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Negativity of curvature on spaces parametrizing Hodge decompositions of reduced first cohomology groups

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Let $\pi : \mathcal{X} \to B$ be a regular holomorphic family of compact Kähler manifolds over a simply-connected complex manifold B and assume that the total space $\mathcal{X}$ is equipped with a fixed Kähler metric g. Let Γ be the fundamental group of a typical fiber X_0. Let $\Phi : \Gamma \to U(H)$ be a unitary representation of Γ on a (separable complex) Hilbert space for which the reduced first cohomology group $H^1_{\rm red}(\Gamma, \Phi)$ is non-zero. Fix an isomorphism between $\pi_1(X_0)$ and Γ. Since B is simply-connected we have a consistent identification between $\pi_1(X_t)$ and Γ for any fiber X_t. We have therefore a consistent family of identifications of $H^1_{\rm red}(\Gamma, \Phi) := V$ with the space of harmonic forms on $(X_t, g|_{X_t})$ with coefficients twisted by Φ. Since X_t is Kähler we have thus obtained a decomposition of $V^{\mathbb{C}} = V \otimes_{\mathbb{R}} \mathbb{C}$ into $W_t \oplus \overline{W}_t$, where W_t corresponds to the space of holomorphic 1-forms with coefficients twisted by Φ. When the representation space is finite-dimensional, one can parametrize the space of Hodge decompositions of $V^{\mathbb{C}}$ by the Siegel upper half-plane, on which the Bergman metric is an invariant Kähler metric of nonpositive holomorphic bisectional curvature and strictly negative holomorphic sectional curvature. If the infinitesimal variation of the Hodge decomposition of $V^{\mathbb{C}}$ is injective, we have an induced Kähler metric on B with the same curvature property. For instance it follows that B is Brody hyperbolic, *i.e.*, it admits no nontrivial entire holomorphic curve. When the representation space is infinite-dimensional, one can no longer associate to the "universal" space of Hodge decompositions a Kähler metric, since infinitesimal deformations are given by bounded operators from W_t to $V^{\mathbb{C}}/W_t \cong \overline{W}_t$. Instead we will in essence construct a canonical complex Finsler metric, which in the finite-dimensional case reduces to the Kobayashi metric. We note that the Kobayashi metric on the Siegel upper half-plane is a continuous complex Finsler metric of holomorphic sectional curvature ≤ -2, and that it agrees with the Carathéodory metric. For the purpose of deducing hyperbolicity properties of parameter spaces of Hodge decompositions of $V^{\mathbb{C}}$, the Kobayashi metric serves the same purpose as the Bergman metric. Since our interest lies only in studying regular holomorphic families of compact Kähler manifolds, we will avoid the technicalities of dealing with infinite-dimensional moduli spaces of Hodge decompositions of $V^{\mathbb{C}}$. Instead,

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we will directly construct a (possibly degenerate) complex Finsler metric on B and show that it is of holomorphic sectional curvature ≤ -2 in the usual sense.

§1 Preliminaries

(1.1) Let $\pi : \mathcal{X} \to B$ be a regular holomorphic family of compact Kähler manifolds over a simply-connected complex manifold B. For $t \in B$ denote by X_t the fiber $\pi^{-1}(t)$. Let Γ be the fundamental group of a typical fiber X_0. Since B is simply-connected there is a canonical way of identifying $\pi_1(X_t)$ with $\pi_1(X_0) = \Gamma$. Let H be a complex Hilbert space, which we will assume to be separable throughout the article. Let $\Phi : \Gamma \to H$ be a unitary representation on H such that the reduced first cohomology group $V := H^1_{\text{red}}(\Gamma, \Phi) \neq 0$. Denote by E_Φ the locally constant bundle of Hilbert spaces on X associated to Φ. For any $t \in B$ we may identify V with the space of harmonic 1-forms $H^1_{\text{harm}}(X_t, E_\Phi)$ with values in E_Φ. For a d-closed smooth 1-form with values in E_Φ we will denote by $[\eta]$ the corresponding element in V.

Let Γ be the fundamental group of some compact Riemannian manifold. By Korevaar-Schoen [KS] and Mok [M1], $H^1_{\text{red}}(\Gamma, \Phi) \neq 0$ for some Φ if and only if Γ violates Property (T) of Kazhdan's (cf. de la Harpe-Valette [HV]). This is the case for instance if Γ is of subexponential growth (cf. Mok [M2]). Another example of a nonvanishing reduced first cohomology group in the Kähler case is given by the left regular representation ρ on a compact Riemann surface S of genus ≥ 2. In this case the harmonic (1,0)-forms on S with values in E_ρ correspond precisely to the square-integrable holomorphic 1-forms on the unit disk as the universal cover of S.

Considering Φ as an orthogonal representation of the underlying space we have a complex conjugation defined on $H^{\mathbb{C}} := H \otimes_{\mathbb{R}} \mathbb{C}$. Write $E_\Phi^{\mathbb{C}}$ for the corresponding locally constant bundle of Hilbert spaces. We have the Hodge decomposition $H^1_{\text{harm}}(X_t, E^{\mathbb{C}}_\Phi) = H^{1,0}_{\text{harm}}(X_t, E^{\mathbb{C}}_\Phi) \oplus H^{0,1}_{\text{harm}}(X_t, E^{\mathbb{C}}_\Phi)$. The space $H^{1,0}_{\text{harm}}(X_t, E^{\mathbb{C}}_\Phi)$ consists of $E^{\mathbb{C}}_\Phi$-valued holomorphic 1-forms. Denote by $W_t \subset V$ the vector subspace corresponding to $H^{1,0}_{\text{harm}}(X_t, E^{\mathbb{C}}_\Phi)$. An E_Φ-valued differential form φ is said to be real if and only if $\overline{\varphi} = \varphi$. For η a real E_Φ-valued harmonic 1-form, we have $\eta = \eta^{1,0} + \eta^{0,1}$ where $\overline{\eta^{0,1}} = \eta^{1,0}$, and $\eta^{1,0}$ is a d-closed E_Φ-valued holomorphic 1-form.

For notational simplicity from now on we will replace $H^{\mathbb{C}}$ by H, and $\Phi \otimes_{\mathbb{R}} id : \Gamma \to U(H^{\mathbb{C}})$ by $\Phi : \Gamma \to U(H)$. In other words, we treat Φ as denoting the complexification of some orthogonal representation so that complex conjugation on H makes sense.

(1.2) In what follows X stands for any of the fibers X_t in the regular holomorphic family $\pi : \mathcal{X} \to B$ of compact Kähler manifolds. By lifting harmonic 1-forms from X to the universal covering space $\widetilde{X}$ and integrating we obtain a canonical map from $H^1_{\text{harm}}(X, E_\Phi)$ to $H^1_{\text{red}}(\Gamma, \Phi)$. We will show that this canonical map is a topological isomorphism, when we endow both sides with structures of Hilbert spaces. The discussion here is valid for compact Riemannian manifolds in general and the Kähler property of X plays no role in the discussion. Since X carries a given Riemannian (Kähler) metric, $H^1_{\text{harm}}(X, E_\Phi)$ carries the usual structure of a Hilbert space when we endow the harmonic forms with global L^2-norms with respect to the given Riemannian metric. Completeness of $H^1_{\text{harm}}(X, E_\Phi)$ follows readily from Schauder estimates on harmonic forms, which work equally well when the underlying Hilbert space H of the unitary representation Φ is separable and infinite-dimensional.

We proceed now to define a Hilbert space norm on $V = H^1_{\text{red}}(\Gamma, \Phi)$. Recall that a 1-cochain $c \in C^1(\Gamma, \Phi)$ is a function $c : \Gamma \to H$. We write c_γ for the value of c at $\gamma \in \Gamma$. The 1-cochain $c \in C^1(\Gamma, \Phi)$ is said to be a 1-cocycle if it satisfies the identity $c_{\gamma\delta} = \Phi(\gamma)(c_\delta) + c_\gamma$. We call this identity the compatibility condition on 1-cocycles.

