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Characterization of Certain Holomorphic Geodesic Cycles on Quotients of Bounded Symmetric Domains in terms of Tangent Subspaces

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Abstract. Let Ω be an irreducible bounded symmetric domain and $\Gamma \subset \mathrm{Aut}(\Omega)$ be a torsion-free discrete group of automorphisms, $X := \Omega/\Gamma$. We study the problem of algebro-geometric and differential-geometric characterizations of certain compact holomorphic geodesic cycles $S \subset X$. We treat special cases of the problem, pertaining to a situation in which S is a compact holomorphic curve, and to the case where Ω is a classical domain dual to the hyperquadric. In both cases we consider algebro-geometric characterizations in terms of tangent subspaces. As a consequence we derive effective pinching theorems where certain complex submanifolds $S \subset X$ are proven to be totally geodesic whenever their scalar curvatures are pinched between certain computed universal constants, *independent of the volume of the submanifold S*, giving new examples of the gap phenomenon for the characterization of compact holomorphic geodesic cycles.

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Let Ω be an irreducible bounded symmetric domain and $\Gamma \subset \mathrm{Aut}(\Omega)$ be a discrete group of automorphisms acting without fixed points. Consider the Hermitian locally symmetric manifold $X := \Omega/\Gamma$. It is interesting, especially in the case when X is compact or of finite volume, to characterize compact holomorphic geodesic cycles $S \subset X$. Such characterizations can either be in algebro-geometric or differential-geometric terms. In this article we study special cases of this problem, pertaining to a situation in which S is a compact holomorphic curve, and to the case where Ω is a classical domain dual to the hyperquadric. In both cases we will be considering algebro-geometric characterizations in terms of tangent subspaces. Such characterizations imply differential-geometric effective pinching theorems where certain complex submanifolds $S \subset X$ are proven to be totally geodesic whenever their scalar curvatures are pinched between certain computed universal constants, *independent of the volume*

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of the submanifold S. These pinching theorems are related to the gap phenomenon discussed in Mok [M3] and Eyssidieux and Mok [EM], which in particular include the characterization of certain totally-geodesic compact holomorphic curves on quotients of the Siegel upper-half-plane.

Identify Ω as a subdomain of its compact dual M, by Borel Embedding. Let G_o be the identity component of the automorphism group of Ω and $K \subset G_o$ be the isotropy subgroup at a point $o \in \Omega$, so that $\Omega = G_o/K$. Write G for the identity component of the automorphism group of M and $P \subset G$ for the isotropy subgroup at o. Consider the action of P on $\mathbb{P}T_o(M)$. There are precisely r orbits $\mathcal{O}_k$, $1 \leqslant k \leqslant r$, such that the topological closures $\bar{\mathcal{O}}_k$ form an ascending chain of subvarieties of $\mathbb{P}T_o(M)$, with $\bar{\mathcal{O}}_r = \mathbb{P}T_o(M)$. We say that a nonzero vector η is of rank k if its projectivization belongs to $\mathcal{O}_k$. For the purpose of this article we will say that η or $[\eta]$ is *generic* if and only if η is of rank r. Given any η of rank k, $k < r$, the closure of the G-orbit of its projectivization $[\eta]$ defines a holomorphic bundle of projective subvarieties $\mathcal{S}_k(M) \subsetneq \mathbb{P}T_M$, called the k-th characteristic bundle, where $\mathcal{S}_{k,o}(M) = \bar{\mathcal{O}}_k$. The restrictions $\mathcal{S}_k(\Omega)$ to Ω are invariant under the action of G_o. Their quotients under $\Gamma \subset G_o$ will be denoted by $\mathcal{S}_k(X)$. For the purpose of this article we will be concerned solely with the case of $k = r - 1$. We call $\mathcal{S}_{r-1}(X)$ the highest characteristic bundle. Thus, η is generic if and only if $[\eta]$ does not lie on the highest characteristic bundle $\mathcal{S}_{r-1}(X)$.

For Ω of rank r, with respect to the Bergman metric there are precisely r equivalence classes of totally-geodesic holomorphic curves on Ω under the action of G_o. The kth equivalence class is represented by a totally-geodesic holomorphic disk Δ_k in Ω whose tangent space at each point is spanned by a vector of rank k. A totally-geodesic holomorphic curve $C \subset X$ will be said to be of type k if and only if C is uniformized by a holomorphic disk in Ω equivalent to Δ_k (said to be of type k) under some holomorphic isometry of Ω.

In [M3] and [EM] we considered compact complex submanifolds of quotients of bounded symmetric domains which are almost totally geodesic in some precise sense and asked whether such submanifolds are necessarily totally geodesic. For a (totally-geodesic) bounded symmetric subdomain $D \subset \Omega$, we say that (Ω, D) exhibits the gap phenomenon if any almost geodesic compact complex submanifold S on some quotient manifold X of Ω modeled on D must necessarily be totally-geodesic. In the special case when Ω is the Siegel upper-half-plane $\mathcal{H}_r$ of rank r, we proved an effective pinching theorem which says in particular that an almost geodesic compact holomorphic curve modeled on a totally-geodesic holomorphic disk of type r is necessarily totally-geodesic. There the proof relies on interpreting the Siegel upper-half-plane as the parameter space for variations of Hodge structures of weight 1. Using variations of Hodge structures and studying Euler characteristics, Eyssidieux [E1] introduced the notion of 0-hyperrigid and 1-hyperrigid subdomains and gave in [E1, 2] algebro-geometric characterizations of certain totally-geodesic complex submanifolds which in particular imply the gap phenomenon for (Ω, D) for Ω an irreducible bounded symmetric domain and $D \subset \Omega$ a 1-hyperrigid subdomains. His results in [E1] cover the cases of $(\mathcal{H}_r, \Delta_r)$, $(D^{\mathrm{I}}_{r,r}, \Delta_r)$, $(D^{\mathrm{II}}_{2r}, \Delta_r)$ and many cases of (Ω, D) in which Ω is irreducible and D is irreducible and of complex dimension > 1. Many new examples of 1-hyperrigid domains $D \subset \Omega$ were discovered in [E2].

Here we solve the problem of characterizing compact totally-geodesic holomorphic curves of type $r = \mathrm{rank}(\Omega)$ in a uniform way whenever the highest characteristic bundle $\mathcal{S}_{r-1}(X)$ is of codimension 1 in $\mathbb{P}T_X$. We say in this case that Ω is of characteristic codimension 1. For the ensuing explanation assume that X is itself compact. Let $C \subset X$ be a nonsingular compact holomorphic curve and $\hat{C}$ be the tautological lifting of C to $\mathbb{P}T_X$. The highest characteristic bundle $\mathcal{S}_X := \mathcal{S}_{r-1}(X) \subset \mathbb{P}T_X$ is a complex-analytic subvariety of codimension 1. The intersection number $\mathcal{S} \cdot \hat{C}$ gives an invariant on C. We show that this invariant is always nonnegative, and is zero if and only if C is a totally-geodesic holomorphic curve of type r. This then leads to an algebro-geometric characterization of totally-geodesic compact holomorphic curves C of type r as the only compact smooth holomorphic curves for which all nonzero tangent vectors are generic (i.e., of rank r). In addition to (sub)-series of classical domains, our result also applies to the 27-dimensional exceptional domain D^{VI} pertaining to E_7. The latter is particularly interesting since an exceptional bounded symmetric domain cannot parametrize a nontrivial homogeneous holomorphic family of variations of Hodge structures of weight 1 [Sa]. We note also that our algebro-geometric characterization is in a sense optimal. In fact, the analogous statement fails on any other irreducible bounded symmetric domain of dimension $\geqslant 2$.

In [M1, 2] we established a rigidity theorem for Hermitian metrics of seminegative curvature on a projective manifold X uniformized by an irreducible bounded symmetric domain of rank $\geqslant 2$. As a consequence, any nontrivial holomorphic map f into a Hermitian manifold Z of seminegative curvature in the sense of Griffiths is necessarily an isometric immersion, totally-geodesic with respect to the Hermitian connection on Z. From the function-theoretic perspective the relevant consequence is that f is a holomorphic immersion. One motivation to study intersection theory on $\mathbb{P}T_X$ is to study holomorphic maps for target complex manifolds Z satisfying much weaker conditions of seminegativity. For instance, if Z is uniformized by a bounded domain in some Stein manifold, then the Carathéodory metric on Z is a continuous complex Finsler metric of seminegative curvature in the generalized sense. For the special case when X is of characteristic codimension 1, we give an an application of our result on totally-geodesic holomorphic curves to show that any nontrivial holomorphic map $f: X \to Z$ must be generically finite.

Our results in the situation of compact holomorphic curves naturally lead to the question of algebro-geometric characterization of certain holomorphic geodesic cycles of higher dimensions. We prove a result of this nature for bounded symmetric domains Ω dual to the hyperquadric. Ω and, hence, X is endowed with a canonical quadric structure, i.e., a conformal class of nondegenerate holomorphic symmetric 2-tensors. In this case, we characterize k-dimensional totally geodesic cycles dual to hyperquadrics in terms of the nondegeneracy of the restriction of the quadric structure. The proof for $k > 1$ relies on the result of Kobayashi–Ochiai [KO] on the integrability of G-structures on compact Kähler–Einstein manifolds modeled on irreducible Hermitian symmetric manifolds of rank > 1, together with the Hermitian metric rigidity theorem of Mok [M1]. The case of $k = 1$ is a special case of our results of Section 2. Our results for Ω dual to the hyperquadric lead also to the differential-geometric characterization of certain holomorphic geodesic cycles under some pinching conditions on their scalar curvatures. It leads to new examples of the gap phenomenon beyond those in [M3], [EM] and [E1, 2].

