f(x) + g_2(x), w, ||z||^2 - \gamma^2||w||^2)
\]
\[
= V_t[f(x) + g_1(x)\alpha_1(x) + g_2(x)\bar{\alpha}_2(x)] + W_{ee}[f_t(x), g_1(x)\alpha_1(x) + g_2(x)\bar{\alpha}_2(x) - g_1(\xi)\alpha_1(\xi) - g_2(\xi)\bar{\alpha}_2(\xi)]
\]
\[
\leq \sum_{i=1}^{q} ||\alpha_2(x_i) - \alpha_2(\xi)||^2 - \gamma^2||\alpha_2(x)||^2 - \gamma^2||\bar{\alpha}_2(x)||^2
\]
(25)

This shows that \( U(x_e) \) is a storage function for the closed-loop system \( \Sigma_{eq} \), with respect to the supply rate \( s = \gamma^2||w||^2 - ||z||^2 \). This further implies that the system \( \Sigma_{eq} \) has an \( L_2 \) gain less than or equal to \( \gamma \). Let \( w = 0 \). Then from inequality (25), we have
\[
\frac{dU(x_e(t))}{dt} = U_{ee}f_t(x_e) \leq -||h_1(x(t))||^2 - ||\bar{\alpha}_2(x(t))||^2.
\]
(26)

Since \( U(x_e) \) is positive definite, it follows that the equilibrium \( (x_e, \xi) = (0, 0) \) of the closed-loop system \( \Sigma_{eq} \) is stable. To prove asymptotic stability, note that any trajectory satisfying \( \frac{dU(x_e)}{dt} = 0 \) is necessarily a trajectory of \( \dot{x} = f(x) + g_2(x)\bar{\alpha}_2(x) \) such that \( x(t) \) is
bounded, and \( h_1(x(t)) = 0 \) and \( \tilde{\sigma}_2(\xi) = 0 \) for all \( t \geq 0 \). From the detectability of the pair \( \{ f, h_1 \} \), it follows that \( \limsup_{t \to \infty} x(t) = 0 \). Thus, the \( \omega \)-limit set of such a trajectory is a subset of
\[
M = \{ (x, \xi) : x = 0, \tilde{\sigma}_2(\xi) = 0 \}.
\]

Any initial condition on this \( \omega \)-limit set yields a trajectory in which \( x(t) = 0 \) for all \( t \geq 0 \), while from (17) and \( \tilde{\sigma}_2(\xi) = 0 \), \( \xi(t) \) is a trajectory of
\[
\dot{\xi} = f_0(\xi) - L(\xi)\tilde{\sigma}_2(\xi) = \tilde{f}(\xi) - L(\xi)\tilde{h}_2(\xi)
\]
where
\[
f_0(\xi) = \begin{bmatrix}
    f(\xi_1) + g_1(\xi_1)\alpha_1(\xi_1) + g_2(\xi_1)\alpha_2(\xi_1) \\
    f(\xi_2) + g_1(\xi_2)\alpha_1(\xi_2) + g_2(\xi_2)\alpha_2(\xi_2) \\
    \vdots \\
    f(\xi_q) + g_1(\xi_q)\alpha_1(\xi_q) + g_2(\xi_q)\alpha_2(\xi_q)
\end{bmatrix}
\]

From Lemma 3.3, it follows that \( \limsup_{t \to \infty} \xi(t) = 0 \) and by the invariance principle, the closed-loop system \( \Sigma_c \) is locally asymptotically stable. \( \square \)

**Remark 3.4:** Theorem 3.1 presents an approach to solve the DHCD problem for nonlinear systems. If \( q = 1 \), then Theorem 3.1 becomes [4, Lemma 3.2], which is a solution to the central \( H_\infty \) control problem for the system \( \Sigma \). It should be noted that the inequality (14) in Theorem 3.1 still contains undetermined observer gains \( L_i(\xi_i) \)'s. The problem of how to design these observer gains will be treated in the next subsection.

### B. Observer Gain Design

Now we show how to design these observer gains \( L_i(\xi_i) \), \( i = 1, 2, \ldots, q \). Two methods will be presented. The first method makes use of the centralized observer design result in [4], and the idea is similar to that of Paz [8] for linear local observers. But our result here is for nonlinear systems. The second method gives a decentralized observer design in terms of solutions to a matrix inequality that implies the Hamilton–Jacobi inequality (14) in condition 3 of Theorem 3.1.