Since Γ is the fundamental group of the compact Kähler manifold X, it is finitely generated. Let $\gamma_1, \cdots \gamma_m$ be a finite set of generators of Γ. We endow the vector space $Z^1(\Gamma, \Phi)$ of 1-cocycles with the norm given by $||c||^2 = ||c_{\gamma_1}||^2 + \cdots + ||c_{\gamma_m}||^2$. Since Γ is generated by $\gamma_1, \cdots, \gamma_m$, any $c \in Z^1(\Gamma, \Phi)$ is completely determined by $c_{\gamma_1}, \cdots, c_{\gamma_m}$ and by the compatibility condition on 1-cocycles. By the latter, $c \equiv 0$ whenever $c_{\gamma_1} = \cdots = c_{\gamma_m} = 0$, so that $|| \cdot ||$ is indeed a norm on $Z^1(\Gamma, \Phi)$. A 1-cocycle $c \in Z^1(\Gamma, \Phi)$ is said to be a 1-coboundary, written $c \in B^1(\Gamma, \Phi)$, if there exists some element $h \in H$ such that $c_\gamma = h - \Phi(\gamma)(h) := \delta h(\gamma)$ for every $\gamma \in \Gamma$. We say that $c \in Z^1(\Gamma, \Phi)$ is a 1-quasi-coboundary if there exists a sequence $\{c^k\}$ of 1-coboundaries such that for every $\gamma \in \Gamma$, we have $c_\gamma = \lim_{k\to\infty} c^k_\gamma$. From the compatibility condition on 1-cocycles the latter is valid if and only if it is valid on a given set of generators. Thus the vector space of 1-quasi-coboundaries can be identified with the closure of $\overline{B^1(\Gamma, \Phi)}$ of $B^1(\Gamma, \Phi)$ in $Z^1(\Gamma, \Phi)$ with respect to the Hilbert space norm $|| \cdot ||$. As a vector space the reduced cohomology group $V = H^1_{\text{red}}(\Gamma, \Phi)$ is defined as $Z^1(\Gamma, \Phi)/\overline{B^1(\Gamma, \Phi)}$. Note that this does not depend on the choice of the finite set of generators of Γ. Fixing this set and hence $|| \cdot ||$, however, we have the induced structure of a Hilbert space on V. We note

LEMMA (1.2.1). *Let $|| \cdot ||_1$ resp. $|| \cdot ||_2$ be two choices of Hilbert space norms on V corresponding to two choices of finite sets of generators $\{\gamma^{(1)}_1, \cdots, \gamma^{(1)}_{m_1}\}$ resp. $\{\gamma^{(2)}_1, \cdots, \gamma^{(2)}_{m_2}\}$. Then, the identity map on V induces a topological isomorphism $\iota : (V, || \cdot ||_1) \to (V, || \cdot ||_2)$.*

PROOF. Let c be an arbitrary element of $Z^1(\Gamma, \Phi)$. For any $\gamma \in \Gamma$, from the compatibility condition on 1-cocycles, we can express c_γ as a linear combination of c_δ, for $\delta \in \{\gamma^{(1)}_1, \cdots, \gamma^{(1)}_{m_1}\}$, with coefficients independent of the 1-cocycle c. It follows that $\iota : (V, || \cdot ||_1) \to (V, || \cdot ||_2)$. is continuous. Applying the same argument to the inverse map we conclude that ι is a topological isomorphism.

Let (X, g) be a compact Riemannian manifold with fundamental group Γ. We can endow $H^1_{\text{harm}}(X, E_\Phi)$ with the structure of a Hilbert space by defining the norm of a harmonic 1-form η by $||\eta||^2 = \int_X ||\eta(x)||^2 dV(x)$, where dV is the volume form on X determined by the Riemannian metric g. We have a canonical map $\lambda : H^1_{\text{harm}}(X, E_\Phi) \to H^1_{\text{red}}(\Gamma, \Phi)$. The main result of this section is

PROPOSITION (1.2.1). *Endow $H^1_{\text{harm}}(X, E_\Phi)$ with the structure of a Hilbert space as determined by the Riemannian metric on X, and $H^1_{\text{red}}(\Gamma, \Phi)$ with the structure of a Hilbert space as determined by a choice of a finite set of generators of Γ. Then the canonical map $\tau : H^1_{\text{harm}}(X, E_\Phi) \to H^1_{\text{red}}(\Gamma, \Phi)$ is a topological isomorphism.*

For the study of E_Φ-valued differential forms on X we have the following simple version of Hodge decomposition on closed forms which remains valid when H is of infinite dimensions.

LEMMA (1.2.2). *Given any smooth d-closed E_Φ-valued 1-form ξ on X, we have $\xi = \eta + \zeta$, where η is a harmonic form, and ζ the L^2-limit of a sequence of d-exact smooth E_Φ-valued 1-forms. Furthermore, ζ is smooth.*

PROOF. Let $(\mathcal{H}, ||\cdot||)$ be the Hilbert space of square-integrable E_Φ-valued differential 1-forms on X and denote by $<\cdot,\cdot>$ the corresponding inner product. Denote by d^* the formal adjoint of d on (X, g). Let $\mathcal{Z}^1(X, E_\Phi)$ be the vector subspace of E_Φ-valued smooth d-closed 1-forms. Let $\mathcal{H}' \subset \mathcal{H}$ be the closure of $\mathcal{Z}^1(X, E_\Phi)$ with respect to $||\cdot||$. For any $\beta \in \mathcal{Z}^1(X, E_\Phi)$ we have $<\beta, d^*\varphi> = 0$ for every smooth E_Φ-valued 2-form φ on X. Taking L^2-limit this remains true for every $\beta \in \mathcal{H}'$. Integrating by parts this implies that β lies in the domain of definition of d and that furthermore $d\beta = 0$ in the sense of distribution. Let $\mathcal{B}^1(X, E_\Phi) \subset \mathcal{H}'$ be the vector subspace of E_Φ-valued smooth d-exact 1-forms, and denote by $\mathcal{H}'' \subset \mathcal{H}'$ the closure of $\mathcal{B}^1(X, E_\Phi)$ with respect to $||\cdot||$. Write $\mathcal{H}' = \mathcal{H}'' \oplus \mathcal{K}$ for the orthogonal decomposition. For any $\eta \in \mathcal{K}$, η is orthogonal to $d\alpha$ for any E_Φ-valued section α on X. It follows by integrating by parts that η belongs to the domain of definition of d^*, and that furthermore $d^*\eta = 0$ in the sense of distribution. Thus η is harmonic in the sense that $d\eta = d^*\eta = 0$, so that η is smooth from local elliptic estimates, even in the case when H is infinite-dimensional. From the orthogonal decomposition $\mathcal{H}' = \mathcal{K} \oplus \mathcal{H}''$, $\mathcal{K} = H^1_{\mathrm{harm}}(X, E_\Phi)$, we have accordingly $\xi = \eta + \zeta$, where η and hence ζ are smooth. $\square$

REMARKS

In the classical case where H is finite-dimensional, we have the deeper statement that $\zeta = d\chi$ for some smooth E_Φ-valued section on X, but the latter is no longer true when H is of countably infinite dimensions.

To relate harmonic 1-forms with the algebraically defined reduced first cohomology group first of all we can associate 1-cocycles to closed E_Φ-valued 1-forms on X by lifting to the universal covering space $\widetilde{X}$ of X and integrating. We fix a base point $o \in \widetilde{X}$. For each $\xi \in \mathcal{Z}^1(X, E_\Phi)$ denote by $\widetilde{\xi}$ its lifting to $\widetilde{X}$. Define for each $p \in \widetilde{X}$, $F_\xi(p) = \int_{Q(o,p)} \widetilde{\xi}$, where $Q(o,p)$ denotes any smooth path joining o to p. Then $F := F_\xi$ satisfies the functional equation

$$F_\xi(\gamma(p)) = \Phi(\gamma)F(p) + c_\gamma(\xi),$$

for any $\gamma \in \Gamma$. We have thus associated to ξ a 1-cocycle $c(\xi) = (c_\gamma)_{\gamma\in\Gamma}$. The 1-cocycle $c(\xi)$ depends on the choice of the base point $o \in \widetilde{X}$, but the class $[c(\xi)] = [c_\gamma]_{\gamma\in\Gamma} \in H^1_{\mathrm{red}}(\Gamma, \Phi)$ is independent of the choice of the base point. We define $K : Z^1(X, E_\Phi) \to Z^1(\Gamma, \Phi)$ by $K(\xi) = c(\xi)$ and $\kappa : Z^1(X, E_\Phi) \to H^1_{\mathrm{red}}(\Gamma, \Phi)$ by $\kappa(\xi) = [c(\xi)]$. We have

LEMMA (1.2.3). *The linear map $\kappa : \mathcal{Z}^1(X, E_\Phi) \to H^1_{\mathrm{red}}(\Gamma, \Phi)$ is continuous if we consider $\mathcal{Z}^1(X, E_\Phi)$ as a dense topological vector subspace of $\mathcal{H}'$. In other words, κ extends to a continuous linear map $\overline{\kappa} : \mathcal{H}' \to H^1_{\mathrm{red}}(\Gamma, \Phi)$.*

PROOF. Let $\{\xi_k\}$ be a sequence of smooth d-closed E_Φ-valued 1-forms which converge in L^2 to a smooth d-closed 1-form ξ_∞. We have to prove that $\kappa(\xi_\infty) = \lim_{k\to\infty} \kappa(\xi_k)$. For each k, ξ_k lifts to a smooth mapping $F_k : \widetilde{X} \to H$ such that

$$F_k(\gamma(p)) = \Phi(\gamma)(F_k(p)) + c^k_\gamma,$$

where we have written F_k for F_{ξ_k}, and c_γ^k for $c_\gamma(\xi_k)$. For ξ_∞ use the same notations with the index k replaced by ∞. We claim that $[c_\gamma^k]_{\gamma\in\Gamma}$ converges to $[c_\gamma^\infty]_{\gamma\in\Gamma}$. This requires a proof, since ξ_k only converges to ξ_∞ in the L^2-norm. We note that the 1-cocycles $c^k = (c_\gamma^k)_{\gamma\in\Gamma}$ may actually diverge. To prove the claim we will have to replace each c^k by cohomologous 1-cocycles $e^k = c^k + \delta h_k$ such that $\{e^k\}$ converges as a sequence of 1-cocycles.

