

A Unified Dimensionless Parameter for Finite Element Mesh for Beams Resting on Elastic Foundation

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Abstract: Discretising a structure into elements is a key step in finite element (FE) analysis. The discretised geometry used to formulate an FE model can greatly affect accuracy and validity. This paper presents a unified dimensionless parameter to generate a mesh of cubic FEs for the analysis of very long beams resting on an elastic foundation. A uniform beam resting on elastic foundation with various values of flexural stiffness and elastic supporting coefficients subject to static load and moving load is used to illustrate the application of the proposed parameter. The numerical results show that (a) Even if the values of the flexural stiffness of the beam and elastic supporting coefficient of the elastic foundation are different, the same proposed parameter “ s ” can ensure the same accuracy of the FE solution, but the accuracy may differ for use of the same element length; (b) The proposed dimensionless parameter “ s ” can indeed be used as a unified index to generate the mesh for a beam resting on elastic foundation, whereas the use of the same element length as a criterion may be misleading; (c) The errors between the FE and analytical solutions for the maximum vertical displacement, shear force and bending moment of the beam increase with the dimensionless parameter “ s ”; and (d) For the given allowable errors for the vertical displacement, shear force and bending moment of the beam under static load and moving load, the corresponding values of the proposed parameter are provided to guide the mesh generation.

Keywords: Beam, cubic finite element, elastic foundation, finite element mesh, unified dimensionless parameter.

1 Introduction

The problem of a beam resting on an elastic foundation is very often encountered in the analysis of building, underground, highway and railroad structures. Various types of foundation models, including the single-parameter Winkler foundation, double-parameter

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elastic foundation, triple-parameter elastic foundation, and semi-infinite elastic continuum foundation, have been provided.

Despite the relative simplicity of the Winkler foundation model, it still has a solid place in today's engineering calculations and remains widely used [Borák and Marcián (2014)]. In various books [Timoshenko (1956); Hetenyi (1961); Frýba (1999); Esveld (2001)], the analytical solutions of beams on an elastic foundation were presented. Froio et al. [Froio and Rizzi (2016, 2017)] presented analytical solution for the elastic bending of beams lying on a linearly variable Winkler support. Froio et al. [Froio, Rizzi, Simoes et al. (2018)] investigated the universal analytical solution of the steady-state response of an infinite beam on a Pasternak elastic foundation under moving load. Dimitrovova [Dimitrovova (2019)] derived and validated the semi-analytical solution for the problem of a uniformly moving oscillator on an infinite beam on a two-parameter visco-elastic foundation.

The finite element (FE) method is used extensively in the analysis of beams on an elastic foundation. It is generally known that discretising a structure into elements is a key step in FE analysis. The discretised geometry used to formulate an FE model can greatly affect the accuracy and validity. Some researchers investigated the effect of the mesh generation and element size on the simulation result. Bowles [Bowles (1977)] formulated a stiffness matrix by combining a conventional beam element based on a cubic function with discrete soil springs at the ends of the beam. The accuracy of this formulation is highly dependent on the number of elements used. Feng et al. [Feng and Cook (1983)] proposed two FE formulations to analyze beams on one- or two-parameter foundation. Numerical studies show that the element based on the exact displacement function can give exact numerical results even if the number of elements is very small, while the element based on a cubic function may require a fine mesh to give acceptable results. Rieker et al. [Rieker, Lin and Trethewey (1996)] investigated the relationship between the model accuracy and the number of elements used to discretize a structure for a moving load analysis, and provided guidance for the development of suitable meshes. Sunitha et al. [Sunitha, Dodagoudar and Rao (2007); Dodagoudar, Rao and Sunitha (2015)] proposed a simple but sufficiently accurate model for beams on elastic foundation using a mesh-free technique, called the element-free Galerkin method, that did not rely on any mesh. Correa et al. [Correa, Costa and Simoes (2018)] presented the FE modeling of a rail resting on a Winkler-Coulomb foundation and subjected to a moving concentrated load.

In view of the lack of guidance for discretization to provide numerical solutions of specific accuracy, the present study therefore attempts to develop a unified dimensionless parameter “ s ” for mesh generation with cubic FE formulation for beams resting on elastic foundation for various scenarios of flexural stiffness and elastic supporting coefficients. Furthermore, for the given allowable errors for calculating the vertical displacement, shear force and bending moment of the beam under static load and moving load, the corresponding values of the proposed parameter are provided to guide mesh generation.