1. Characteristic Bundles

(1.1) Let Ω be an irreducible bounded symmetric domain. Let G_o be the identity component of the automorphism group of Ω and $K \subset G_o$ be the isotropy subgroup at a point $o \in \Omega$, so that $\Omega = G_o/K$. Ω is a Hermitian symmetric space with respect to the Bergman metric. K acts faithfully on the real tangent space $T_o^{\mathbb{R}}(\Omega)$. Let $\mathfrak{a}$ be a maximal Abelian subspace of the real tangent space, whose dimension r is the rank of Ω as a Riemannian symmetric space. Then, $T_o^{\mathbb{R}}(\Omega) = \bigcup_{k \in K} k\mathfrak{a}$. We have $\mathfrak{a} \cap J\mathfrak{a} = 0$ for the canonical complex structure J on Ω. The complexification $(\mathfrak{a} + J\mathfrak{a}) \otimes_{\mathbb{R}} \mathbb{C}$ decomposes into $\mathfrak{a}^+ \oplus \overline{\mathfrak{a}^+}$, where $\mathfrak{a}^+ \subset T_o(\Omega)$, the holomorphic tangent space at o, so that $T_o(\Omega) = \bigcup_{k \in K} k\mathfrak{a}^+$. Thus, any holomorphic tangent vector ξ at o is equivalent under the action of the isotropy group to some $\eta \in \mathfrak{a}^+$. By the Polydisk Theorem (cf. [W]), there exists a totally-geodesic (holomorphic) polydisk $D \cong \Delta^r$ passing through o, $D \subset \Omega$, such that $T_o(D) = \mathfrak{a}^+$. With respect to Euclidean coordinates on $D \cong \Delta^r$, we can write $\eta = (\eta_1, \ldots, \eta_r)$. For $\eta \neq 0$ we will say that η is of rank k, $1 \leqslant k \leqslant r$, if and only if exactly k of the coefficients η_j are nonzero. The automorphisms of D, including those which permute the individual disk factors, extend to global automorphisms of Ω belonging to G_o. Thus, any $\eta \in T_o(\Omega)$ is equivalent under K to a vector $\eta = (\eta_1, \ldots, \eta_r)$, such that each coefficient is real and $\eta_1 \geqslant \eta_2 \cdots \geqslant \eta_r \geqslant 0$. η is, furthermore, uniquely determined by ξ. We call η the normal form of ξ under K.

Consider now Ω as a subdomain of its compact dual M, by Borel Embedding. Write G for the identity component of the automorphism group of M and $P \subset G$ for the isotropy subgroup at o. $G \supset G_o$ is a complexification of G_o. Consider the action of P on $\mathbb{P}T_o(M)$. Let L be a Levi subgroup of P, which can be taken to be a complexification of K. We write $L := K^{\mathbb{C}}$. We have $P = K^{\mathbb{C}} \cdot M^-$, where M^- is the unipotent radical of P. M^- is Abelian and acts trivially on $T_o(M) = T_o(\Omega)$. Let H_o be the identity component of the automorphism group of the polydisk D, $H_o \subset G_o$. Then, $H_o \cong \mathrm{SU}(1,1)^r$. Its complexification H inside G, $H \cong \mathrm{SL}(2,\mathbb{C})^r$, acts transitively on a polysphere $\Sigma \cong \mathbb{P}_1^r$ such that $(D;\Sigma)$, $D \subset \Sigma$, is a dual pair of Hermitian symmetric spaces. Since H contains $\mathbb{C}^* \times \cdots \times \mathbb{C}^*$ (r times), any $\eta = (\eta_1, \ldots, \eta_r)$ in the normal form under K must be equivalent under $K^{\mathbb{C}}$ to a vector of the form $\eta^{(k)} = (1, \ldots 1; 0, \ldots 0)$, with exactly the first k entries being equal to 1, where k is the rank of η. In particular, two nonzero vectors of the same rank in $T_o(\Omega) = T_o(M)$ are equivalent under $K^{\mathbb{C}}$ and, hence, under P. Moreover, using the action of $(\mathbb{C}^*)^r$, for $k < r$ any nonzero vector of rank k is a limit of vectors of rank $k+1$. We note furthermore, that nonzero vectors of different rank cannot be equivalent under P, i.e., under $K^{\mathbb{C}}$. To see this, denote by $\mathfrak{g}$ the Lie algebra of G, etc. and write $\mathfrak{g} = \mathfrak{m}^+ + \mathfrak{k}^{\mathbb{C}} + \mathfrak{m}^-$ for the Harish-Chandra decomposition of $\mathfrak{g}$, where $\mathfrak{m}^+$ is identified with $T_o(M)$. For $\xi \in T_o(M)$ let $\delta(\xi)$ be the dimension of all $\zeta \in \mathfrak{m}^-$ such that $[\xi, \zeta] = 0$. Then, $\delta(\xi)$ is invariant under the action of $K^{\mathbb{C}}$ on ξ. Writing δ_k for $\delta(\eta^{(k)})$ one deduces readily from root space decompositions that $\delta_1 > \delta_2 > \cdots > \delta_r = 0$, which shows that vectors of different rank cannot be $K^{\mathbb{C}}$-invariant. (The relations on δ_k can be deduced from the fact that $\mathfrak{a}^+$ is spanned by root vectors belonging to a maximal set of strongly orthogonal positive roots with respect to some Cartan subalgebra of $\mathfrak{g}$ lying in $\mathfrak{k}^{\mathbb{C}}$.) We note that any $\eta \in T_o(\Sigma)$ of rank k is tangent to a rational curve $C \subset \Sigma$ of degree k with respect to the positive generator of $\mathrm{Pic}(M) \cong Z$.

For a nonzero $\eta \in T_o(\Omega)$ of rank k we will also say that its projectivization $[\eta] \in \mathbb{P}T_o(\Omega)$) is of rank k. In this article we will say that η or $[\eta]$ is *generic* if and only if η is of rank r. For the action of P on $\mathbb{P}T_o(M)$ it follows from the above that there are precisely r orbits $\mathcal{O}_k$, $1 \leqslant k \leqslant r$, such that the topological closures $\overline{\mathcal{O}}_k$ form an ascending chain of subvarieties of $\mathbb{P}T_o(M)$, with $\overline{\mathcal{O}_r} = \mathbb{P}T_o(M)$. Here $\mathcal{O}_k = P[\eta]$ for any $[\eta] \in \mathbb{P}T_o(M)$ of rank k. Given any η of rank k, $k < r$, the closure of the G-orbit of its projectivization $[\eta]$ defines a holomorphic bundle of projective subvarieties $\mathcal{S}_k(M) \subseteq \mathbb{P}T_M$, called the kth characteristic bundle, where $\mathcal{S}_{k,o}(M) = \overline{\mathcal{O}}_k$. The restrictions $\mathcal{S}_k(\Omega)$ to Ω are invariant under the action of G_o. Their quotients under $\Gamma \subset G_o$ will be denoted by $\mathcal{S}_k(X)$. In this article we will be concerned solely with the case of $k = r - 1$. We call $\mathcal{S}_{r-1}(X)$ the highest characteristic bundle. It consists precisely of projectivizations of nongeneric vectors η.

(1.2) Write $\mathcal{S}_M$ for $\mathcal{S}_{r-1}(M)$, etc. We say that M, Ω and X are of characteristic codimension q, whenever $\mathcal{S}_M \subset \mathbb{P}T_M$ is of codimension q. We consider henceforth, the case when $q = 1$. Then, $\mathcal{S}_M \subset \mathbb{P}T_M$ defines a divisor line bundle, to be denoted by $[\mathcal{S}_M]$. Let $\pi\colon\ \mathbb{P}T_M \longrightarrow M$ be the canonical projection and L be the tautological line bundle over $\mathbb{P}T_M$. We adopt the convention for the tautological line bundle so that the restriction of L to $\mathbb{P}T_o(M)$ is negative. The Picard group of $\mathbb{P}T_M$ is a free Abelian group of rank 2, generated by L and $\pi^*(\mathcal{O}(1))$, where $\mathcal{O}(1)$ denotes the positive generator of $\mathrm{Pic}(M)$.

Let m be the degree $\mathcal{S}_{M,o}$ as a subvariety of $\mathbb{P}T_o(M)$. Then $L^m \otimes [\mathcal{S}_M]$ is a holomorphic line bundle which is trivial on every fiber of $\pi\colon\ \mathbb{P}T_M \longrightarrow M$. It follows that $L \otimes [\mathcal{S}_M] \cong \pi^*\mathcal{O}(\ell)$ for some integer ℓ. In other words, $[\mathcal{S}_M] \cong L^{-m} \otimes \pi^*\mathcal{O}(\ell)$. Let now s be a nontrivial holomorphic section of $L^{-m} \otimes \pi^*\mathcal{O}(\ell)$ whose zero set is precisely $\mathcal{S}_M$, of multiplicity 1. s is equivalent to a holomorphic section $\sigma \in \Gamma(M, S^m T^*_M \otimes \mathcal{O}(\ell))$, i.e., a twisted symmetric holomorphic covariant tensor. Let ρ: $G' \to G$ be the universal cover of the simple Lie group G. The action of G on $\mathbb{P}T_M$ lifts in an obvious way to the action of G'. Both $S^m T^*_M$ and $\pi^*\mathcal{O}(\ell)$ are homogeneous vector bundles on M, so that there is a well-defined holomorphic action of G' on $S^m T^*_M \otimes \mathcal{O}(\ell)$, compatible with the action of G' on $\mathbb{P}T_M$. As $\mathcal{S}_M \subset \mathbb{P}T_M$ is a G'-invariant cycle, for any $\gamma \in G'$, $\gamma^*\sigma$ must be a nonzero multiple of σ. We write

$\gamma^*\sigma = c_\gamma \sigma$. Thus, $c\colon G' \longrightarrow \mathbb{C}^*$ is a homomorphism. Since G' is simple we conclude that c is trivial, i.e., σ is a G'-invariant section.

Recall that $\Sigma \subset M$ denotes a polysphere of maximal dimension, isomorphic to $\mathbb{P}_1^r$, $r = \mathrm{rank}(M)$, as given by the Polysphere Theorem. Σ has the homological property that each $\mathbb{P}_1$ factor represents the positive generator of $H_2(M, \mathbb{Z}) \cong \mathbb{Z}$. Furthermore, it is totally geodesic with respect to some canonical Kähler metric g_c defining a Hermitian symmetric structure on M. We write G_c for the identity component of the isometry group of (M, g_c). Recall that $\eta^{(k)} = (1, \dots, 1; 0, \dots, 0)$, with precisely the first k entries equal to 1, with respect to coordinates given by $\Sigma \cong \mathbb{P}_1^r$. Then $\eta^{(k)}$ is tangent to a rational curve C_k of degree k in M totally geodesic with respect to g_c. For $k = r$ we obtain a totally-geodesic rational curve C_r such that the holomorphic tangent space at each point is generated by a generic vector. This means that the tautological lifting $\hat{C}_r$ is disjoint from the highest characteristic bundle $\mathcal{S}_M$.