For the integral F_ξ of any smooth E_Φ-valued 1-form ξ on X, normalized by $F_\xi(o) = 0$, and for any $\gamma \in \Gamma$, write the functional equation for F_ξ in the form $c_\gamma(\xi) = F_\xi(\gamma p) - \Phi(\gamma)F_\xi(p)$. Fix $\gamma \in \Gamma$. Recall that $F_\xi(p) = \int_{P(o,p)} \widetilde{\xi}$. If we could estimate F_ξ on compact sets in terms of ξ, then we would have bounds on $G_\gamma(\xi)$ for γ belonging to a finite set of generators of Γ in terms of L^2-norms of ξ, and Lemma (1.2.2) would fall out easily. Unfortunately the desired estimates on F_ξ are in general not possible. Estimates on F_ξ depend on the choice of the base point $o \in \widetilde{X}$. A change of base point will result in replacing $c(\xi) \in Z^1(\Gamma, \Phi)$ by a cohomologous element $c(\xi) + \delta h \in Z^1(\Gamma, \Phi)$. From now on $o \in \widetilde{X}$ will denote a variable base point. To indicate dependence on the base point, we will write F_ξ^o, $c(\xi, o)$, etc. Let $E \subset \widetilde{X}$ be a bounded measurable subset. Then, the average of $c(\xi, o)$ as o ranges over E makes sense. It will give a 1-cocycle, to be denoted by $c(\xi, E)$ whose class $[c(\xi, E)] \in H^1_{\text{red}}(X, E_\Phi)$ is the same as $[c(\xi, o)]$. This is what we are going to do, with an aim to finding a suitable average such that $c(\xi, E)$ can be estimated. For this purpose we are going to establish

(†) Let $\xi \in \mathcal{Z}^1(X, E_\Phi)$ and $D \subset \widetilde{X}$ be a bounded domain. For $o \in \widetilde{X}$ let $F_\xi^o : \widetilde{X} \to H$ be such that $dF_\xi^o = \widetilde{\xi}$, the lifting of ξ to $\widetilde{X}$. For $p \in \widetilde{X}$ write $F_\xi^D(p)$ for $\frac{1}{Vol(D,g)} \int_D F_\xi^o(p) dV(o)$, where dV denotes the volume element of (X, g), defining $F_\xi^D : \widetilde{X} \to H$ such that $dF_\xi^D = \widetilde{\xi}$. Then, given any compact measurable subset $M \subset \widetilde{X}$ we have $\int_M ||F_\xi^D|| \leq C_M ||\xi||$ for some constant C_M depending on M but independent of ξ.

We note that Lemma (1.3.2) follows readily from (†). To see this using F_ξ^D in place of $F_\xi = F_\xi^o$ we have analogously $c_\gamma^D(\xi)(p) = F_\xi^D(\gamma p) - \Phi(\gamma) F_\xi^D(p)$. From (†) we conclude that $||c_\gamma^D|| \leq C_\gamma ||\xi||$ for some constant C_γ independent of ξ. Recall that $\xi_k \in \mathcal{Z}^1(X, E_\Phi)$ converges to $\xi_\infty \in \mathcal{Z}^1(X, E_\Phi)$ in the L^2-norm. For each $\gamma \in \Gamma$, applying (†) to the differences $\xi_k - \xi_\infty$, we see that $c_\gamma^D(\xi_k - \xi_\infty)$ converges to 0. Since $||\cdot||$ on $H^1_{\text{red}}(\Gamma, \Phi)$ is defined by a finite set of generators of Γ, $[c_\gamma^D(\xi_k)] = \kappa(\xi_k)$ converges to $[c_\gamma^D(\xi_\infty)] = \kappa(\xi_\infty)$, giving the continuity of $\kappa : \mathcal{Z}^1(X, E_\Phi) \to H^1_{\text{red}}(\Gamma, \Phi)$.

For the proof of (†) F_ξ^o is obtained by integrating $\tilde{\xi}$ with initial point o, and F_ξ^D is obtained from F_ξ^o by averaging as the initial point o varies over D. The L^1-norm of F_ξ^o over a closed ball disjoint from o can in an obvious way be controlled by the L^1-norm of ξ. This gives readily (†) for the special case when $\overline{M}$ and $\overline{D}$ are disjoint. However, when $o \in \overline{M}$, we have to take care of singular weight functions arising from the use of polar coordinates centred at the initial point o. For $r > 0$ and $o \in \widetilde{X}$ write $B(o; r)$ for the geodesic ball on $\widetilde{X}$ of radius r centred at o. By covering arguments it is clear that (†) follows from

(♭) Let $\epsilon > 0$, $\epsilon < 1$ be such that 4ϵ is less than the injectivity radius $i(X, g)$ of (X, g). Let $b \in \widetilde{X}$ be arbitrary. For $r > 0$ write B_r for the geodesic ball $B(b; r)$.

Then, in the notations of (†) we have

$$\int_{B_\epsilon}\left(\int_{B_{2\epsilon}}||F^o_\xi(p)||dV(p)\right)dV(o)\le C||\xi||$$

for some constant C independent of $\xi\in\mathcal{Z}^1(X,E_\Phi)$.

To complete the proof of Lemma (1.2.2) we proceed to prove ($\flat$). For each $o\in B_\epsilon$ we can use normal geodesic coordinates on the geodesic ball $B(o;4\epsilon)$. Note that the closed geodesic ball $\overline{B_{2\epsilon}}$ is contained in $B(o;3\epsilon)$. For such o we will obtain F^o_ξ on $B(o;3\epsilon)$ by integrating along geodesics emanating from o. Write n for the complex dimension of X. Using normal geodesic coordinates estimates of F^o_ξ as stated in ($\flat$) in terms of ξ can be reduced to the same problem for Euclidean space $\mathbb{R}^m$ in place of $\widetilde{X}$. We have

$$\int_{B_\epsilon}\left(\int_{B_{2\epsilon}}||F^o_\xi(p)||dV(p)\right)dV(o)\le C\int_{B_\epsilon}\left(\int_{B_{2\epsilon}}\frac{1}{d(o;p)^{2n-1}}||\widetilde{\xi}(p)||dV(p)\right)dV(o)$$

for some positive constant C. Here and in what follows C is a generic symbol for a positive constant independent of ξ. The choice of a single base point o gives F^o_ξ. For estimates of the L^1-norm of F^o_ξ over $B_{2\epsilon}$ in terms of ξ we have to introduce the singular factor $d(o;p)^{2n-1}$ which corresponds to blowing-up at the centre o. The latter is integrable at o as a function in p. On the other hand, when we average the estimates as o varies over B_ϵ, the singular factor $d(o,p)^{2n-1}$ is also integrable as a function in o at p. This translates immediately to the estimate

$$\int_{B_\epsilon}\left(\int_{B_{2\epsilon}}||F^o_\xi(p)||dV(p)\right)dV(o)\le C\int_{B_{3\epsilon}}||\widetilde{\xi}(p)||dV(p)$$

$$\le C\left(\int_{B_{3\epsilon}}||\widetilde{\xi}(p)||^2dV(p)\right)^{\frac{1}{2}}\le C||\xi||,$$

where we used in the second last step the Cauchy-Schwarz inequality. The proof of Lemma (1.2.3) is complete. □

We turn now to the proof of our main result in this section.

PROOF OF PROPOSITION (1.2.1). The canonical map $\lambda : H^1_{\text{harm}}(X,E_\Phi)\to H^1_{\text{red}}(\Gamma,\Phi)$ is precisely the restriction of $\kappa:\mathcal{Z}^1(X,E_\Phi)\to H^1_{\text{red}}(\Gamma,\Phi)$. By Lemma (1.2.3) λ is continuous. Actually, if we denote by L the restriction of $K:\mathcal{Z}^1(X,E_\Phi)\to Z^1(\Gamma,\Phi)$ to $H^1_{\text{harm}}(X,E_\Phi)$, then L is already continuous from Schauder estimates on harmonic forms.

Let now $[c]=[c_\gamma]_{\gamma\in\Gamma}$ be a class in the reduced first cohomology group. We are going to find a harmonic form η in $H^1_{\text{harm}}(X,E_\Phi)$ such that $\lambda(\eta)=[c]$. We can find a finite cover $\mathcal{U}=\{U_\alpha\}$ of X by open coordinate balls and associate in a canonical way each 1-cocycle $[c]\in H^1_{\text{red}}(\Gamma,\Phi)$ to a Cech 1-cocycle on X relative to the cover $\mathcal{U}$. By the standard method of passing from Cech 1-cocycles to closed 1-forms by partition of unity, there exists a linear map $T:Z^1(\Gamma,\Phi)\to\mathcal{Z}^1(X,E_\Phi)\subset\mathcal{H}'$ such that $c\big(T((c_\gamma)_{\gamma\in\Gamma})\big)=(c_\gamma)_{\gamma\in\Gamma}$, i.e., $K(T(c))=c$. From the construction by partition of unity T is continuous with respect to Hilbert space norms. Furthermore, $T(\delta h)\in\mathcal{B}^1(X,E_\Phi)$.