2 A proposed dimensionless parameter

In the cubic FE analysis of very long beams resting on Winkler foundation, the accuracy of the solution mainly depends on the element size. For beams with the same flexural stiffness and the elastic supporting coefficients, the accuracy of the solution is higher for finer

meshes or smaller elements. However, for beams with different flexural stiffness and elastic supporting coefficients, this rule may not hold, i.e., the accuracy for a finer mesh may not necessarily be higher than that for a coarser mesh.

To generate the FE mesh for beam on Winkler foundation with different flexural stiffness EI and elastic supporting coefficient k_w with specific accuracy, in which E and I are, respectively, the elastic modulus and sectional moment of inertia of the beam, a dimensionless parameter “ s ” is proposed as

$$s = l / L_c \tag{1}$$

where l denotes the element length and L_c is the characteristic length, which is a general parameter related to the properties of the beam and foundation and is calculated as [Esveld (2001)]

$$L_c = \sqrt[4]{4EI / k_w} \tag{2}$$

3 Formulae for calculating vertical displacement, shear force and bending moment

To compare with the calculation results for very long beams on elastic foundation, the corresponding analytical and FE solutions are given below.

3.1 Analytical solutions for the cases of static and moving concentrated forces

3.1.1 Case of static concentrated forces

The analytical solutions for the vertical displacement $y(x)$, bending moment $M(x)$ and shear force $Q(x)$ of an infinite-length beam at abscissa x on Winkler foundation subject to a static concentrated force P at $x = 0$ can be found in various references [Timoshenko (1956); Hetenyi (1961); Frýba (1999); Esveld (2001)] and are given as follows

$$y(x) = \frac{P}{2k_w L_c} e^{-|x|/L_c} \left(\cos \frac{x}{L_c} + \sin \frac{|x|}{L_c} \right) \tag{3}$$

$$M(x) = \frac{PL_c}{4} e^{-|x|/L_c} \left(\cos \frac{x}{L_c} - \sin \frac{|x|}{L_c} \right) \tag{4}$$

$$Q(x) = -\text{sign}(x) \cdot \frac{P}{2} e^{-|x|/L_c} \cos \frac{x}{L_c} \tag{5}$$

where $\text{sign}(x)$ is the signum function defined as

$$\text{sign}(x) = \begin{cases} 1 & x > 0 \\ 0 & x = 0 \\ -1 & x < 0 \end{cases} \tag{6}$$

It should be noted that Eqs. (3)-(5) only apply to the case of a single concentrated force. If there are multiple concentrated forces, $y(x)$, $M(x)$ and $Q(x)$ can be obtained by superposition.

3.1.2 Case of moving concentrated forces

The deflection, bending moment and shear force underneath the concentrated force P moving on an infinite-length beam on elastic foundation for the dimensionless speed parameter $\alpha < 1$ and the light damping of elastic foundation can be written as [Frýba (1999)]

$$y(x=0) = \frac{P}{2k_w L_c} \cdot \frac{1}{(1-\alpha^2)^{1/2} \cdot [1+0.5\alpha^2\beta^2/(1-\alpha^2)^2]} \quad (7)$$

$$M(x=0) = \frac{PL_c}{4} \cdot \frac{1}{(1-\alpha^2)^{1/2} \cdot [1+0.5\alpha^2\beta^2/(1-\alpha^2)^2]} \quad (8)$$

$$Q(x=\pm 0) = \mp \frac{P}{2} \cdot \left[1 \pm \frac{\alpha\beta}{2(1-\alpha^2)^{3/2} + \alpha^2\beta^2(1-\alpha^2)^{-1/2}} \right] \quad (9)$$

where the dimensionless speed parameter α is the ratio of the actual speed to the critical speed (resonance), the dimensionless damping parameter β is the ratio of the actual damping to the critical damping, and α and β can be expressed as

$$\alpha = v/v_{cr} \quad (10)$$

$$\beta = c_w/c_{cr} \quad (11)$$

$$v_{cr} = 2 \cdot \sqrt{EI/\bar{m}}/L_c \quad (12)$$

$$c_{cr} = 2 \cdot \bar{m}/\sqrt{\bar{m}/k_w} \quad (13)$$

in which v denotes the actual speed of the moving concentrated force; v_{cr} denotes the critical speed (resonance) of the moving concentrated force on elastic foundation; \bar{m} denotes the mass of per length of beam; and c_w and c_{cr} denote, respectively, the actual and critical damping of elastic foundation.