We now measure the holomorphic section $s \in \Gamma(\mathbb{P}T_M, L^{-m} \otimes \pi^*\mathcal{O}(\ell))$ by the Hermitian metric induced by g_c, denoting the pointwise norms by $||s||_c$. The Hermitian metric on the tautological line bundle L induced by g_c will be denoted by $\hat{g}_c$, while the induced Hermitian metric on $\mathcal{O}(1)$ will be denoted by h_c. Here the choice of g_c induces a Hermitian metric on the anticanonical line bundle K_M^{-1} and, hence, on $\mathcal{O}(1)$ by taking roots from the isomorphism $K_M \cong \mathcal{O}(a)$ for some positive integer a. Then, $||s||_c$ is invariant under the isometry group G_c of (M, g_c). By the Poincaré–Lelong equation we have

$$\sqrt{-1}\partial\bar{\partial} \log ||s||_c = -mc_1(L, \hat{g}_c) + \ell c_1(\mathcal{O}(1), h_c) + [\mathcal{S}_M], \tag{$*$}$$

where $[\mathcal{S}_M]$ denotes here the closed positive (1,1)-current defined by the reduced irreducible subvariety $\mathcal{S}_M \subset \mathbb{P}T_M$.

Since $\hat{C}_r \cap \mathcal{S}_M = \emptyset$, C_r is the orbit under some compact subgroup $H_c \subset G_c$, $H_c \cong \mathbb{P}\mathrm{SU}(2)$, and $||s||_c$ is invariant under G_c, hence constant on $\hat{C}_r$, we conclude from the Poincaré–Lelong equation that

$$-mc_1(L, \hat{g}_c)|_{\hat{C}_r} + \ell c_1(\pi^*\mathcal{O}(1), \pi^* h_c)|_{\hat{C}_r} \equiv 0. \tag{$*'$}$$

In particular, once m, the degree of $\mathcal{S}_{M,o}$ in $\mathbb{P}T_o(M)$, is known, ℓ is completely determined by restricting to a totally-geodesic holomorphic curve in (M, g_c) of maximal degree.

(1.3) We are now ready to formulate the characterization of compact totally-geodesic holomorphic curves of maximal Gaussian curvature on quotients of bounded symmetric domains of characteristic codimension 1. Such bounded symmetric domains abound. We state here without proof a complete listing of these domains which follows by a case-by-case checking from the classification of irreducible bounded symmetric domains.

PROPOSITION 1. *Let Ω be an irreducible bounded symmetric domain of rank $r > 1$. Then Ω is of characteristic codimension $q(\Omega) = 1$ if and only if it belongs to one of the following classes:*

(1) Ω *of Type* $\mathrm{I}_{m,n}$ *with* $m = n > 1$;
(2) Ω *of Type* II_n *with* n *even*, $n \geqslant 4$;
(3) Ω *of type* $\mathrm{III}_n, n \geqslant 2$;
(4) Ω *of Type* $\mathrm{IV}_n, n \geqslant 3$; *or*
(5) Ω *of Type* VI (*the* 27-*dimensional exceptional domain pertaining to* E_7).

Here Case (3) consists of bounded symmetric domains biholomorphic to the Siegel upper-half-plane via the Cayley transform. Case (4) consists of domains dual to the hyperquadric $Q_n, n \geqslant 3$. A description of the corresponding hypersurface $\mathcal{S}_o \subset \mathbb{P}T_o(\Omega)$ will be given in the proof of Proposition 3 in (2.2).

2. Characterization of Totally-Geodesic Holomorphic Curves of Maximal Gaussian Curvature in the Case of Characteristic Codimension 1

(2.1) Recall that Ω is an irreducible bounded symmetric domain of rank $r \geqslant 2$ and of characteristic codimension 1, embedded into its compact dual M, by Borel Embedding. Let g_o be a canonical metric on Ω, and g_c be a canonical metric on M, so that $((\Omega, g_o); (M, g_c))$ constitutes a dual pair of Riemannian symmetric spaces. Thus, G_o is a noncompact real form of G, G_c is a compact real form of G, and $g_o|_{T_o(\Omega)}$ agrees with $g_c|_{T_o(M)}$, so that K acts as a group of holomorphic isometries both on Ω and on the compact dual M. The canonical metric g_o on Ω induces a canonical Hermitian metric $\hat{g}_o$ on $L|_\Omega$. For the holomorphic line bundle $\mathcal{O}(1)$, to avoid confusion we denote its restriction to Ω by E. Denoting by $G'_o = \rho^{-1}(G_o) \subset G'$, etc., we have a natural action of G'_o on E, so that, denoting by h_o the Hermitian metric on E induced by g_o, G'_o acts isometrically on (E, h_o).

Recall that $D \subset \Sigma$, $D \cong \Delta^r$, is a totally-geodesic polydisk in (Ω, g_o). For the totally-geodesic rational curve $C_k \subset \Sigma \subset M$ of degree k, $1 \leqslant k \leqslant r$, the open subset $\Delta_k = C_k \cap \Omega$ gives a totally-geodesic holomorphic disk in (Ω, g_o), said to be of type k. For $k = r$ we have the dual relation about Chern forms on the tautological lifting $\hat{\Delta}_r$, as follows:

$$-mc_1(L, \hat{g}_o)|_{\hat{\Delta}_r} + \ell c_1(\pi^* E, \pi^* h_o)|_{\hat{\Delta}_r} \equiv 0.$$

The latter identity is related to the dual case of C_r as follows. $(\Delta_r, g_o|_{\Delta_r})$ is invariant under a group of isometries $H_o \subset G_o$, $H_o \cong \mathbb{P}\mathrm{SU}(1,1)$ and dual to $H_c \cong \mathbb{P}\mathrm{SU}(2)$. In both identities the closed (1,1) forms are invariant under H_o resp. H_c so that it suffices to check at the origin o. Let $\tau_c : C_r \to \mathbb{P}T_M|_{C_r}$ be the holomorphic section which defines the tautological lifting, and denote its restriction to Δ_r by τ_o. Then, $\tau_c^* c_1(L, g_c)$ is the Gauss curvature form of $(C_r, g_c|_{C_r})$, while $\tau_o^* c_1(L, g_o)$ is the Gauss curvature form of $(\Delta_r, g_o|_{\Delta_r})$. At the origin o the two curvature forms are opposite to each other, by duality. For the same reason the curvature form of $(\mathcal{O}(1), h_c)$ on M and (E, h_o) on Ω are opposite to each other. (The ath power gives the Hermitian anticanonical line bundles with opposite curvature forms at the origin.)

For irreducible bounded symmetric domains of rank $r \geqslant 2$ and characteristic codimension 1 as given in the listing in (1.3), Proposition 1, we prove the following theorem on characterizing compact totally-geodesic holomorphic curves of type r.

THEOREM 1. *Let Ω be an irreducible bounded symmetric domain of rank $r \geqslant 2$ and of characteristic codimension* 1. *Let Γ be a torsion-free discrete group of biholomorphic automorphisms of Ω, and write $X := \Omega/\Gamma$. Let $C \subset X$ be a compact smooth holomorphic curve such that for any $x \in C$, the holomorphic tangent space $T_x(C)$ is spanned by a generic tangent vector. Then, C is totally geodesic in X.*

Proof. Replacing Γ by some subgroup of finite index, we may assume without loss of generality that the homogeneous holomorphic line bundle E dual to $\mathcal{O}(1)$ on M descends to a holomorphic line bundle on X. (This assumption is just for convenience in notations.) We will denote the Hermitian holomorphic line bundles obtained by descent by the same notations $(L, \hat{g}_o)$ resp. (E, h_o) on $\mathbb{P}T_X$ resp. X, etc. Let $\hat{C}$ be the tautological lifting of C to $\mathbb{P}T_X$. From the assumption, for any $x \in C$, $T_x(C) = \mathbb{C}\eta$ for some generic tangent vector η, so that $\hat{C} \cap \mathcal{S}_X = \emptyset$. Thus, by the Poincaré–Lelong equation we have

$$\sqrt{-1}\partial\bar{\partial} \log ||s||_o = -mc_1(L, \hat{g}_o) + \ell c_1(\pi^* E, \pi^* h_o) + [\mathcal{S}_X]. \tag{1}$$

Restricting to the compact holomorphic curve $\hat{C}$ satisfying $\hat{C} \cap S_X = \emptyset$, and integrating over $\hat{C}$, we have

$$\int_{\hat{C}} -mc_1(L, \hat{g}_o) + \ell c_1(\pi^* E, \pi^* h_o) = 0, \tag{2}$$

which is the same as

$$\int_C -mc_1(T_C, g_o|_C) + \ell c_1(E, h_o) = 0. \tag{3}$$

For convenience we normalize our choice of canonical metric g_o (a constant multiple of the Bergman metric) to be such that the Gaussian curvature of $(\Delta_r, g_o|\Delta_r)$ is identically -1. Denote by R the curvature tensor of (Ω, g_o). We note that for any $\eta \in T_o(\Sigma) \subset T_o(\Omega)$ of unit length,

$$\eta = (\eta_1, \ldots, \eta_r), \qquad R_{\eta\bar{\eta}\eta\bar{\eta}} = -r \sum_k |\eta_k|^4 \leqslant -r \cdot \frac{1}{r} = -1,$$

with equality if and only if $|\eta_1| = \cdots = |\eta_r| = 1/\sqrt{r}$. This is precisely the case if and only if η is tangent to some totally-geodesic holomorphic disk of type k.