#### B.1 The First Method

**First, we review the result for nonlinear centralized observer design of [4].** Suppose that the assumptions 1) and 2) of Theorem 3.1 hold. Suppose that there exists a \( C^2 \) positive definite function \( S(x) \), locally defined in a neighborhood of \( x = 0 \) and vanishing at \( x = 0 \), which satisfies the Hamilton–Jacobi inequality
\[
H_c(x, S_c^T) = S_c[f(x) + g_1(x)\alpha_1(x) + g_2(x)\alpha_2(x) - \frac{1}{4}\gamma^2 S_c g_1(x)S_c^T \leq 0
\]

and the Hessian matrix of \( H_c(x, S_c^T) \) is nonsingular at \( x = 0 \). Suppose also that there exists a \( C^2 \) matrix-valued function \( L(x) \in R^{n \times p} \) with \( p = \sum_{i=1}^q p_i \) such that
\[
S_c L(x) = 2\gamma^2 h_2^T(x).
\]

Then, from [4, Th. 3.1], the centralized controller
\[
i = f(\eta) + g_1(\eta)\alpha_1(\eta) + g_2(\eta)\alpha_2(\eta) + L(\eta)(y - h_2(\eta))
\]
\[
u = \alpha_2(\eta)
\]
locally stabilizes the system \( \Sigma \) and guarantees that the resulting closed-loop system has an \( L_2 \) gain less than or equal to \( \gamma \).

For decentralized nonlinear control, we have the following result.

Note that, in the decentralized case, \( L(x) = [L_1(x) \ L_2(x) \ldots \ L_q(x)] \) with \( L_i(x) \in R^{n \times p_i}, \ i = 1, \ldots, q \), and condition (27) becomes
\[
S_c L_i(x) = 2\gamma^2 h_2^T(x), \quad i = 1, \ldots, q.
\]

Then by using Theorem 3.1, we have the following corollary which is an extension of Theorem 2.1 in [8] for the linear DHCD problem.

**Corollary 3.5:** Suppose that assumptions 1) and 2) of Theorem 3.1 hold, and that the local observer gains \( L_i(\xi_i), i = 1, 2, \ldots, q \) given in (28) are such that the Hamilton–Jacobi inequality (14) has a \( C^2 \) local positive definite solution \( T(\xi) \) in a neighborhood of \( \xi = 0 \), vanishing at \( \xi = 0 \), and the Hessian matrix of \( H_{L_i}(\xi_i, Q_i^T) \) is nonsingular at \( \xi = 0 \). Then the controller given by (15) and (16) with the above observer gains solves the DHCD problem for the nonlinear system \( \Sigma \).

#### B.2 The Second Method

Next, we present the second approach to the observer gain design. Since \( f(x) \) and \( h_2(x) \) \( (i = 1, \ldots, q) \) are smooth functions with \( f(0) = 0 \) and \( h_2(0) = 0 \), there exist smooth matrix-valued functions \( A(x) \) and \( C_i(x) \) \( (i = 1, \ldots, q) \) such that
\[
f(x) = A(x)x, \quad h_2(x) = C_i(x)x.
\]

**Theorem 3.6:** Under conditions 1) and 2) of Theorem 3.1, we assume that \( V_c = 2x^T P(x) \) with \( P(x) \) being a \( C^2 \) matrix-valued function. Furthermore, we assume that \( [1] \) of Theorem 3.1 there exists a \( C^2 \) matrix-valued functions \( T(\xi) \), locally defined and nonsingular in a neighborhood of \( \xi = 0 \), of the form

\[
T(\xi) = \begin{bmatrix}
    T_{11}(\xi) & T_{12}(\xi) & \cdots & T_{1q}(\xi) \\
    T_{21}(\xi) & T_{22}(\xi) & \cdots & T_{2q}(\xi) \\
    \vdots & \vdots & \ddots & \vdots \\
    T_{q1}(\xi) & T_{q2}(\xi) & \cdots & T_{qq}(\xi)
\end{bmatrix}, \quad T_i(\xi) \in R^{n \times n}
\]

that satisfies the matrix inequality
\[
T(\xi)A(\xi) + A^T(\xi)T(\xi) + T(\xi)K(\xi)T(\xi) - \gamma^2 T(\xi)C^T(\xi)C(\xi)T(\xi) < 0
\]

where \( \gamma_i(\xi) \) is as defined in (9), and there exists a positive definite function \( Q(\xi) \) with \( Q(0) = 0 \) such that \( Q(\xi) = 2\gamma_i^2 T^T(\xi) \) where

\[
T_i(\xi) = \text{diag}\{T_{11}(\xi), T_{22}(\xi), \cdots, T_{qq}(\xi)\}
\]