We proceed to prove that $\lambda : H^1_{\text{harm}}(X, E_\Phi) \to H^1_{\text{red}}(\Gamma, E_\Phi)$ is a topological isomorphism. Let $c \in Z^1(\Gamma, \Phi)$ and write $T(c) = \xi$ and $\xi = \eta + \zeta, \eta \in H^1_{\text{harm}}(X, E_\Phi), \zeta \in \mathcal{H}'$, according to the Hodge decomposition as given by Lemma (1.2.2). Since $T(B^1(\Gamma, \Phi)) \subset \mathcal{B}^1(X, E_\Phi)$ from the continuity of T it descends to a continuous linear map $\tau : H^1_{\text{red}}(\Gamma, \Phi) \to \mathcal{H}'/\mathcal{H}'' \cong \mathcal{K}$, where $\cong$ denotes a linear isometry. We have $K(T(c)) = c$ so that $\kappa(T(c)) = [c]$, giving $\kappa(\eta) + \kappa(\zeta) = [c]$. Since $\kappa(\mathcal{B}^1(X, E_\Phi)) = 0$ it follows from Lemma (1.2.3) that $\kappa(\zeta) = 0$, so that $\kappa(\tau([c])) = [c]$, i.e., $\lambda(\tau[c]) = [c]$.

Conversely let $\eta \in H^1_{\text{harm}}(X, E_\Phi)$. $\xi := T(K(\eta)) \in \mathcal{Z}^1(X, E_\Phi)$ is such that $K(\xi) = K(\eta)$. Write $\xi = \tilde{\eta} + \zeta$, $\tilde{\eta} \in H^1_{\text{harm}}(X, E_\Phi)$, $\zeta \in \mathcal{H}''$, according to Hodge decomposition. Then, $K(\eta - \tilde{\eta}) = K(\xi - \tilde{\eta}) = K(\zeta)$. Thus, $\eta - \tilde{\eta} = \zeta + d\alpha$ for some smooth section α of E_Φ over X. Since $\zeta + d\alpha \in \mathcal{H}''$ and $H^1_{\text{harm}}(X, E_\Phi) = \mathcal{K}$ is orthogonal to $\mathcal{H}''$, we conclude $\tilde{\eta} = \eta$, i.e., $T(K(\eta)) = \eta$. Passing to quotient spaces $\tau(\kappa(\eta)) = \eta$, i.e., $\tau(\lambda(\eta)) = \eta$. Since both λ and τ are continuous, we have proven that λ and τ are inverses of each other and that they are topological isomorphisms, as desired. □

(1.3) Recall that by assumption the total space of $\pi : \mathcal{X} \to B$ is equipped with a Kähler metric g and denote by ω the Kähler form of $(\mathcal{X}, g)$. Beyond this our considerations are local over B and rely only on restrictions of $\pi : \mathcal{X} \to B$ to local holomorphic curves on B. For $r > 0$ write $\Delta(r) = \{z \in \mathbb{C} : |z| < r\}$, $\Delta = \Delta(1)$. Let $\epsilon > 0$ be arbitrary and $\pi : \mathcal{X} \to \Delta(1+\epsilon)$ be a regular family of compact Kähler manifolds diffeomorphic to the trivial family $X_0 \times \Delta(1+\epsilon) \to \Delta$. Thus, we have a smooth family of diffeomorphisms $f_t : X_t \cong X_0$. From now on we will only consider the family restricted to the unit disk $\Delta \Subset \Delta(1+\epsilon)$. Any smooth differential form φ on X_0 can then be identified with $f_t^*\varphi$ on X_t.

Let $h : H \otimes H \to \mathbb{C}$ be the bilinear pairing such that $h(v \otimes \overline{w}) = \langle v, w \rangle$ for the Hermitian inner product on H. We will use the same symbol h for the induced bilinear pairing on E_Φ, and extend h in an obvious way to $E_\Phi \otimes E_\Phi$-valued differential forms. For $t \in \Delta$, and E_Φ-valued smooth 1-forms μ, ν; consider the skew-symmetric pairing

$$S(\nu, \mu) = \int_{X_t} \sqrt{-1} h(\nu \wedge \mu) \wedge \omega^{n-1}.$$

When ν is replaced by $\nu' + d\varphi$ for some smooth section φ of E_Φ over X_t, clearly $S(\nu', \mu) = S(\nu, \mu)$. Furthermore, if $\nu' - \nu$ is only the L^2-limit of a sequence of d-exact E_Φ-valued 1-forms $d\varphi_i$ it remains true that $S(\nu', \mu) = \lim_{i \to \infty} S(\nu + d\varphi_i, \mu) = S(\nu, \mu)$. It follows that S induces a skew-symmetric pairing on $H^1_{\text{red}}(\Gamma, \Phi)$. We will use the same notation S for the latter and call it the symplectic form on $H^1_{\text{red}}(\Gamma, \Phi)$. Note that with the identification $H^1_{\text{red}}(\Gamma, \Phi) \cong H^1_{\text{harm}}(X_t, E_\Phi)$ as given in Proposition (1.2.1) we have $S(a, b) = S(\nu, \mu)$ whenever ν resp. μ is the harmonic representative of a resp. b; $a, b \in H^1_{\text{red}}(\Gamma, \Phi)$. Now for $\nu \in H^{1,0}_{\text{harm}}(\Gamma, E_\Phi)$ we have

$$S(\nu, \overline{\nu}) = \int_{X_t} \sqrt{-1} h(\nu \wedge \overline{\nu}) \wedge \omega^{n-1} = \|\nu\|^2,$$

while

$$S(\overline{\nu}, \nu) = -\|\nu\|^2,$$

so that S is positive-definite on W_t and negative-definite on $\overline{W}_t$. Furthermore, for $\nu_1, \nu_2 \in H^{1,0}_{\text{harm}}(\Gamma, E_\Phi)$ we have $S(\nu_1, \nu_2) = S(\overline{\nu_1}, \overline{\nu_2}) = 0$. From $V = H^1_{\text{red}}(\Gamma, \Phi) \cong H^1_{\text{harm}}(X_t, E_\Phi) = H^{1,0}_{\text{harm}}(X_t, E_\Phi) \oplus H^{0,1}_{\text{harm}}(X_t, E_\Phi)$ we can give V the structure of a Hilbert space by using norms on harmonic forms. $W_t, \overline{W}_t \subset V$ are orthogonal to each other with respect to this Hilbert space structure. The Hermitian inner product on V may depend on t, although by Proposition (1.2.1) they are all equivalent to (but not necessarily identical to) Hilbert space structures on $V = H^1_{\text{red}}(\Gamma, \Phi)$ defined via 1-cocycles as given in (1.2). On the other hand, we have

LEMMA (1.3.1). *The symplectic form S on $V \cong H^1_{\text{harm}}(X_t, E_\Phi)$ is independent of t.*

PROOF. Recall the smooth family of diffeomorphisms $f_t : X_t \cong X_0$. For ν, μ d-closed E_Φ-valued 1-forms on X_0, we have

$$S(f_t^*\nu, f_t^*\mu) = \int_{X_t} \sqrt{-1}h(f_t^*\nu \wedge f_t^*\mu) \wedge \omega^{n-1}.$$

Since ω is a Kähler form defined on the total space $\mathcal{X}$, $f_t^*(\omega\,|_{X_0})$ is cohomologous to $\omega\,|_{X_t}$, so that

$$\begin{aligned} S(f_t^*\nu, f_t^*\mu) &= \int_{X_t} \sqrt{-1}h(f_t^*\nu \wedge f_t^*\mu) \wedge (f_t^*\omega)^{n-1} \\ &= \int_{X_0} \sqrt{-1}h(\nu \wedge \mu) \wedge \omega^{n-1} \\ &= S(\nu, \mu). \end{aligned}$$

Since f_t induces the identity map on $V = H^1_{\text{red}}(\Gamma; \Phi)$ by definition this means that $S : V \times V \to \mathbb{C}$ does not depend on the choice of $t \in \Delta$, as desired.

(1.4) We are going to measure how the Hodge decomposition $V = W_t \oplus \overline{W}_t$ varies as t ranges over Δ. In the finite-dimensional case $\Gamma(X_t, \Omega(E_\Phi))$ constitutes a holomorphic vector bundle as t varies, by the Direct Image Theorem. For the infinite-dimensional case this remains true. To see this we start with examining the variation of $H^1_{\text{harm}}(X_t, E_\Phi)$. Let $\eta \in H^1_{\text{harm}}(X_0, E_\Phi)$. For $f_t^*\eta$ on X_t we have $[f_t^*\eta] = [\eta_t]$ for a unique $\eta_t \in H^1_{\text{harm}}(X_t, E_\Phi)$. Write $f_t^*\eta = \eta_t + \rho_t$, where η_t and ρ_t are orthogonal. It follows that $\|\eta_t\| \le \|f_t^*\eta\| \le (1 + C|t|)\|\eta\|$ for some positive constant C. Here and in what follows $\|\cdot\|$ will denote *global* L^2-norms on some X_t unless otherwise specified, and C will be a generic symbol for positive constants. Starting with η_t we also have $\|\eta\| \le \|(f_t^{-1})^*\eta_t\| \le (1 + C|t|)\|\eta_t\|$, which gives $(1 - C|t|)\|\eta\| \le \|\eta_t\| \le (1 + C|t|)\|\eta\|$. Since $\|f_t^*\eta\|^2 = \|\eta_t\|^2 + \|\rho_t\|^2$ it follows that $\|f_t^*\eta - \eta_t\| = \|\rho_t\| \le C|t|\|\eta\|$. Denote by $\Delta_t = dd_t^* + d_t^*d$ the d-Laplacian on $(X_t, \omega\,|_{X_t})$. We have,