We denote by ω the Kähler form of (Ω, g_o) and also the induced Kähler form on X. Any local holomorphic curve on (X, g_o) is of strictly negative Gaussian curvature. In the integral identity on C, we can write the first term as $-m\kappa_C\omega$, where κ_C denotes the Gaussian curvature of (X_o, g_o), and the second term as $-c\omega$ for some $c > 0$. Thus

$$\int_C -m\kappa_C\omega = c \int_C \omega. \tag{4}$$

The positive constant c is uniquely determined by the fact that $-m\kappa_{\Delta_r} \equiv c$; i.e., $c = m$ under our normalization. By the Gauss equation for submanifolds, for any $x \in X$, we have $\kappa_C(x) = R_{\eta\bar{\eta}\eta\bar{\eta}} - ||\sigma(x)||^2$ where $\eta \in T_x(C)$ is of unit length, Σ denotes the second fundamental form of C in X, $||\cdot||$ denotes the norm induced by g_o. As remarked, $R_{\eta\bar{\eta}\eta\bar{\eta}} \leqslant -1$, so that $\kappa_C \leqslant -1$. Plugging in (4) we see immediately that

$$\int_C -m\kappa_C\omega \geqslant m\int_C \omega, \tag{5}$$

unless $\kappa_C \equiv -1$, and the second fundamental form σ vanishes identically on C, i.e., unless C is a totally-geodesic holomorphic curve of type r. The proof of Theorem 1 is complete. □

Remarks. Theorem 1 for domains Ω of types I, II and III is a special case of Eyssidieux [E1,Proposition 9.3.6], by a method involving variations of Hodge structures and Gauss-Manin connections. Here the proof applies to all irreducible domains Ω of characteristic codimension 1 by a uniform intersection-theoretic method.

The following proposition shows that Theorem 1 is in a sense optimal.

PROPOSITION 2. *Let Ω be an irreducible bounded symmetric domain of dimension $\geqslant 2$ other than those listed in Proposition* 1. *Then, there exists a torsion-free discrete subgroup Γ of biholomorphic automorphisms of Ω such that on the quotient manifold $X := \Omega/\Gamma$, there is a nongeodesic compact smooth holomorphic curve $C \subset X$ whose tangent spaces are spanned by generic vectors.*

Proof. When Ω is of rank 1, i.e., biholomorphic to the unit ball B^n in $\mathbb{C}^n$, any nonzero tangent vector is of rank 1 and, hence, generic. Thus, for $n \geqslant 2$ and for any choice of Γ, any nongeodesic compact smooth holomorphic curve $C \subset X := \Omega/\Gamma$ gives an example as desired. For Ω of rank $\geqslant 2$ and of characteristic codimension $\geqslant 2$, from the listing as given in (1.3) Proposition 1, we have the following possible cases:

(1) Ω is of Type $\mathrm{I}_{m,n}$ with $m > n > 1$;
(2) Ω is of Type II_n with n odd, $n \geqslant 5$;
(3) Ω is of Type V, pertaining to E_6.

In Case (1), in standard notations $\Omega = D^{\mathrm{I}}_{m,n}$, and there exists a totally-geodesic complex subdomain $D \subset D^{\mathrm{I}}_{m,n}$, such that $D \cong \Delta^{n-1} \times B^{m-n+1}$, $m - n + 1 \geqslant 2$, and such that all nonzero vectors tangent to D are generic, i.e., of rank n. We may choose the torsion-free discrete subgroup $\Gamma \subset \mathrm{Aut}(\Omega)$ to be such that each $\gamma \in \Gamma$ fixes D as a set, and that the restrictions $\gamma|_D$ gives a discrete subgroup of the form $\Gamma_1 \times \Gamma_2 \times \cdots \times \Gamma_{n-1} \times \Gamma_n$, where each $\Gamma_i \subset \mathrm{Aut}(\Delta)$; $1 \leqslant i \leqslant n-1$; and $\Gamma_n \subset \mathrm{Aut}(B^{n-m+1})$ are torsion-free cocompact discrete subgroups. We are in fact free to choose Γ_i, $1 \leqslant i \leqslant n$ with the latter property. Choose now any Γ_n and let

$C_o \subset B^{n-m+1}/\Gamma_n$ be a nongeodesic compact smooth holomorphic curve. Choose now $\Gamma_1 = \cdots = \Gamma_{n-1}$ such that $\Delta/\Gamma_1 = \cdots = \Delta/\Gamma_{n-1} \cong C_o$. Then, $D/\Gamma \cong C_o^{n-1} \times B^{n-m+1}/\Gamma_n$ contains a copy of C_o^n, whose diagonal C is an example of a nongeodesic smooth compact holomorphic curve whose tangent space at each point is generated by a generic vector.

In general, for an irreduible bounded symmetric domain Ω of rank $r \geqslant 2$, there exists a (totally-geodesic) bounded symmetric subdomain $D \subset \Omega$ such that $D \cong \Delta^{r-1} \otimes B^s$, where s is the dimension of rank-1 boundary components of Ω (cf. [W]). By the same proof as in [M1, Chapter 6, Proposition 4, p.105-6], s is the same as the codimension q of $\mathcal{S}_{r-1,o} = \mathcal{S}_o$ in $\mathbb{P}T_o(\Omega)$. Thus the preceding argument yields examples of curves $C \subset \Omega/\Gamma$ with the desired properties whenever $q = s > 1$. We note from [W] that $s = 3$ for Case (2); $\Omega = D^{\mathrm{II}}_{2m+1}$; while $s = 5$ for Case (3), $\Omega = D^{\mathrm{V}}$. □

In the proof of Theorem 1, where $[\mathcal{S}] \cong L^{-m} \otimes \pi^* E^{\ell}$, the precise values of the positive integers m and ℓ were unimportant. There we derived the curvature inequality $\kappa_C \leqslant -1$ on Gauss curvatures and showed that it is sharp if and only if C is totally geodesic and of type $r = \mathrm{rank}(\Omega)$. We have formulated the proof to show that the inequality, which is local, and its sharpness follow by duality from the existence of totally-geodesic (rational) curves of degree r on the compact dual M of Ω. Nonetheless, it is interesting to note the following uniform description of $[\mathcal{S}]$, which involves case-by-case checking.

PROPOSITION 3. *Let Ω be an irreducible bounded symmetric domain of rank $r \geqslant 2$ and of characteristic codimension* 1. *Let Γ be a torsion-free discrete group of biholomorphic automorphisms Ω of $X := \Omega/\Gamma$. Assume furthermore, that the negative Hermitian holomorphic line bundle (E, h_o) on Ω, which is dual to $(\mathcal{O}(1), h_c)$ on M, descends to X (which is always the case at the expense of passing to a subgroup of finite index of Γ). Let $\mathcal{S} \subset \mathbb{P}T_X$ be the highest characteristic bundle on X. Denote by L the tautological line bundle on $\mathbb{P}T_X$. Then, the divisor line bundle $[\mathcal{S}]$ on $\mathbb{P}T_X$ is isomorphic to the holomorphic line bundle $L^{-r} \otimes \pi^* E^2$.*

Proof. From the proof of Theorem 1 and in the notations used there it remains to prove that $\mathcal{S}_o \subset \mathbb{P}T_o(\Omega)$ is of degree m exactly equal to the rank r of Ω, and that $\ell = 2$. We use the complete listing of cases as given in Proposition 1. For Case (4), $\mathcal{S}_o \subset \mathbb{P}T_o(\Omega)$ corresponds to the zero locus of a nondegenerate symmetric bilinear form on $T_o(\Omega)$. It is, hence, of degree 2 equal to the rank of $\Omega = D^{\mathrm{IV}}_n, n \geqslant 3$. For Cases (1), (2) and (3), identifying $T_o(\Omega)$ as a vector space of n-by-n matrices as usual $\mathcal{S}_o \subset \mathbb{P}T_o(\Omega)$ corresponds set-theoretically to the zero-locus of the determinant, which gives an element of $\Gamma(\mathbb{P}T_o(\Omega), L^{-n})$. For Case (1) consisting of Type-I domains $\mathrm{I}_{n,n}$, the determinant vanishes simply at a generic point of $\mathcal{S}_o$, so that the degree of S_o is precisely n, which is the same as the rank r. The same is true for Case (3) consisting of Type-III domains $D^{\mathrm{III}}_n, n \geqslant 3$, of rank n, with $T_o(\Omega)$ identified with the vector space of symmetric n-by-n matrices. In the remaining Case (2) consisting of Type-II domains D^{II}_n, n even, of rank $n/2$, with $T_o(\Omega)$ identified with the vector space of skew-symmetric n-by-n matrices, the determinant vanishes to the second order at a generic point of $\mathcal{S}_o$. This can be seen by expressing the determinant of a skew-symmetric n-by-n matrix, n even, as the square of the Pfaffian, which itself vanishes simply at a generic point of $\mathcal{S}_o$. Thus, the reduced subvariety $\mathcal{S}_o \subset \mathbb{P}T_o(\Omega)$ is of degree $n/2 = \text{rank}(\Omega)$.

For Case (5), where Ω is the 27-dimensional exceptional bounded symmetric domain D^{VI}, of rank 3, by [Za, Theorem 4.8, p.90] the tangent space $T_o(\Omega)$ and its (higher) characteristic vectors admit an algebraic description, as follows. Let $\mathfrak{J}$ be the Jordan algebra of Hermitian (3×3)-matrices over the Cayley numbers $\mathbb{O}$, of complex dimension $3(\dim_{\mathbb{C}}\mathbb{O} + 1) = 3(8 + 1) = 27$. Then, $T_o(\Omega)$ can be identified with $\mathfrak{J}$ in such a way that the second characteristic variety $\mathcal{S}_{2,o} = \mathcal{S}_o \subset \mathbb{P}T_o(\Omega)$ is precisely the cubic hypersurface corresponding to matrices A of rank $\leqslant 2$, defined by the cubic polynomial $\det(A) = 0$. Thus, $m = r = 3$ in this case. This completes the proof that $m = r$ in all cases.

It remains to show that $\ell = 2$. ℓ can be determined using the compact dual (M, g_c) by means of [(1.2), $(*')$], which results from the Poincaré–Lelong equation. Recall in the notations there

$$-mc_1(L, \hat{g}_c)|_{\hat{C}_r} + \ell c_1(\pi^*\mathcal{O}(1), \pi^* h_c)|_{\hat{C}_r} \equiv 0. \tag{$*'$}$$

Integrating over $\hat{C}_r$ we have

$$m(2g(C_r) - 2) + \ell \deg(C_r) = 0. \tag{1}$$

Since $C_r \subset X$ is a rational curve of degree r and we have established $m = r$, it follows from (1) that

$$-2r + \ell r = 0; \quad \text{i.e.,}\ \ell = 2, \tag{2}$$

as desired. The proof of Proposition 3 is complete. $\square$

(2.3) An immediate differential-geometric consequence of Theorem 1 is the following optimal effective pinching theorem.