Denote
\[
\tilde{L}(\xi) = \gamma^2 T_D(\xi)C^T(\xi).
\]

Then the controller given by (15) and (16) with the observer gain as specified by (37) and (32) solves the DHCD problem for the system \( \Sigma \).
Proof: From inequality (32), and definitions (32) and (37), it follows that
\[
T(\xi)[\tilde{A}(\xi) - \tilde{L}(\xi)\tilde{C}(\xi)]^T + [\tilde{A}(\xi) - \tilde{L}(\xi)\tilde{C}(\xi)T(\xi) + T(\xi)K_{\xi}(\xi)K_{\xi}(\xi)T(\xi) + \frac{1}{\gamma^2} \tilde{g}_1(\xi)\tilde{g}_1^T(\xi)
\]

which further implies that
\[
\left[A_{f}(\xi) - \tilde{L}(\xi)\tilde{C}(\xi)\right]^T[T^{-1}(\xi)]^T + T^{-1}(\xi)[\tilde{A}_{f}(\xi) - \tilde{L}(\xi)\tilde{C}(\xi)][T^{-1}(\xi)]^T
\]

By using definitions (29) and (33)–(36), and by comparing with
which further implies that
\[
\left[A_{f}(\xi) - \tilde{L}(\xi)\tilde{C}(\xi)\right]^T[T^{-1}(\xi)]^T + T^{-1}(\xi)[\tilde{A}_{f}(\xi) - \tilde{L}(\xi)\tilde{C}(\xi)][T^{-1}(\xi)]^T
\]

IV. AN EXAMPLE
Consider the following nonlinear system:
\[
x = \begin{bmatrix} -2x_1 + x_1x_2^2 \\ x_2^2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}u_0 + \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}
\]

TABLE I
Here “—” indicates that the system becomes unstable

<table>
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<tr>
<th>\gamma_{\text{max}}</th>
<th>A = 0.1</th>
<th>A = 0.17</th>
<th>A = 0.18</th>
<th>A = 0.23</th>
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<td>0.1732</td>
<td>—</td>
<td>—</td>
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<td>Nonlinear</td>
<td>0.1726</td>
<td>0.1728</td>
<td>0.1728</td>
<td>0.1767</td>
</tr>
</tbody>
</table>

z = \begin{bmatrix} x_1 \\ x_2^2 \end{bmatrix}

g = \begin{bmatrix} 2x_1 + 2x_2 \\ 2x_1 \\ 2x_2 \end{bmatrix}

which is of the form \( \Sigma \) in (1)–(3). Hence, \( g = 2 \). We will design a
decentralized controller for the above system by using Theorem 3.1 and
Corollary 3.7.

For this example, it is easy to check that \( \{ f, h_1 \} \) is locally
detectable. By solving (13), the following positive definite solutions are
obtained with a minimum \( \gamma \) value of \( \gamma_{d} = 0.48 \):
\[
V(x) = 0.3642x_1^2 - 0.03706x_1x_2 + 0.04331x_2^2
\]
\[
-0.04442x_2^3 + 0.1324x_2x_1^2 - 0.088x_2x_1x_2^2
\]
\[
+0.04536x_2^3 + 0.1612x_2x_1^2 - 0.7842x_2x_2^2
\]
\[
+2.1051x_2^3 - 2.1938x_1x_2^2 + 1.1772x_2^4.
\]

The observer gain is taken as linear by solving (38) of Corollary 3.7
\[
\hat{L} = \begin{bmatrix} L_1 & 0 \\ 0 & L_2 \end{bmatrix} = \begin{bmatrix} 0.3645 & 0.4029 & 0 & 0 \\ 0 & 0.6231 & -0.2519 \end{bmatrix}
\]

Finally the controller is given by (15) and (16) with \( \alpha_1(\xi) \) and
\( \alpha_2(\xi) \) as defined in (6) and (7).

As there is no viable method for computing the \( H_{\infty} \)-norm of an
affine nonlinear system, the following approximate \( L_2 \)-gain \( \gamma_{\text{max}} \)
\[
\gamma_{\text{max}} = \max_{\theta \geq 0} \sqrt{\theta + \frac{1}{\theta}}
\]

If the \( H_{\infty} \)-norm of the closed-loop system is \( \gamma_r \), then obviously
\( 0 \leq \gamma_{\text{max}} \leq \gamma_r \leq \gamma_{d} \).