$$\|\Delta_t(f_t^*\eta)(x)\| \le C|t|\,\|\eta\|$$

for some constant C independent of $x \in X_t$, from Schauder estimates on harmonic forms. Here note that the norm $\|\cdot\|$ on the left is pointwise and that on the right is global. Cover X_0 by a finite number of coordinate neighborhoods $\{U^\alpha\}$, which leads to a covering $\{U_t^\alpha\}$ of X_t via the diffeomorphisms f_t. We may take U^α to be domains with smooth boundaries. Solve now $\Delta_t u_t^\alpha = \Delta_t(f_t^*\eta)$ on U_t^α with Dirichlet boundary conditions. We have $\|u_t^\alpha(x)\| \le C|t|\,\|\eta\|$ for every $x \in U_t^\alpha$ and every α. Then, $\Delta_t(f_t^*\eta - u_t^\alpha) = 0 = \Delta_t\eta_t$. Since $\|f_t^*\eta - \eta_t\| \leq C|t|\,\|\eta\|$ we conclude that $\|f_t^*\eta - u_t^\alpha - \eta_t\|_{U_t^\alpha} \leq C|t|\,\|\eta\|$ for all α, t. Elliptic estimates then give uniform bounds for the harmonic forms $f_t^*\eta - u_t^\alpha - \eta_t$ and hence for $f_t^*\eta - \eta_t$, giving

LEMMA (1.4.1). *There exists a constant C such that*

$$\|(f_t^*\eta - \eta_t)(x)\| \leq C|t|\,\|\eta\| \quad \text{for every} \quad t \in \Delta,\ x \in X_t.$$

By Lemma (1.2.1) $V = H^1_{\mathrm{red}}(\Gamma, \Phi)$ carries a well-defined structure of a topological vector space. By Proposition (1.2.1) this structure agrees with that of $H^1_{\mathrm{harm}}(X_t, E_\Phi)$ for any $t \in \Delta$ via the canonical isomorphism $H^1_{\mathrm{harm}}(X_t, E_\Phi) \cong H^1_{\mathrm{red}}(\Gamma, \Phi)$, so that the canonical map $H^1_{\mathrm{harm}}(X_t, E_\Phi) \cong H^1_{\mathrm{harm}}(X_{t'}, E_\Phi)$ for $t, t' \in \Delta$ is a topological isomorphism. Lemma (1.4.1) implies this latter statement without using 1-cocycles.

LEMMA (1.4.2). *For $t \in \Delta$, let $\Xi_t : H^1_{\mathrm{harm}}(X_0, E_\Phi) \to H^1_{\mathrm{harm}}(X_t, E_\Phi)$ be the linear map defined by $\Xi_t(\eta) = \eta_t$. Then, Ξ_t is an inverted bounded linear operator such that $\Xi_t(\eta)$ is uniformly Lipschitz in (t, η) for $t \in \Delta$ and for every $\eta \in H^1_{\mathrm{harm}}(X_0, E_\Phi)$ of unit norm.*

PROOF. Lemma (1.4.1) shows that for every $\eta \in H^1_{\mathrm{harm}}(X_0, E_\Phi), \Xi_t(\eta)$ is Lipschitz in t at $t = 0$. For $t_0 \in \Delta$ fixed and $t \in \Delta$ arbitrary write $f_t^*\eta = (f_t f_{t_0}^{-1})^*(f_{t_0}^*\eta)$. Then $f_t^*\eta = (f_t f_{t_0}^{-1})^*\eta_{t_0} + (f_t f_{t_0}^{-1})^*(f_{t_0}^*\eta - \eta_{t_0})$. Write $g_t = f_t f_{t_0}^{-1}$. Then, $g_{t_0} = id$, and $f_t^*\eta = g_t^*\eta_{t_0} + g_t^*(f_t^*\eta - \eta_{t_0})$. Since $f_{t_0}^*\eta - \eta_{t_0}$ lies in the closure of d-exact E_Φ-valued 1-form on $X_{t_0}, g_t^*(f_{t_0}^*\eta - \eta_{t_0})$ lies on the closure of d-exact E_Φ-valued 1-form on X_t. As a consequence $\Xi_t(\eta) = \eta_t$ satisfies

$$\|(g_t^*\eta_{t_0} - \eta_t)(x)\| \leq C|t - t_0|\,\|\eta_{t_0}\| \quad \text{for every} \quad t \in \Delta, x \in X_t,$$

by the same arguments as in Lemma (1.4.1). Here C may be taken to be the same positive constant for any $t_0 \in \Delta$, since the regular family is defined on a neighborhood of the closed disk $\overline{\Delta}$. As a consequence $\Xi_t(\eta)$ is uniformly Lipschitz in (t, η) for $t \in \Delta$ and for $\eta \in H^1_{\mathrm{harm}}(X_0, E_\Phi)$ of unit norm.

We proceed now to show

PROPOSITION (1.4.1). *For $t \in \Delta(\epsilon), \epsilon > 0$ sufficiently small, there exist invertible bounded linear operators $\Theta_t : H^{1,0}_{\mathrm{harm}}(X_0, E_\Phi) \to H^{1,0}_{\mathrm{harm}}(X_t, E_\Phi)$ such that for every $\eta \in H^{1,0}_{\mathrm{harm}}(X_0, E_\Phi)$, $\Theta_t(\eta)$ is holomorphic in t.*

PROOF. We first show that there exists $\Psi_t : H^{1,0}_{\mathrm{harm}}(X_0, E_\Phi) \to H^{1,0}_{\mathrm{harm}}(X_t, E_\Phi)$ with similar properties as Θ_t except that Ψ_t is only uniformly Lipschitz in t. Fix now $\eta \in H^{1,0}_{\mathrm{harm}}(X_0, E_\Phi)$. For an E_Φ-valued 1-form φ write $\varphi = \varphi^{1,0} + \varphi^{0,1}$ for the decomposition of φ into components of types $(1,0)$ and $(0,1)$. We have

$$\eta_t^{0,1} = (f_t^*\eta)^{0,1} + (\eta_t - f_t^*\eta)^{0,1}.$$

By Lemma (1.4.1) we have uniform bounds $\|(\eta_t - f_t^*\eta)(x)\| \leq C|t|\,\|\eta\|$ for every $x \in X_t$. The component $(f_t^*\eta)^{0,1}$ may be interpreted as taking the $(0,1)$-component of a fixed differential 1-form with twisted coefficients with respect to a variable family of complex structures. Since η is of type $(1,0)$ if follows that $\|(f_t^*\eta)^{0,1}\| \leq C|t|\,\|\eta\|$. As a consequence $\|\eta_t^{0,1}(x)\| \leq C|t|\,\|\eta\|$ for every $x \in \mathcal{X}$. Define now $\Psi_t : H^{1,0}_{\mathrm{harm}}(X_0, E_\Phi) \to H^{1,0}_{\mathrm{harm}}(X_t, E_\Phi)$ by $\Psi_t(\eta) = \eta_t^{1,0}$. Then Ψ_t has the required properties.

For an E_Φ-valued smooth d-closed 1-form φ_t on X_t we denote by $[\varphi_t] \in V = H^1_{\mathrm{red}}(\Gamma, \Phi)$ its class in the reduced cohomology group. For two Hilbert spaces H' and H'' we denote by $\mathfrak{B}(H', H'')$ the Banach space of bounded operators T from H' and H'' equipped with the operator norm $\|T\| = \sup\{\|Tx\| : x \in H', \|x\| = 1\}$. In case $H' = H''$ we will also write $\mathfrak{B}(H')$ for the Banach algebra $\mathfrak{B}(H'; H')$. For each $t \in \triangle$ we have $\Psi_t \in \mathfrak{B}(H^{1,0}_{\mathrm{harm}}(X_0, E_\Phi), H^{1,0}_{\mathrm{harm}}(X_t, E_\Phi))$. We will identify each $H^1_{\mathrm{harm}}(X_t, E_\Phi)$ with $H^1_{\mathrm{red}}(\Gamma, \Phi) = V$ and by abuse of notation consider Ψ_t as an element of $\mathfrak{B}(W_0, W_t) \subset \mathfrak{B}(W_0, V)$. By Lemma (1.4.2) the continuous map $\Psi : \triangle \to \mathfrak{B}(W_0, V)$ is Lipschitz. It follows that for almost every $t_0 \in \triangle$, $\frac{\partial \Psi}{\partial t}(t_0)$ and $\frac{\partial \Psi}{\partial \bar{t}}(t_0)$ are defined.