THEOREM 2. *Let Ω be an irreducible bounded symmetric domain of rank $r \geqslant 2$ and of characteristic codimension* 1. *Let Γ be a torsion-free discrete group of biholomorphic automorphisms of Ω and write $X := \Omega/\Gamma$. Normalize the canonical Kähler metric on Ω such that the Gaussian curvature of a totally-geodesic holomorphic disk of type r is equal to* –1. *Let $C \subset X$ be a compact smooth holomorphic curve such that for any $x \in C$, the Gaussian curvature $K(x)$ satisfies the pinching condition*

$$-\left(1 + \frac{1}{r-1}\right) < K(x)\ (\leqslant -1).$$

Then, C is a totally-geodesic holomorphic curve of maximal Gaussian curvature.

The proof of Theorem 2, which we omit, is obtained by a curvature computation analogous to the proof of [EM, (2.2), Theorem 2, p. 91].

Remarks. Under our normalization the Gaussian curvature of a totally-geodesic holomorphic disk Δ_k of type k is $-(r/k)$. Since there exist torsion-free discrete subgroups $\Gamma \subset \mathrm{Aut}(\Omega)$ on which there are compact totally-geodesic holomorphic curves of type $r-1$, Theorem 2 is optimal. A simple example of such a Γ can be obtained as follows. Let $D \subset \Omega$ be a totally-geodesic polydisk of maximal dimension, $D \cong \Delta^r$. We have an embedding $i\colon \mathbb{P}\mathrm{SU}(1,1)^r = \mathrm{Aut}_o(D) \hookrightarrow \mathrm{Aut}_o(\Omega)$. Let $\Gamma_o \subset \mathbb{P}\mathrm{SU}(1,1)$ be a torsion-free cocompact discrete subgroup and $\Gamma = i(\Gamma_o^r) \subset \mathrm{Aut}_o(\Omega)$. Then, writing $C = \Delta/\Gamma_o$, we have $D/\Gamma_o^r \cong (\Delta/\Gamma_o)^r = C^r \subset X = \Omega/\Gamma$ and there exists on C^r, hence on X, a compact totally-geodesic holomorphic curve of type k for any k, $1 \leqslant k \leqslant r$.

3. Generic Finiteness of Holomorphic Mappings onto Compact Complex Manifolds Carrying Continuous Complex Finsler Metrics of Seminegative Curvature

(3.1) Let X be a projective manifold uniformized by an irreducible bounded symmetric domain of rank $\geqslant 2$. In [M1, 2] we prove that up to normalizing constants, the canonical Kähler metric on X is the unique Hermitian metric of seminegative curvature in the sense of Griffiths. From this metric rigidity theorem it follows that any nontrivial holomorphic mapping f of X into a Hermitian manifold Z of seminegative curvature in the sense of Griffiths is necessarily an isometric immersion totally-geodesic with respect to the Hermitian connection on Z. One motivation to develop intersection theory on $\mathbb{P}T_X$ is to study holomorphic maps for target complex manifolds Z satisfying much weaker conditions of seminegativity. As a consequence of Theorem 1 and [M1, 2], we establish the following result on holomorphic mappings onto compact complex manifolds carrying continuous complex Finsler metrics of seminegative curvature. Here a complex Finsler metric is equivalently a Hermitian metric on the tautological line bundle of the projectivization $\mathbb{P}T_X$ of the holomorphic tangent bundle T_X. For a Hermitian line bundle (L, h) where h is only assumed to be continuous, we say that it is of seminegative curvature if and only if for any local holomorphic basis e of L, $h(e, \overline{e})$ is given by e^{φ} for some continuous plurisubharmonic function φ.

THEOREM 3. *Let Ω be an irreducible bounded symmetric domain of rank $r \geqslant 2$ and of characteristic codimension* 1. *Let Γ be a torsion-free cocompact discrete group of biholomorphic automorphisms of Ω and write $X := \Omega/\Gamma$. Let Z be a complex manifold carrying a continuous complex Finsler metric of seminegative curvature. Then, any nonconstant holomorphic map $f\colon X \to Z$ is necessarily an immersion at a generic point.*

We note that for a bounded domain D on a Stein manifold, the Carathéodory metric is a continuous complex Finsler metric of seminegative curvature. As a consequence of Theorem 3, we have

COROLLARY 1. *Let Ω be an n-dimensional irreducible bounded symmetric domain of rank $r \geqslant 2$ and of characteristic codimension* 1. *Let Γ be a torsion-free cocompact discrete group of biholomorphic automorphisms of Ω and write $X := \Omega/\Gamma$. Let Z be a compact complex manifold of dimension $m < n$ carrying a continuous complex Finsler metric of seminegative curvature. Then, there exists no nontrivial holomorphic mapping from X into Z.*

For the proof of Theorem 3 we will need the proof of Hermitian metric rigidity of Mok [M1, 2]. There are two modifications. First of all, the complex Finsler metric we have is assumed only to be continuous and of seminegative curvature in the sense of currents. Secondly, in place of integrating over the first characteristic bundle as we did in both [M1, 2] we need to consider instead the highest characteristic bundle. For the adaptation it is more convenient to make use of the later proof in [M2], where we employed Moore's Ergodicity Theorem. For easy reference we state

MOORE'S ERGODICITY THEOREM (cf. Zimmer [Z, Thm.(2.2.6), p.19]). *Let G be a simple Lie group and Γ be a lattice on G, i.e., $\Gamma\backslash G$ is of finite volume in the left invariant Haar measure. Suppose $H \subset G$ is a closed subgroup. Consider the action of H on $\Gamma\backslash G$ by multiplication on the right. Then, H acts ergodically if and only if H is noncompact.*

For the proof of Theorem 3 we establish first of all the following rigidity result which is valid for an irreducible bounded symmetric domain Ω of rank $\geqslant 2$ without any assumption on the characteristic codimension.

PROPOSITION 3. *Let Ω be an irreducible bounded symmetric domain of rank $r \geqslant 2$. Let Γ be a torsion-free cocompact discrete group of biholomorphic automorphisms of Ω, $X := \Omega/\Gamma$. Let Z be a complex manifold which admits a continuous complex Finsler metric of seminegative curvature. Let $f: X \to Z$ be a nonconstant holomorphic mapping. Then, for any $x \in X$, and any nonzero tangent vector $\eta \in T_x(X)$ of rank $< r$, we have* $\mathrm{d}f(\eta) \neq 0$.

Proof. We adapt the proof of [M2, p.113ff]. Consider first of all the case when Ω is of rank 2. In this case the highest characteristic bundle $\mathcal{S}$ is the same as the characteristic bundle used there. Denote by $\pi: \mathbb{P}T_X \to X$ the canonical projection. Fix a canonical Kähler metric g on X and write ω for its Kähler form. Write $\hat{g}$ for the canonical Hermitian metric on the tautological line bundle L induced by g. The closed (1,1)-form $\nu = -c_1(L, \hat{g}) + \pi^*\omega$ on $\mathbb{P}T_X$ is strictly positive. Let p be the complex dimension of $\mathcal{S}_o$, $q = n - 1 - p$ the codimension of $\mathcal{S}_o$ in $\mathbb{P}T_o(\Omega)$, i.e.,q is the characteristic codimension of Ω. For $[\alpha] \in \mathcal{S}_o$, let $\mathcal{N}_{[\alpha]} \subset T_{[\alpha]}(\mathbb{P}T(X))$ be the kernel of $-c_1(L, \hat{g})[\alpha] \geqslant 0$ on $T_{[\alpha]}(\mathbb{P}T(X))$. By [M1], $\mathcal{N}_{[\alpha]}$ is of complex dimension q, and $\mathcal{N}_{[\alpha]} \subset T_{[\alpha]}(\mathcal{S})$, i.e., $\mathcal{N}_{[\alpha]} = \mathrm{Ker}(-c_1(L, \hat{g})[\alpha]|_{\mathcal{S}})$. Then, on the characteristic bundle $\mathcal{S}$ we have

$$\int_{\mathcal{S}} -c_1(L,h) \wedge (-c_1(L,\hat{g}))^{2n-2q-1} \wedge v^{q-1}$$
$$= \int_{\mathcal{S}} (-c_1(L,\hat{g}))^{2n-2q} \wedge v^{q-1} = 0, \tag{1}$$

where for the continuous complex Finsler metric h of seminegative curvature, $-c_1(L,h)$ is understood to be the closed positive $(1,1)$ current, which is $-1/2\pi$ times the curvature current. Thus, $-c_1(L,h)$ has coefficients which are measures. The integrand of the left-hand side of (1) is then a nonnegative measure, and Equation (1) forces the identical vanishing

$$-c_1(L,h) \wedge (-c_1(L,\hat{g}))^{2n-2q-1} \equiv 0 \quad \text{on } \mathcal{S}. \tag{2}$$

Since the sum of two nonnegative log-plurisubharmonic functions remain log-plurisubharmonic, $\hat{g}+h$ is a continuous complex Finsler metric of seminegative curvature, and (2) remains valid with h replaced by $\hat{g}+h$. Write now $\hat{g}+h = e^u\hat{g}$. Then,

$$c_1(L,\hat{g}+h) = c_1(L,\hat{g}) - \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}u,$$

and (2) gives the identity

$$\sqrt{-1}\partial\bar{\partial}u \wedge (-c_1(L,\hat{g}))^{2n-2q-1} \equiv 0 \quad \text{on } \mathcal{S}, \tag{3}$$

as currents. Since u is almost plurisubharmonic, $\bar{\partial}u$ is integrable, and coefficients of $\partial\bar{\partial}u$ are measures. By local smoothing and partition of unity there exists a sequence (u_k) of smooth functions such that u_k converges to u uniformly, $\mathrm{d}u_k$ converges to $\mathrm{d}u$ in L^1 and $\partial\bar{\partial}u_k$ converges to $\partial\bar{\partial}u$ as distributions of order 0 (i.e., their coefficients converge as measures). Write on $\mathcal{S}$

$$\begin{aligned} T_k &= \sqrt{-1}\ \bar{\partial}u_k \wedge (-c_1(L,\hat{g}))^{2n-2q-1}, \\ T &= \sqrt{-1}\ \bar{\partial}u \wedge (-c_1(L,\hat{g}))^{2n-2q-1}. \end{aligned} \tag{4}$$