In computer simulations, the plant disturbance \( w_0(t) \) is taken as
(\( A = 0.1 \)), the linear and the nonlinear controllers give the same
closed-loop disturbance attenuation \( \gamma_{\text{max}} \). Also the \( L_2 \)-attenuation
\( \gamma_{\text{max}} \) is less than the desired value of \( \gamma_{d} = 0.48 \). The disturbance/noise
is increased to \( A = 0.18 \), the linear controller fails to
stabilize the system, but the nonlinear controller not only stabilizes
the system but also provides adequate disturbance attenuation. In
fact, the nonlinear controller can meet the design specification for a
disturbance level of up to \( A = 0.23 \).

Of course, the nonlinear method presented in this paper is a
local one. This is confirmed by the simulation result that if the
disturbance/noise level is increased beyond \( A = 0.23 \), the nonlinear
controller fails to stabilize the system.
V. Conclusion

This paper presents a sufficient condition for the DHCD problem for nonlinear systems. The resulting controller guarantees local asymptotic stability and provides a predetermined $L_2$-gain bound on the closed-loop system. Two design methods of the local observers are given: one is based on the centralized observer gain and another one is related to the solution of the matrix inequalities. The results are extensions of those in [8] and [13] for the case of linear systems.

REFERENCES


A Methodology for the Design of Optimal Traffic Shapers in Communication Networks

Venkat Anantharam and Takis Konstantopoulos

Abstract—The authors consider the problem of optimally regulating the source traffic in a communication network to simultaneously satisfy a finite number of affine burstiness constraints. They prove that an optimal solution is a series connection of correspondingly dimensioned “leaky buckets.” They propose a simple “fork-join” implementation of the optimal solution and study extensions to the problem of optimally shaping the traffic flow to meet a burstiness constraint specified by a concave increasing function. A consequence of their optimality results is that permutations of leaky buckets in a series connection are input–output equivalent.

Index Terms—Communication networks, flow control, Skorokhod reflection mapping.

I. PRELIMINARIES

In this paper we consider the problem of designing flow control schemes in a communication network. Flow control is necessary for the regulation and shaping of a source traffic stream, which must interact and share network resources with other traffic streams after it is admitted. Therefore, one normally requires the admitted flow to satisfy certain “burstiness” or “shaping” constraints. It is also desirable that the controller be optimal, in that the offered traffic is transmitted as quickly as possible.

A general model for a traffic process is a nonnegative sigma-finite Borel measure $A$ on the time axis $\mathbb{R}_+$. This is represented by an increasing right-continuous process $\{A_t, t \geq 0\}$; the interpretation is that for $0 \leq s \leq t$, $A_t - A_s$ gives the volume of traffic (in cells) on the time interval $[s, t]$. Write $M$ for the collection of such processes. We write $A^N$ for the restriction of $A$ on $\mathbb{R}_+ \subseteq \mathbb{R}_+$, defined by $A_s^N := \int_{s \leq t \leq N} dA_t$. We also define a partial ordering on $M$ by $A \leq B \iff A_t \leq B_t$ for all $t \geq 0$. We say that $A \in M$ is $(\sigma, \sigma, \rho, n)$ constrained iff, for all $0 \leq s \leq t$,

$$A_t \leq \min_{1 \leq i \leq n} \{\sigma_i + \rho_i t\}, \quad A_t - A_s \leq \min_{1 \leq i \leq n} \{\sigma_i + \rho_i (t - s)\}. \quad (1)$$

Here, $\sigma_i, \sigma_i \geq 0, \rho_i \geq 0$, for all $i$. For $n = 1$ we simply say that $A$ is $(\sigma, \sigma, \rho)$ constrained. The above definitions are discussed in Anantharam [1] and Cruz [4], [5], and they also closely match the standard shaping descriptors that have been adopted in practice for high-speed networks. More generally, for $f_0, f$ arbitrary concave increasing functions from $\mathbb{R}_+$ into $\mathbb{R}_+$, we say that $A$ is $(f_0, f)$ constrained iff

$$A_t \leq f_0(t), \quad A_t - A_s \leq f(t - s). \quad (2)$$

Of course, (1) is a convenient special case of (2). A traffic regulator, or flow controller, is simply a map $\varphi : M \to M$. Some properties that such a map may possess are as follows.

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