To modify Ψ_t to Θ_t the key point is to show that whenever $\frac{\partial \Psi}{\partial \bar{t}}(t_0)$ is defined, $\frac{\partial \Psi}{\partial \bar{t}}(t_0) \in \mathfrak{B}(W_0, W_{t_0})$. To this end without loss of generality we assume that there exists a holomorphic section $\sigma : \triangle \to \mathcal{X}$ of the regular family $\pi : \mathcal{X} \to \triangle$. Integrating $\eta_t^{1,0}$ on $\widetilde{X}_t$ there exists a smooth function $F : \widetilde{\mathcal{X}} \to H$ such that

$$F(\gamma(p)) = \Phi(\gamma)F(p) + c_\gamma(t)$$

for every $x \in \widetilde{\mathcal{X}}$ and $\gamma \in \Gamma$ and such that F is holomorphic on each $\widetilde{X}_t$. Define $\mu : \widetilde{X}_0 \to H$ as follows. For any $t_0 \in \triangle$ and $p \in \widetilde{X}_{t_0}$, let $v \in T_p(\widetilde{\mathcal{X}})$ be such that v projects to $\frac{\partial}{\partial t}$ on $\triangle$. Since F is holomorphic on $\widetilde{X}_0$, $dF(\overline{v})$ does not depend on the choice of v. The function $\mu : \widetilde{X}_{t_0} \to H$ defined by $\mu(p) = dF(\overline{v})$ is holomorphic, and it transforms according to

$$\mu(\gamma(p)) = \Phi(\gamma)\mu(p) + \frac{\partial c_\gamma}{\partial \bar{t}}(t_0).$$

Thus, μ is the integral of an E_Φ-valued holomorphic 1-form $\nu = \nu_\eta(t_0)$ on X_{t_0}, and $\left(\frac{\partial c_\gamma}{\partial \bar{t}}(t_0)\right)_{\gamma \in \Gamma}$ is an associated 1-cycle.

We have the Hodge decomposition $V = W_t \oplus \overline{W_t}$. In the finite-dimensional case, $\dim(V) = 2p$, the assignment of $W_t \subset V$ corresponds to a mapping ρ of Δ into the Grassmannian $Gr(V; p)$ of n-dimensional vector subspaces of V. From the Lipschitz property of Ψ we conclude that $\rho : \Delta \to Gr(V;p)$ is Lipschitz. From the holomorphicity of $\nu_\eta(t)$ for almost all t the Lipschitz map ρ satisfies the Cauchy-Riemann equation $\overline{\partial}\rho = 0$ in the sense of distribution. Hence, ρ is holomorphic. To give an argument that also works in the infinite-dimensional situation we will instead produce a holomorphic trivialization of $\mathcal{W}$ in a neighborhood of 0 by solving differential equations. We will assume that V is infinite-dimensional, although obviously the same argument works in the finite-dimensional case.

Write $\mathcal{V} = V \times \Delta$ for the trivial bundle over Δ with fibers isomorphic to V. By the Lipschitz property of Ψ there is a Lipschitz trivialization of $\mathcal{W}$ on a neigborhood of 0. Thus, there exist Lipschitz sections $s_1(t), s_2(t), \cdots s_n(t), \cdots$ of $\mathcal{W}$ over a neighborhood $\Delta(\epsilon)$ of 0, such that $(s_1(0), s_2(0), \cdots, s_n(0), \cdots)$ is an orthonormal basis of W_0, and such that $(s_n(t))$ constitutes a uniformly Lipschitz family of sections over $\Delta(\epsilon)$. Shrinking ϵ if necessary we may assume that for each $t \in \Delta(\epsilon)$, $(s_1(t), s_2(t), \cdots s_n(t), \cdots)$ is a topological basis of W_t. Write $s(t)$ for the infinite column vector $[s_1(t), s_2(t), \cdots, s_n(t), \cdots]^T$. For almost all $t \in \Delta$, $\nu_\eta(t)$ is defined and holomorphic for all $\eta \in W_0$, which means that s satisfies some differential equation $\overline{\partial}s = As \otimes d\bar{t}$ in the sense of distribution. Here A is taken to be a Lipschitz map from $\Delta(\epsilon)$ into the Banach algebra $\mathfrak{B}(H)$ of a Hilbert space H of countably infinite dimensions. To produce a trivialization of $\mathcal{W}$ in a neighborhood of 0 we are going to replace s by some Zs, where $Z : \Delta(\epsilon) \to \mathfrak{B}(H)$.

We proceed to solve $\frac{\partial}{\partial \bar{t}}(Zs) = 0$, i.e., $\frac{\partial Z}{\partial \bar{t}} s + Z \frac{\partial s}{\partial \bar{t}} = 0$, giving $(ZA + \frac{\partial Z}{\partial \bar{t}})s = 0$. We impose the initial condition $Z(0) = I$, the identity map, and proceed to solve $\frac{\partial Z}{\partial \bar{t}} = -ZA$ on a neighborhood of 0. Using the Cauchy kernel we have a solution of $\frac{\partial Y}{\partial \bar{t}} = -A$, where $Y : \Delta(\epsilon) \to \mathfrak{B}(H)$ with $Y(0) = 0$. Let now $Z = \exp(Y)$. Then $\frac{\partial Z}{\partial \bar{t}} = \frac{\partial}{\partial \bar{t}}(exp(Y)) = \exp(Y)\frac{\partial Y}{\partial \bar{t}} = -ZA$; $Z(0) = I$, as desired. The proof of Proposition (1.4.1) is now completed by defining $\Theta_t(\eta) = Z\Psi_t(\eta)$ for any $\eta \in W_0$ and for t sufficiently small. □

REMARKS

It is possible to prove *a priori* that the harmonic forms η_t representing the same class in the reduced cohomology group $V = H^1_{\text{red}}(\Gamma, \Phi)$ varies smoothly in t even when the Hilbert space H is of countably infinite dimension, by the same method of boot-strapping and Schauder estimates as in the finite-dimensional case. For the sake of presenting a self-contained proof we have instead given a proof of Proposition 1.4.1 basing on the much easier property that the variation of η_t is at least Lipschitz.

§2 The canonical complex Finsler metric

(2.1) We are going to derive from $\pi : \mathcal{X} \to B$ and the variation of Hodge decompositions $V = H^1_{\text{red}}(\Gamma, \Phi) = W_t \oplus \overline{W}_t$, a canonical complex Finsler metric on B. Again take $B = \Delta$. Fix any $t_0 \in \Delta$. For any $[\eta] \in W_{t_0}, \eta \in H^{1,0}_{\text{harm}}(X_{t_0}, E_\Phi)$, let $\eta(t)$ be a holomorphic 1-parameter family of d-closed holomorphic 1-forms on X_t with values in E_Φ, defined for $|t - t_0|$ sufficiently small, such that $\eta(t_0) = \eta$. Then, $[\eta]'(0) \in V$. Suppose $\eta_1(t)$ and $\eta_2(t)$ are 2 holomorphic families with the same property. Then, the difference $[\eta_1(t) - \eta_2(t)]$ is a germ of holomorphic section of V vanishing at t_0. Writing $\xi(t) = \eta_1(t) - \eta_2(t), \xi(t_0) = 0$ and $[\xi]'(t_0) \in W_0$. Thus $[\eta]'(t_0) \mod W_{t_0} \in V/W_{t_0} \xrightarrow{\cong} \overline{W}_{t_0}$ is independent of the choice of $\eta(t)$ extending η. By the arguments of Proposition (1.4.1) the map $\eta \to [\eta]'(t_0) \mod W_{t_0}$ can be realized as a bounded linear operator $\kappa_{t_0} : W_{t_0} \to \overline{W}_{t_0}$. We call κ_{t_0} the Kodaira-Spencer class at t_0. Recall that $\mathfrak{B}(W_{t_0})$ denotes the Banach algebra of bounded linear operators on the Hilbert space W_{t_0} and that $\|T\|$ denotes its operator norm. Denote by $\mathcal{W} \subset V \times \Delta$ the subbundle with fiber W_t over $t \in \Delta$. We may consider $\mathcal{W} \subset V \times \Delta$ as a holomorphic vector bundle in an obvious way, by Proposition (1.4.1). Let now $\alpha \in W_{t_0}$, and $\alpha(t)$ be a germ of holomorphic section of $\mathcal{W}$ at t_0. Write $\alpha'(t_0) = \xi + \overline{\mu}$, where $\xi, \mu \in W_{t_0}$. Since S vanishes identically on W_{t_0}, we have

$$S(\alpha, \alpha'(t_0)) = S(\alpha, \xi) + S(\alpha, \overline{\mu}) = S(\alpha, \overline{\mu}).$$

As explained $\alpha(t)$ can be modified to $\widetilde{\alpha}(t)$ with $\widetilde{\alpha}'(0) = \overline{\mu} \in \overline{W}_{t_0}$. Since $S(\cdot, \bar{\cdot})$ is a Hermitian inner product on W_{t_0}, by the Cauchy-Schwarz Inequality

$$|S(\alpha, \overline{\mu})|^2 \leq |S(\alpha, \overline{\alpha})| \, |S(\mu, \overline{\mu})|,$$

so that

$$\frac{|S(\alpha, \overline{\mu})|^2}{|S(\alpha, \overline{\alpha})|^2} \leq \frac{|S(\mu, \overline{\mu})|}{|S(\alpha, \overline{\alpha})|} \leq \|\kappa_{t_0}\|$$

for the Kodaira-Spencer class $\kappa_{t_0} \in \mathfrak{B}(W_{t_0})$. From this we can define

DEFINITION (2.1.1). *For $t_0 \in \Delta$, letting $\mathcal{O}_{t_0}(\mathcal{W})$ be the space of germs of holomorphic sections of $\mathcal{W}$ at t_0, we define a semi-norm $\|\cdot\|$ on the holomorphic tangent bundle $T(\Delta)$ by*

$$\left\|\frac{\partial}{\partial t}\right\|^2(t_0) = \sup\left\{\frac{|S(\alpha,\alpha'(t_0))|^2}{|S(\alpha,\overline{\alpha})|^2} : 0 \neq \alpha \in W_{t_0}, \alpha(t) \text{ is an extension of } \alpha \text{ to a germ in } \mathcal{O}_{t_0}(W)\right\}.$$

We call $\|\cdot\|$ the canonical complex Finsler metric on Δ induced by $\pi : \mathcal{X} \to \Delta$.