Then $\mathrm{d}T_k$ converges to $\mathrm{d}T \equiv 0$ as distributions of order 0. Consider now

$$\sqrt{-1}\,\mathrm{d}(u_kT_k) = \sqrt{-1}\,\mathrm{d}u_k \wedge T_k + \sqrt{-1}u_k\,\mathrm{d}T_k. \tag{5}$$

Integrating over $\mathcal{S}$ we conclude that

$$\int_{\mathcal{S}} \sqrt{-1}\,\mathrm{d}u_k \wedge T_k \wedge v^{q-1} = \int_{\mathcal{S}} \sqrt{-1}\,\partial u_k \wedge \bar{\partial}u_k \wedge (-c_1(L,\hat{g}))^{2n-2q-1} \wedge v^{q-1} \to 0. \tag{6}$$

Consider now the Hermitian bilinear form B on smooth $(1,0)$-forms φ on $\mathcal{S}$ given by

$$B(\varphi,\psi) = \int_{\mathcal{S}} \sqrt{-1}\varphi \wedge \bar{\psi} \wedge (-c_1(L,\hat{g}))^{2n-2q-1} \wedge v^{q-1}. \tag{7}$$

B is positive semi-definite. Note that $B(\varphi,\psi)$ can also be defined for φ of class L^1 and ψ smooth. From (6) we have $B(\partial u_k, \partial u_k) \to 0$. Fix any smooth $(1,0)$ form ψ on $\mathcal{S}$.

By the Cauchy–Schwarz inequality $B(\partial u_k, \psi)^2 \leqslant B(\partial u_k, \partial u_k)B(\psi, \psi)$. By (6) $B(\partial u_k, \partial u_k) \to 0$ as $k \to \infty$. It follows that $B(\partial u_k, \psi) \to 0$ as $k \to \infty$. Since $\partial u_k \to \partial u$ as distributions we have $B(\partial u, \psi) = 0$ for any smooth ψ.

For each $[\alpha] \in \mathcal{S}$, recall that $\mathcal{N}_{[\alpha]} \subset T_{[\alpha]}(\mathcal{S})$. Since $-c_1(L, \hat{g}) \geqslant 0$ and, hence, its restriction $-c_1(L, \hat{g})|_{\mathcal{S}} \geqslant 0$ is closed, the assignment $[\alpha] \mapsto \mathrm{Re}\,\mathcal{N}_{[\alpha]}$ defines a smooth integrable distribution on $\mathcal{S}$ of real rank $2q$ whose integral leaves are q-dimensional complex submanifolds. The fact that $B(\partial u, \psi) = 0$ for any smooth ψ means that ∂u, regarded as an L^1 section of $\mathrm{Hom}(T(X), \mathbb{C})$, vanishes almost everywhere on the subbundle $\mathcal{N}$. It follows that the continuous function u is constant on almost every local leaf and, hence, on every leaf of $\mathcal{N}$.

The leaves of $\mathcal{N}$ can be described as follows. For any $[\eta] \in \mathbb{P}T_o(\Omega)$ let $N_\eta = \{\zeta \in T_o(\Omega) = R_{\eta\bar{\eta}\zeta\bar{\zeta}} = 0\}$ be the null-space of η. For any $[\alpha] \in \mathcal{S}_o$ we have $\dim_{\mathbb{C}} N_\alpha = q$. Let $\Delta \subset \Omega$ be the unique minimal disk passing through o such that $T_o(\Delta) = \mathbb{C}\alpha$. Then, there exists a unique totally-geodesic bounded symmetric domain Ω_o passing through o such that $T_o(\Omega_o) = N_\alpha$. Moreover, $\mathbb{C}\alpha + N_\alpha$ is tangent to a unique totally-geodesic $(q+1)$-dimensional subdomain which can be identified with $\Delta \times \Omega_o$. Identify $\{o\} \times \Omega_o$ with Ω_o. For every $z \in \Omega_o$ write $[\alpha(z)] := \mathbb{P}T_z(\Delta \times \{z\}) \in \mathcal{S}_z(\Omega)$. As z runs over Ω_o, this defines a lifting of Ω_o to a complex submanifold $F \subset \mathcal{S}(\Omega)$ which is precisely the leaf of the lifting of $\mathcal{N}$ to $\mathcal{S}(\Omega)$ passing through $[\alpha]$. Note that G_o acts transitively on $\mathcal{S}$. Let $H \subset G_o$ be the closed subgroup which fixes Ω_o as a set. The leaf space of the foliation $\mathcal{N}$ on $\mathcal{S}$ can be identified as the homogeneous space G_o/H.

Recall that the continuous function u is constant on every leaf of $\mathcal{N}$ on $\mathcal{S}$. Note that $\mathcal{S}(\Omega) \subset \mathbb{P}T(\Omega)$ is homogeneous under the action of $G_o = \mathrm{Aut}_o(\Omega)$. Write $\mathcal{S}(\Omega) = G_o/K_{[\alpha]}$ for $K_{[\alpha]} \subset K$ the isotropy subgroup of any $[\alpha] \in \mathcal{S}_o$. Then, $\mathcal{S} = \Gamma\backslash G_o/K_{[\alpha]}$. Lift u to a continuous function $\hat{u}$ on $\Gamma\backslash G_o$. Then, $\hat{u}$ is invariant under multiplication on the right by elements of H, which is noncompact. By Moore's Ergodicity Theorem, $\hat{u}$ is constant on $\Gamma\backslash G_o$, hence u is constant on $\mathcal{S}$. Let now θ be a continuous complex Finsler metric on Z of seminegative curvature, and h be $f^*\theta$. h is possibly degenerate but the preceding discussion still applies to $\hat{g} + h = e^u\hat{g}$. Now $u([\alpha]) = 1$ if and only if $h([\alpha]) = 0$, i.e., $\mathrm{d}f(\alpha) = 0$. Since u is constant on $\mathcal{S}$, $\mathrm{d}f(\alpha_o) = 0$ for some $[\alpha_o] \in \mathcal{S}$ implies $\mathrm{d}f(\alpha) = 0$ for any $[\alpha] \in \mathcal{S}$. As $\mathcal{S}_x \subset \mathbb{P}T_x(X)$ is linearly nondegenerate at every point $x \in X$, we conclude that $\mathrm{d}f \equiv 0$, i.e., f is a constant mapping, contradicting the hypothesis. This proves Proposition 3 for the case of rank 2.

We now adapt the proof to the case of rank $r > 2$. Let now $\mathcal{S} = \mathcal{S}_{r-1}$ be the highest characteristic bundle. We use the same integral formula as (1), with q denoting the characteristic codimension of Ω. For $r > 2$, $\mathcal{S}_x \subset \mathbb{P}T_x(X)$ is singular and the integration is performed over the smooth part $\mathrm{Reg}(\mathcal{S})$ of $\mathcal{S}$. The formula (1) concerns integrals of restrictions of smooth forms on $\mathbb{P}T(X)$, and Stokes' Theorem can be justified by passing to a desingularization $\rho\colon \hat{\mathcal{S}} \to \mathcal{S} \subset \mathbb{P}T(X)$ and pulling back smooth differential forms on $\mathbb{P}T(X)$. For $[\gamma] \in \mathcal{S}_o$ there is a unique totally-geodesic $(r-1)$-dimensional polydisk $D \subset \Omega$ passing through o such that $\gamma \in T_o(D)$. Note that

$\dim_{\mathbb{C}} N_\gamma = q$. Then, $T_o(D) + N_\gamma$ is tangent to a unique bounded symmetric subdomain which can be identified with $D \times \Omega_o$ for some q-dimensional bounded symmetric subdomain $\Omega_o \subset \Omega$ such that $T_o(\Omega_o) = N_\gamma$. For the distribution $\mathcal{N}$ on $\mathrm{Reg}(\mathcal{S})$ given by $\mathcal{N}_{[\gamma]} = \mathrm{Ker}(c_1(L, \hat{g})[\gamma]|_{\mathcal{S}})$, its leaves F are similarly described as before in terms of Ω_o.

The same argument as in the case of rank 2 shows that for u defined by $\hat{g} + f^*\theta = e^u \hat{g}$, u is constant on the G_o-orbit of any $[\gamma] \in \mathrm{Reg}(\mathcal{S})$. We note however, that for $r > 2$, $\mathrm{Reg}(\mathcal{S})$ is not homogeneous under the action of G_o.

Suppose now $\mathrm{d}f(\gamma) = 0$ for some $[\gamma] \in \mathcal{S}$. Lift $f : X \to Z$ to $\tilde{f} : \Omega \to Z$. The set $\mathcal{D} := \mathrm{Ker}[\mathrm{d}\tilde{f}] \cap \mathrm{Reg}(\mathcal{S}(\Omega))$ is complex-analytic. On the other hand, it corresponds to the subset of $\mathrm{Reg}(\mathcal{S}(\Omega))$ on which the lifting $\hat{u}$ of u takes the value 1. The complex-analytic subvariety $\mathcal{D} \subset \mathrm{Reg}(\mathcal{S}(\Omega))$ is therefore, nonempty and G_o-invariant. Its fiber over each $z \in \Omega$ must therefore be invariant under the complexification $K_z^{\mathbb{C}}$ of the isotropy subgroup $K_z \subset G_o$ at z. Since $K_z^{\mathbb{C}}$ acts transitively on $\mathrm{Reg}(\mathcal{S}_z(\Omega))$ we conclude that $\mathcal{D} = \mathrm{Reg}(\mathcal{S}(\Omega))$, so that again $\tilde{f}$ and, hence, $f: X \to Z$ is a constant mapping, contradicting with the hypothesis. The proof of Proposition 3 is complete. □

We are now ready to apply Proposition 3 to the special case of characteristic codimension 1.

Proof of Theorem 3. Let now $X = \Omega/\Gamma$ where Ω is of characteristic codimension 1 with Γ torsion-free and cocompact, and $f: X \to Z$ be a nonconstant holomorphic mapping into a complex manifold Z equipped with a continuous complex Finsler metric θ of seminegative curvature (in the generalized sense). Suppose $f: X \to Z$ is of maximal rank $< \dim_{\mathbb{C}} X$. Then, for a generic point y of $f(X) := Y$, the fiber $f^{-1}(y)$ is a smooth p-dimensional manifold for some $p \geqslant 1$, $p = \dim_{\mathbb{C}} X - \dim_{\mathbb{C}} Y$. By Proposition 3, $\mathrm{d}f(\gamma) \neq 0$ for any nonzero tangent vector γ, $[\gamma] \in \mathcal{S}$. For $x \in X$, let F^x be the fiber $f^{-1}(f(x))$. For $x \in X$ generic, $\mathbb{P}T_x(F^x) \cong \mathbb{P}^{p-1}$ must be disjoint from $\mathcal{S}_x \subset \mathbb{P}T_x$. Since $\mathcal{S}_x \subset \mathbb{P}T_x(X)$ is a hypersurface, we must have $p = 1$, so that each irreducible component F_k^x of F^x must lift tautologically to $\hat{F}_k^x \subset \mathbb{P}T(X)$ such that $\hat{F}_k^x \cap \mathcal{S} = \emptyset$. By Theorem 1, $F_k^x \subset X$ is a compact totally-geodesic holomorphic curve of type r. For such a compact holomorphic curve the normal bundle $N_{F_k^x|X}$ is strictly negative. Varying x we obtain positive-dimensional holomorphic families of such compact holomorphic curves. On the other hand, since $N_{F_k^x|X}$ is strictly negative, $F_k^x \subset X$ is an exceptional curve, and must be the unique compact holomorphic curve on some tubular neighborhood U of F_k^x in X, a plain contradiction. This means that $f: X \to Z$ must be of maximal rank $= \dim_{\mathbb{C}} X$ at some point, i.e., f is a holomorphic immersion at a generic point, as desired. □

4. Characterization of Certain Totally-Geodesic Submanifolds of Bounded Symmetric Domains Dual to the Hyperquadric in Terms of Quadric Structures

(4.1) For the bounded symmetric domains Ω of Type IV_n, $n \geqslant 3$, dual to hyperquadrics, there is on Ω an invariant quadric structure, i.e., an invariant holomorphic nondegenerate symmetric 2-tensor with twisted coefficients. Theorem 1 in the special case of Type-IV domains says that compact totally-geodesic holomorphic curves of maximal Gaussian curvature on quotients of Ω are characterized by the nondegeneracy of the restriction of the canonical quadric structure. This leads naturally to the question of characterizing higher-dimensional totally-geodesic complex submanifolds in terms of restrictions of the quadric structure. We answer this in the affirmative, using for higher-dimensional submanifolds results from Hermitian metric rigidity of [M1, 2] and from the characterization of compact Kähler–Einstein manifolds with G-structures modeled on irreducible bounded symmetric domains of rank $\geqslant 2$ by Kobayashi and Ochiai [KO]. We prove

THEOREM 4. *Let $D_n^{\mathrm{IV}}, n \geqslant 3$, be the irreducible bounded symmetric domain dual to the hyperquadric $Q_n, n \geqslant 3$. Let Γ be a torsion-free discrete group of biholomorphic automorphisms of D_n^{IV} and write $X := D_n^{\mathrm{IV}}/\Gamma$. Let $S \subset X$ be a compact complex submanifold of any dimension d, $1 \leqslant d < n$, such that for any $x \in C$, the restriction of the canonical quadric structure on D_n^{IV} is nondegenerate. Then, S is totally geodesic in X.*

Proof. For the canonical quadric structure Q on X, and for any nonzero vector $\eta \in T_x(X)$, $x \in X$, $Q(\eta, \eta) = 0$ if and only if $[\eta] \in \mathcal{S}_x$. Hence, for $S \subset X$ of dimension $d = 1$, $Q|_S$ is nondegenerate if and only if $\hat{S} \cap \mathcal{S} = \phi$, so that $S \subset X$ is a totally-geodesic holomorphic curve of type 2, by Theorem 1. For $d \geqslant 2$ we will make use of Kähler–Einstein metrics and quadric structures. Any complex submanifold $S \subset X$ has ample canonical bundle, so that there must exist a Kähler–Einstein metric h of negative Ricci curvature, by Aubin [A] and Yau [Y]. Consider first the case $d \geqslant 3$, in which D_d^{IV} is irreducible. By [KO], the quadric structure $Q|_S$ on the Kähler–Einstein manifold (S, h) must be integrable, so that S is uniformized by D_d^{IV}. Write $S \cong D_d^{\mathrm{IV}}/\Gamma_o$ for some torsion-free cocompact discrete subgroup $\Gamma_o \subset \mathrm{Aut}(D_d^{\mathrm{IV}})$. By the Hermitian metric rigidity theorem of Mok [M1, 2], the holomorphic embedding $S = D_d^{\mathrm{IV}}/\Gamma_o \hookrightarrow X = D_n^{\mathrm{IV}}/\Gamma$ must be totally-geodesic, i.e., $S \subset X$ is a totally-geodesic holomorphic cycle, as desired. $\square$

Consider finally the case $d = 2$. Note that D_2^{IV} is isomorphic to the bidisk. The assumption that $Q|_S$ is nondegenerate means that for any $x \in S$, $\mathbb{P}T_x(S) \cap \mathcal{S}_x$ consists of two distinct simple points. It follows that $T(S) = L_1 \oplus L_2$ is the direct sum of two holomorphic line bundles L_1 and L_2. Since the Hermitian holomorphic vector bundle $(T(S), h)$ is Hermitian–Einstein, the holomorphic direct sum $L_1 \oplus L_2$ must also be an isometric direct sum of Hermitian holomorphic line bundles such that the direct summands $L_1, L_2 \subset T(S)$ are parallel holomorphic subbundles (cf. Siu [Si, proof of Proposition (1.6), pp.19–20]). This gives a local de Rham decomposition of S as a product of two Riemann surfaces of constant negative Gaussian curvature, implying that S is uniformized by the bidisk Δ^2. Thus, $S \cong \Delta^2/\Gamma_o$ for some torsion-free cocompact discrete subgroup $\Gamma_o \subset \mathrm{Aut}(\Delta^2)$.

It remains to show that the holomorphic embedding $f: \Delta^2/\Gamma_o \hookrightarrow D_n^{\mathrm{IV}}/\Gamma = X$, $f(\Delta^2/\Gamma_o) = S$, must be totally geodesic. Lift $f: \Delta^2/\Gamma_o \to X$ to $F: \Delta^2 \to D_n^{\mathrm{IV}}$ and use Euclidean coordinates (z, w) for $\Delta^2 = \Delta \times \Delta$. Denote by g the canonical

Kähler–Einstein metric on D_n^{IV}. From integral formulas as in the proof of Proposition 3 we have $R^{\tilde{S}}_{\mu\bar{\mu}\eta\bar{\eta}} = 0$ for $\mu_p = \mathrm{d}F_p(\partial/\partial z)$, $\eta_p = \mathrm{d}F_p(\partial/\partial w)$ at a point $p \in \Delta^2$, where $R^{\tilde{S}}$ denote the curvature tensor of $(\tilde{S}, g|_{\tilde{S}})$. Write $C := F(\Delta \times \{p_2\})$ for any $p_2 \in \Delta$. Then, $\eta|_C$ is a nowhere zero holomorphic section of $T(D_n^{\mathrm{IV}})$ over C. From $R^{\tilde{S}}_{\mu\bar{\mu}\eta\bar{\eta}} \equiv 0$ it follows that the holomorphic line subbundle L spanned by η is parallel on C. To proceed we prove

LEMMA 1. *Let $C \subset D_n^{\mathrm{IV}}$ be a (local) holomorphic curve such that $T_p(C)$ is spanned by a characteristic vector at any $p \in C$. Suppose there exists on C a parallel holomorphic line subbundle L spanned at each point $p \in C$ by a characteristic vector η_p normal to C. Then, C is an open subset of a minimal disk.*

Proof. Parametrize C locally as the image of a holomorphic curve $f\colon \Delta \to D_n^{\mathrm{IV}}, f(o) = o, f'(z) = \mu_{f(z)} \in T_{f(z)}(C)$. Let V be the holomorphic vector bundle on C such that $T(C) \subset V \subset T(D_n^{\mathrm{IV}})|_C$ and such that $V_p/T_p(C) = T_{[\mu_p]}(\mathcal{S}_p)$ for any $p \in C$, $T_p(C) = \mathbb{C}\mu_p$. Denote by $\mathcal{S}'_p \subset T_p(D_n^{\mathrm{IV}})$ the set of all nonzero vectors whose projectivization lie in $\mathcal{S}_p$. (We call $\mathcal{S}'_p$ the cone over $\mathcal{S}_p$.) Then $T_{\mu_p}(\mathcal{S}'_p) = V_p$. V_p is the orthogonal complement of $L_p = \mathbb{C}\eta_p$. Since L is parallel, V is a parallel holomorphic subbundle of $T(D_n^{\mathrm{IV}})|_C$. To show that $C \subset D_n^{\mathrm{IV}}$ is totally geodesic it suffices to show that $\nabla_\mu \mu$ is proportional to μ at every point $p \in C$. For the computation we may take p to be the origin o in D_n^{IV}, in its standard Harish-Chandra realization as a bounded symmetric domain in $\mathbb{C}^n$. At $o \in D_n^{\mathrm{IV}}$ the Euclidean coordinates are normal geodesic coordinates and we have $\nabla_\mu \xi(o) = \partial_\mu \xi(o)$ for any smooth section ξ of V over C. The fact that V is parallel over C means that

$$\partial_\mu \xi(o) \in V_o \quad \text{for any choice of } \xi. \tag{$*$}$$

On the other hand, in terms of Harish-Chandra coordinates, the characteristic bundle $\mathcal{S}(D_n^{\mathrm{IV}})$ agrees with the constant bundle $\mathcal{S}_o \times D_n^{\mathrm{IV}}$, the statement $(*)$ amounts to saying that, writing σ for the second fundamental form of $\tilde{\mathcal{S}_o}$ as a submanifold of the Euclidean space $T_p(D_n^{\mathrm{IV}})$, we have

$$\sigma_{\mu_o}(f''(o), \xi_o) = 0 \quad \text{for all } \xi_o \in V_o. \tag{$*'$}$$

Denote by $\bar{\sigma}$ the projective second fundamental form of $\mathcal{S}_o \subset \mathbb{P}T_o(D_n^{\mathrm{IV}})$. Then, by the finiteness of the Gauss map on (the homogeneous and nonlinear projective submanifold) $\mathcal{S}_o$, the kernel of the $\bar{\sigma}_{[\mu_o]}$ is trivial. Equivalently, this means that the kernel of σ_{μ_o} agrees with $\mathbb{C}\mu_o$. From $(*')$ it follows that $f''(o)$ is proportional to μ_o. But $f''(o) = \nabla_\mu \mu(o)$, implying that C is tangent at o to a minimal disk to the second order. As a consequence, C is totally geodesic and, hence, itself an open subset of a minimal disk, as desired. □

We are now ready to complete the proof of Theorem 4.