We have $\|\frac{\partial}{\partial t}\|(t_0) \leq \|\kappa_{t_0}\|$ for the Kodaira-Spencer class, $\kappa_{t_0} \in \mathfrak{B}(W_{t_0})$. The two norms actually agree, but we will not need this fact here. For our purpose it is sufficient to establish

LEMMA (2.1.1). *Suppose $t_0 \in \Delta$ is such that $\|\frac{\partial}{\partial t}\|(t_0) = 0$. Then, the Kodaira-Spencer class $\kappa_{t_0} : W_{t_0} \to \overline{W}_{t_0}$ vanishes.*

PROOF. The canonical complex Finsler metric $\|\cdot\|$ vanishes at $t_0 \in \Delta$ if and only if $S(\alpha, \alpha'(t_0)) = 0$ for every $\alpha \in W_{t_0}$ and for every holomorphic extension $\alpha(t)$ of α to a neighborhood of t_0. We argue that this implies $S(\alpha, \beta'(t_0)) = 0$ for every $\alpha, \beta \in W_{t_0}$ and for every holomorphic extension $\alpha(t)$ of α to a neighborhood of t_0. To see this recall that $\alpha(t)$ and $\beta(t)$ are of type $(1,0)$ at t, so that $S(\alpha(t), \beta(t)) = 0$. Differentiating we have

$$S(\alpha'(t), \beta(t)) + S(\alpha(t), \beta'(t)) = 0. \tag{1}$$

Assume now $\|\frac{\partial}{\partial t}\|(t_0) = 0$. Then,

$$S(\alpha + s\beta, \alpha' + s\beta')(t_0) = 0 \quad \text{for any} \quad s \in \mathbb{C},$$

so that

$$S(\alpha, \alpha') + s(S(\alpha, \beta') + S(\beta, \alpha')) + s^2 S(\beta, \beta') = 0 \quad \text{at} \quad t_0 \tag{2}$$

for every $s \in \mathbb{C}$, showing that

$$S(\alpha(t_0), \beta'(t_0)) + S(\beta(t_0), \alpha'(t_0)) = 0. \tag{3}$$

Comparing with (1) at $t = t_0$ and using the skew-symmetry of S, we conclude that

$$S(\alpha(t_0), \beta'(t_0)) = 0 \tag{4}$$

for any choice of $\alpha = \alpha(t_0), \beta = \beta(t_0) \in W_{t_0}$. In other words, $S(\alpha, \kappa_{t_0}(\beta)) = 0$ for any $\alpha, \beta \in W_{t_0}$. Fixing β and choosing $\alpha = \overline{\kappa_{t_0}(\beta)} \in W_{t_0}$, we conclude from (4) that

$$S(\overline{\kappa_{t_0}(\beta)}, \kappa_{t_0}(\beta)) = \|\kappa_{t_0}(\beta)\|^2 = 0 \tag{5}$$

for any choice of $\beta \in W_{t_0}$. It follows that $\|\frac{\partial}{\partial t}\|(t_0) = 0$ if any only if $\kappa_{t_0} = 0$, as desired. □

(2.2) We proceed to make use of the canonical Finsler metric as defined in (2.1) to prove hyperbolicity properties of base spaces. We are going to establish

MAIN THEOREM. *Let $\pi : \mathcal{X} \to B$ be a regular holomorphic family of compact Kähler manifolds over a complex manifold B and assume that the total space $\mathcal{X}$ is equipped with a fixed Kähler metric g. For a typical fiber $X_0 = \pi^{-1}(0)$ let $\Phi : \pi_1(X_0) \to U(H)$ be a unitary representation such that the reduced first cohomology group $H^1_{\mathrm{red}}(\Gamma, \Phi) \neq 0$; $\Gamma := \pi_1(X_0)$. Then, either the Hodge decomposition of $H^1_{\mathrm{red}}(\Gamma, \Phi) \otimes_{\mathbb{R}} \mathbb{C}$ on X_t is locally constant in t or it induces in a canonical way a possibly degenerate continuous complex Finsler metric of holomorphic sectional curvature ≤ -2 on the universal cover $\widetilde{B}$ of B. In particular, if B is a compact Riemann surface C then either the Hodge decomposition is locally constant, or C is of genus ≥ 2.*

PROOF. Fix $t_0 \in \Delta, 0 \neq \alpha_0 \in W_{t_0}$, and choose an extension of α_0 to a germ $\alpha(t)$ in $\mathcal{O}_{t_0}(\mathcal{W})$ so that $\alpha'(t_0) \in \overline{W}_{t_0}$. Consider the germ of real-analytic function h_0 at t_0 defined by

$$h_0(t) = \frac{|S(\alpha(t), \alpha'(t))|^2}{|S(\alpha(t), \overline{\alpha(t)})|^2}.$$

We have

$$\log h_0(t) = \log |S(\alpha(t), \alpha'(t))|^2 - \log |S(\alpha(t), \overline{\alpha(t)})|^2.$$

Since $\alpha(t)$ and $\alpha'(t)$ are holomorphic in $t, \log |S(\alpha(t), \alpha'(t))|^2$ is pluriharmonic in t, and we have

$$\frac{\partial^2}{\partial t\, \partial \bar{t}} \log h(t_0) = -\frac{\partial^2}{\partial t\, \partial \bar{t}} \log |S(\alpha, \overline{\alpha})|^2(t_0).$$

Here we use α to denote the choice of a germ in $\mathcal{O}_{t_0}(\mathcal{W})$ extending $\alpha_0 \in W_{t_0}$, as described in the above. We have $S(\alpha, \overline{\alpha}) > 0$ and

$$\frac{\partial}{\partial t} S(\alpha, \overline{\alpha}) = S(\alpha', \overline{\alpha}).$$

By the choice of $\alpha, \alpha'(t_0) \in \overline{W}_{t_0}$. Since S vanishes on $\overline{W}_{t_0}$ we have

$$\frac{\partial}{\partial t} S(\alpha, \overline{\alpha})(t_0) = 0.$$

Now

$$\frac{\partial^2}{\partial t\, \partial \bar{t}} \log S(\alpha, \overline{\alpha}) = \frac{1}{S(\alpha, \overline{\alpha})} \frac{\partial^2}{\partial t\, \partial \bar{t}} S(\alpha, \overline{\alpha}) - \frac{1}{S(\alpha, \overline{\alpha})^2} \left| \frac{\partial}{\partial t} S(\alpha, \overline{\alpha}) \right|^2,$$

so that

$$\frac{\partial^2}{\partial t\, \partial \bar{t}} \log |S(\alpha, \overline{\alpha})|^2(t_0) = \frac{2}{S(\alpha, \overline{\alpha})} \frac{\partial^2}{\partial t\, \partial \bar{t}} S(\alpha, \overline{\alpha})(t_0).$$

Now

$$\frac{\partial^2}{\partial t\, \partial \bar{t}} S(\alpha, \overline{\alpha}) = \frac{\partial}{\partial \bar{t}} S(\alpha', \overline{\alpha}) = S(\alpha', \overline{\alpha}').$$

At $t_0, \alpha'(t_0) \in \overline{W}_{t_0}$, so that

$$S(\alpha', \overline{\alpha}')(t_0) = -S(\overline{\alpha}', \alpha')(t_0) = -\|\alpha'(t_0)\|^2$$

with respect to the canonical norm on W_{t_0}. It follows that

$$\frac{\partial^2}{\partial t\, \partial \bar{t}} \log h(t_0) = -2 \frac{\partial^2}{\partial t\, \partial \bar{t}} \log S(\alpha, \overline{\alpha})(t_0) = 2 \frac{S(\overline{\alpha}', \alpha')}{S(\alpha, \overline{\alpha})}(t_0) = 2 \frac{\|\alpha'(t_0)\|^2}{\|\alpha(t_0)\|^2}.$$