Proof of Theorem 4 (Continued). For the remaining case where $d = 2$ we know now that $S \cong \Delta^2/\Gamma_o$, and the embedding $f\colon S \to X$ lifts to $F\colon \Delta^2 \to D_n^{\mathrm{IV}}$, such that the factors $\Delta \times \{p_2\}$ and similarly $\{p_1\} \times \Delta$ are embedded under F onto minimal disks in D_n^{IV}. Let now τ be the second fundamental form of $\Sigma = F(\Delta^2)$ in D_n^{IV}. Recall that $\mu_p = \mathrm{d}F_p(\partial/\partial z)$, $\eta_p = \mathrm{d}F_p(\partial/\partial w)$. From $R_{\mu\bar{\mu}\eta\bar{\eta}} \equiv 0$ we have $\tau_{\mu\eta} \equiv 0$. Since $F(\Delta \times \{p_2\})$ and $F(\{p_1\} \times \Delta)$ are minimal disks, which are totally geodesic, we have $\tau_{\mu\mu}, \tau_{\eta\eta} \equiv 0$. Thus, the second fundamental form $\tau \equiv 0$, and $\tilde{S} \subset D_n^{\mathrm{IV}}$ is a complex two-dimensional totally-geodesic submanifold of rank 2, i.e., a totally-geodesic bidisk. As a consequence, $S = f(\Delta^2/\Gamma_o)$ is totally geodesic, as desired. The proof of Theorem 4 is complete. □

(4.2) The algebro-geometric characterization of certain totally-geodesic complex submanifolds of $X = D_n^{\mathrm{IV}}/\Gamma$, $n \geqslant 3$, as given in Theorem 4, leads to a differential-geometric pinching theorem in analogy with Theorem 2 in (2.3). We have

THEOREM 5. *Let D_n^{IV}, $n \geqslant 3$, be the irreducible bounded symmetric domain dual to the hyperquadric Q_n, $n \geqslant 3$. Let Γ be a torsion-free cocompact discrete group of biholomorphic automorphisms of D_n^{IV} and write $X = D_n^{\mathrm{IV}}/\Gamma$. Normalize the canonical Kähler metric on D_n^{IV} so that the Gaussian curvature of a totally-geodesic holomorphic disk of type $2 = \mathrm{rank}\,(D_n^{\mathrm{IV}})$) is equal to -1. Let $S \subset X$ be a compact complex submanifold of any dimension d, $1 \leqslant d < n$, such that for any $x \in C$, the scalar curvature $K(x)$ satisfies the pinching condition*

$$-(d^2+1) < K(x) \quad (\leqslant -d^2).$$

Then, $S \subset X$ is totally geodesic.

Proof. In terms of the standard realization of D_n^{IV} as a bounded domain, given by (even for $n = 1, 2$)

$$D_n^{\mathrm{IV}} = \left\{ (z_1, \ldots, z_n) \in \mathbb{C}^n : \|z\|^2 < 2 \quad \text{and} \quad \|z\|^2 < 1 + \left| \frac{1}{2} \sum_i z_i^2 \right|^2 \right\}.$$

The curvature tensor of the normalized canonical Kähler metric g_o is given by

$$R_{i\bar{j}k\bar{\ell}}(0) = -\left(\delta_{ij}\delta_{k\ell} + \delta_{i\ell}\delta_{jk} - \delta_{ik}\delta_{j\ell}\right) \tag{1}$$

(cf. [M2, pp. 86ff.]). Let $\xi = \sum \xi^i\, \partial/\partial z_i$ and $\eta = \sum \eta^i\, \partial/\partial z_i$ be tangent vectors of type $(1,0)$ at $o \in D_n^{\mathrm{IV}}$. Then

$$R_{\xi\bar{\xi}\eta\bar{\eta}} = -\left(\left| \sum_i \xi^i \bar{\eta^i} \right|^2 + \sum_{i,j} \left|\xi^i\right|^2 \left|\eta^j\right|^2 - \left| \sum_i \xi^i \eta^i \right|^2 \right). \tag{2}$$

Write Q for the symmetric complex bilinear form on $T_o(D_n^{\mathrm{IV}})$ given by $Q(\xi, \eta) = \sum \xi^i \eta^i$. Up to a conformal factor Q is the canonical quadric structure at o. Write $\langle \cdot, \cdot \rangle$ for the Hermitian inner product of $\left(D_n^{\mathrm{IV}}, g_o\right)$ at o and $\| \cdot \|$ for the corresponding norm. Then,

$$R_{\xi\bar{\xi}\eta\bar{\eta}} = -\|\xi\|^2 \|\eta\|^2 - |\langle \xi, \eta \rangle|^2 + |Q(\xi, \eta)|^2. \tag{3}$$

Write $\Theta_{i\bar{j}k\bar{\ell}} = -(\delta_{ij}\delta_{k\ell} + \delta_{i\ell}\delta_{jk})$. Then, Θ agrees with the curvature tensor of the n-dimensional complex hyperbolic space form of constant holomorphic sectional curvature -2 with the metric tensor $g_{i\bar{j}} = \delta_{ij}$ at the point of reference. Writing

$$R_{\xi\bar{\xi}\eta\bar{\eta}} = \Theta_{\xi\bar{\xi}\eta\bar{\eta}} + \left|Q(\xi,\eta)\right|^2, \tag{4}$$

for a local d-dimensional complex submanifold S' passing through o, by the Gauss equation the scalar curvature $K(o)$ satisfies

$$K(o) \leqslant -d(d+1) + \sum_{i=1}^{d} \left|Q(e_i, e_j)\right|^2, \tag{5}$$

where $\{e_i\}$ is any orthonormal basis of $T_o(S')$. The complex bilinear form $Q' = Q|_{T_o(S')}$ can be diagonalized by a unitary matrix (cf. [M2, (2.3), Lemma 1, p. 70]). Thus, we may assume $Q'\left(\sum \xi^i e_i, \sum \xi^i e_i\right) = \sum \lambda_i (\xi^i)^2$ where λ_i are nonnegative real numbers, $0 \leqslant \lambda_i \leqslant 1$. When S' corresponds to $D_d^{\mathrm{IV}} \subset D_n^{\mathrm{IV}}$, $\lambda_1 = \cdots = \lambda_d = 1$. S' is totally geodesic and the scalar curvature $K(o) = -d^2$. From (6) we have in general

$$K(o) \leqslant -d(d+1) + \sum_{i=1}^{d} \lambda_i^2 \leqslant -d^2. \tag{6}$$

If $\lambda_k = 0$ for some k, $1 \leqslant k \leqslant d$, we have $K(o) \leqslant -(d^2+1)$ since $0 \leqslant \lambda_i \leqslant 1$ for all i, $1 \leqslant i \leqslant d$. The assumption $K(x) > -(d^2+1)$ in Theorem 5 therefore, forces $\lambda_i \neq 0$ for $1 \leqslant i \leqslant d$, i.e., the restriction of the canonical quadric structure to $S \subset X$ is everywhere nondegenerate. By Theorem 4, $S \subset X$ is a totally-geodesic complex submanifold of scalar curvature $-d^2$, i.e., of maximal scalar curvature, as desired. □

Remarks. Unlike Theorem 2, for $d \geqslant 2$ Theorem 5 is not known to be optimal. We note however that for $d < n/2$, D_n^{IV} contains a totally-geodesic complex submanifold $\Sigma' \cong B^d$, of rank 1. For suitably chosen $\Gamma \subset \mathrm{Aut}(D_n^{\mathrm{IV}})$ there exists a totally-geodesic d-dimensional submanifold S' uniformized by Σ' such that the restriction of the canonical quadric structure to S is totally degenerate. It is of scalar curvature $-(d^2+d)$ while any S in Theorems 4 and 5 is of scalar curvature $-d^2$. For $1 < d < n/2$ the optimal lower bound of $K(x)$ may be $-(d^2+d)$, but our proof only shows that under the assumption $-(d^2+d) < K$ everywhere on S, the restriction of the canonical quadric structure to S is nowhere totally degenerate.

(4.3) For Ω a bounded symmetric domain and $D \subset \Omega$ a (totally geodesic) bounded symmetric subdomain, in [EM] we say that (Ω, D) exhibits the gap phenomenon if any almost geodesic (in some precise sense) compact complex submanifold S on some quotient manifold X of Ω modeled on D must necessarily be totally geodesic. The simplest example of the gap phenomenon is given by $\Omega = \Omega_0 \times \cdots \times \Omega_0$, and $D \subset \Omega$ the diagonal, and the proof as given in [EM] resulted from the uniqueness of Kähler–Einstein metrics. Using variations of Hodge structures, Eyssidieux [E1] and [E2] proved the gap phenomenon for many examples of (Ω, D) in which D is irreducible and $D \subset \Omega$ is a 1-hyperrigid subdomain in the sense of [E1]. Theorem 4 implies the gap phenomenon for the pair $(D_n^{\mathrm{IV}}; D_d^{\mathrm{IV}})$, in a more precise way as formulated in Theorem 5. Theorem 4, including the case of $d = 1$, appears to go beyond the method of variations of Hodge structures. For $d > 1$ it brings back the role of Kähler–Einstein metrics, as the proof relies essentially on the existence of Kähler–Einstein metrics on compact Kähler manifolds with ample canonical line bundle.

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