On the other hand, by the Cauchy-Schwarz Inequality,

$$h(t_0) = \frac{|S(\alpha(t_0), \alpha'(t_0))|^2}{|S(\alpha(t_0), \alpha(t_0))|^2} \le \frac{\|\alpha(t_0)\|^2 \|\alpha'(t_0)\|^2}{\|\alpha(t_0)\|^4} = \frac{\|\alpha'(t_0)\|^2}{\|\alpha(t_0)\|^2},$$

and we have proven the fundamental inequality

$$\frac{\partial^2}{\partial t\, \partial \bar{t}} \log h(t_0) = 2\frac{\|\alpha'(t_0)\|^2}{\|\alpha(t_0)\|^2} \ge 2h(t_0)$$

which amounts to saying that, if $h(t_0) \neq 0$, then $h|dt|^2$ is a Hermitian metric of negative curvature ≤ -2 at t_0. By taking suprema in the definition of $\|\cdot\|$ on $T(\triangle)$ we expect to get a Hermitian metric with the same property everywhere in the generalized sense, i.e., defining the canonical complex Finsler metric on $\triangle$ by $h(t)|dt|^2$ we expect $\frac{\partial^2}{\partial t\, \partial \bar{t}} \log h \ge 2h$ everywhere on $\triangle$ in the generalized sense. This needs a justification when $V = H^1_{\mathrm{red}}(\Gamma, \Phi)$ is infinite-dimensional. The justification comes from the uniformity of extension of α_0 to a neighborhood, as given in the proof of Proposition (1.4.1), according to which for some $\epsilon > 0$ there exists $\Theta_t : H^{1,0}_{\mathrm{harm}}(X_{t_0}, E_\Phi) \to H^{1,0}_{\mathrm{harm}}(X_t, E_\Phi)$ such that Θ_t are invertible bounded linear operators, with uniform bounds, and such that for $\eta \in H^{1,0}_{\mathrm{harm}}(X_{t_0}, E_\Phi), \Theta_t(\eta) = \eta_t$ is holomorphic in t. From the identification $H^{1,0}_{\mathrm{harm}}(X_{t_0}, E_\Phi) \cong W_t \subset V$, we obtain $\Theta_t : W_{t_0} \to W_t \subset V$, such that $\Theta_t([\eta])$ is holomorphic in t. Let now $\alpha_0 \in W_{t_0}, \alpha_0 = [\eta]$ and define $\alpha(t)$ on $\triangle_\epsilon(t_0)$ by $\alpha(t) = \Theta_t([\eta]) = [\Theta_t(\eta)]$. The Hermitian metric $h(t)|dt|^2$ depends on the choice of $\alpha_0 \in W_{t_0}$ and the extension α. We will fix the extension α as in the above. The definition of $\|\cdot\|$ on $\triangle_\epsilon(t_0)$ can be given by

$$\left\|\frac{\partial}{\partial t}\right\|^2 (t) = \sup_\alpha \left\{ \frac{|S(\alpha(t), \alpha'(t))|^2}{|S(\alpha(t), \alpha(t))|^2} : \alpha(t_0) \in W_{t_0}, \alpha(t) = \Theta_t(\alpha) \right\}.$$

Note that it is not necessary (and in general not possible) to require that $\alpha'(t) \in \overline{W}_t$ for a general point $t \in \triangle_\epsilon(t_0)$. For $\alpha_0 \in W_{t_0}$ and $0 \neq \alpha \in \Gamma(\triangle_\epsilon(t), \mathcal{W})$ as defined we will write h_α for h_0, so that on $\triangle_\epsilon(t)$,

$$h(t) = \sup_{\alpha_0 \neq 0} h_\alpha(t).$$

Here we note that it is sufficient to consider only those α for which $\alpha_0 = \alpha(t_0)$ is of unit length in W_{t_0}, since $h_{\lambda\alpha} = h_\alpha$ for any $\lambda \in \mathbb{C}^*$. Noting that derivatives of α for $\|\alpha_0\| = 1$ are uniformly bounded on compact subsets of $\triangle_\epsilon(t)$ from the uniform boundedness of Θ_t and from Cauchy estimates, it follows readily that h is a continuous function. Moreover, the uniform bounds on derivatives also imply that these exists $C > 0$ independent of α_0 such that

$$\frac{\partial^2}{\partial t\, \partial \bar{t}} \log h_\alpha(t) \ge 2h_\alpha(t) - C|t - t_0|.$$

We conclude that given any $\delta > 0$, there exists $\epsilon(\delta) > 0$, such that

$$\frac{\partial^2}{\partial t\, \partial \bar{t}} \left(\log h_\alpha + \delta |t - t_0|^2\right)(t) \ge 2h_\alpha(t)$$

on $\triangle_{\epsilon(\delta)}(t_0)$ for every $\alpha_0 \in W_{t_0}$, $\|\alpha_0\| = 1$. Taking suprema, we conclude that

$$\frac{\partial^2}{\partial t\, \partial \bar{t}}\big(\log h + \delta|t - t_0|^2\big)(t) \geq 2h(t)$$

on $\triangle_{\epsilon(\delta)}(t_0)$ in a generalized sense. Passing to limits at t_0 as δ decreases to 0 we conclude that

$$\frac{\partial^2}{\partial t\, \partial \bar{t}} \log h(t_0) \geq 2h(t_0).$$

Since the same remains true when t_0 is replaced by $t \in \triangle_\epsilon(t_0)$ with obvious modifications of the notations, we have shown that $h(t)|dt|^2$ is a possibly degenerate continuous complex Finsler metric of Gaussian curvature ≤ -2.

Let now $\pi : \mathcal{X} \to B$ be the regular family of compact Kähler manifolds as in the hypothesis of the Main Theorem and denote by $\rho : \widetilde{B} \to B$ the universal cover. Write $\widetilde{\pi} : \widetilde{\mathcal{X}} \to \widetilde{B}$, for the induced family of compact Kähler manifolds. $\pi : \widetilde{\mathcal{X}} \to \widetilde{B}$ is diffeomorphically trivial, i.e., $\widetilde{\mathcal{X}} \cong X_0 \times \widetilde{B}$ diffeomorphically. Given a representation $\Phi : \pi_1(X_0) \to U(H)$ as in the hypothesis of the Main Theorem such that $H^1_{\text{red}}(\Gamma, \Phi) \neq 0, \Gamma := \pi_1(X_0)$, we have a trivial bundle $V \times \widetilde{B}$ over $\widetilde{B}, V = H^1_{\text{red}}(\Gamma, \Phi) \cong H^1_{\text{harm}}(X_0, E_\Phi)$, and a Hodge decomposition $V = W \oplus \overline{W}$ over $\widetilde{B}$. The construction of a canonical complex Finsler metric as described gives a (possible degenerate) continuous complex Finsler metric of holomorphic curvature ≤ -2 on $\widetilde{B}$. In the special case when B is a compact Riemann surface, $\widetilde{B}$ must be conformally equivalent to the unit disk, by the Uniformization Theorem and the Ahlfors-Schwarz Lemma. The proof of the Main Theorem is complete. □

From the proof of the Main Theorem the assumption that the total space of $\pi : \mathcal{X} \to B$ carries a Kähler metric can be considerably weakened. It suffices to assume that the fibers $X_t = \pi^{-1}(t)$ are equipped with Kähler metrics with Kähler forms ω_t, varying continuously in t, such that, in the notations of the proof of Lemma (1.3.1), $f_t^*\omega_0$ is cohomologous to ω_t. This is the same as saying that $[\omega_t] \in H^2(X_t, \mathbb{R})$ is locally constant in t.

We turn to the special case of regular families of polarized manifolds. By a polarized projective manifold we mean a pair (X, η) where X is a projective manifold and $\eta \in H^2(X, \mathbb{Z})$ is the first Chern class some ample line bundle on X. By a regular family $\pi : (\mathcal{X}, \eta) \to B$ of polarized projective manifolds we mean a regular family $\pi : \mathcal{X} \to B$ of projective manifolds, equipped with $\eta_t \in H^2(X_t, \mathbb{Z})$ for the fiber $X_t = \pi^{-1}(t)$, such that η_t varies continuously in t and (X_t, η_t) is a polarized projective manifold. We have

Corollary to the Proof of the Main Theorem. *Let $\pi : (\mathcal{X}, \eta) \to B$ be a regular family of polarized projective manifolds over a complex manifold B. Then the analogue of the Main Theorem holds for the Hodge decomposition of $H^1_{\text{red}}(\Gamma, \Phi) \otimes_{\mathbb{R}} \mathbb{C}$, without assuming that there exists a Kähler metric on the total space $\mathcal{X}$.*

Remarks

The Hodge decomposition of $H^1_{\text{red}}(\Gamma, \Phi) \otimes_{\mathbb{R}} \mathbb{C}$ on X_t does not depend on the specific choice of a Kähler metric on X_t.

Proof of Corollary. As $t \in B$ varies, $H^2(X_t, \mathbb{Z})$ constitutes a locally constant bundle $\mathbb{H}$ of Abelian groups with discrete fibers. Since $\eta_t \in H^2(X_t, \mathbb{Z})$ varies continuously, it defines a locally constant section of $\mathbb{H}$ over B. For any $x \in B$ there exists an open neighborhood U of x in B together with a smooth family of Kähler metrics ω_t on X_t, $t \in U$, such that $[\omega_t] = \eta_t$. The arguments of Lemma (1.3.1) then work over U to define the symplectic form S on $V = H^1_{\mathrm{red}}(\Gamma, \Phi) \cong H^1_{\mathrm{harm}}(X_t, E_\Phi)$ and to prove that it is independent of t. We note again that the Hodge decomposition $H^{1,0}_{\mathrm{harm}}(X_t, E_\Phi) = H^{1,0}_{\mathrm{harm}}(X_t, E_\Phi) \oplus H^{0,1}_{\mathrm{harm}}(X_t, E_\Phi)$ is actually independent of the choice of Kähler metric on X_t, since the two direct summands consist of twisted holomorphic resp. anti-holomorphic 1-forms. Over X_t the symplectic form S is defined using ω_t, but it only depends on its Kähler class η_t, which is given. As the rest of the arguments leading to the Main Theorem are local over B, the local choices of Kähler forms ω_t, $t \in U$, suffices. $\square$

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