3.2 Formulae of FE solutions

3.2.1 Formulae for vertical displacement

In the cubic FE analysis for very long beams on viscoelastic foundation subjected to moving concentrated forces, in order to obtain the vertical displacement of any section of the beam, the nodal displacement vector of the beam needs to be calculated firstly by solving the equation

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{C}\dot{\mathbf{q}} + \mathbf{K}\mathbf{q} = \mathbf{F} \quad (14)$$

where \mathbf{M} , \mathbf{C} and \mathbf{K} denote the global mass, damping and stiffness matrices, respectively; $\ddot{\mathbf{q}}$, $\dot{\mathbf{q}}$ and \mathbf{q} denote the global acceleration, velocity and displacement

vectors, respectively; and \mathbf{F} denotes the global nodal force vector. The vertical displacement y_A at a certain point A in an element can then be calculated by

$$y_A = \mathbf{N}_{\xi=\xi_A} \mathbf{q}^e \quad (15)$$

where ξ denotes the local coordinate measured from the left node of the element; ξ_A denotes the distance between the left node of the element and point A; and \mathbf{N} and \mathbf{q}^e denote the shape function matrix and the nodal displacement vector of the element, respectively. \mathbf{N} can be written as

$$\mathbf{N} = [N_1 \quad N_2 \quad N_3 \quad N_4] \quad (16)$$

$$\text{with } N_1 = 1 - 3(\xi/l)^2 + 2(\xi/l)^3, \quad N_2 = \xi[1 - 2(\xi/l) + (\xi/l)^2]$$

$$N_3 = 3(\xi/l)^2 - 2(\xi/l)^3, \quad N_4 = \xi[(\xi/l)^2 - (\xi/l)]$$

3.2.2 Formulae for shear force and bending moment

For a very long beam resting on a viscoelastic foundation subjected to several concentrated forces, the detailed formulae based on cubic FE for calculating the shear force and bending moment can be found in Lou [Lou (2008)]. For convenience, the key formulae are listed as follows.

Formulae for shear force and bending moment at nodes

From the dynamic equilibrium of each beam element [Paz (1997)], the nodal element force vector \mathbf{f}^e of the two nodes of a typical beam element at time t can be expressed as

$$\mathbf{f}^e = \mathbf{m}\ddot{\mathbf{q}}^e + \mathbf{c}_w\dot{\mathbf{q}}^e + \mathbf{k}_b\mathbf{q}^e + \mathbf{k}_w\mathbf{q}^e - \mathbf{f}_E \quad (17)$$

where $\mathbf{f}^e = [Q_1^e \quad M_1^e \quad Q_r^e \quad M_r^e]^T$ is the nodal element force vector at the two nodes of the beam element, Q_1^e and M_1^e are the shear force and bending moment at the left node of the beam element, respectively; Q_r^e and M_r^e are the shear force and bending moment at the right node of the beam element, respectively; the positive directions of Q_1^e , M_1^e , Q_r^e and M_r^e are shown in Fig. 1; $\ddot{\mathbf{q}}^e$, $\dot{\mathbf{q}}^e$ and \mathbf{q}^e are the nodal acceleration, velocity and displacement vectors of the beam element, respectively; \mathbf{c}_w is the element damping matrix due to the viscous damping foundation supporting the beam element; \mathbf{k}_b is the element stiffness matrix of the beam element itself; \mathbf{k}_w is the element stiffness matrix due to the elastic foundation supporting the beam element; and \mathbf{f}_E is the equivalent nodal force vector of the beam element due to all concentrated forces acting on the beam element.

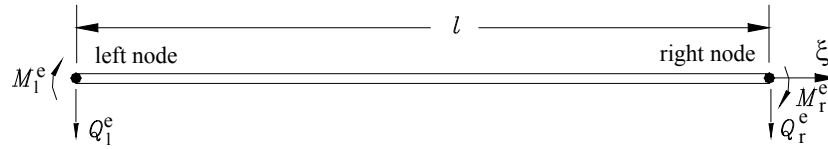


Figure 1: Shear forces Q_1^e and Q_r^e , and bending moments M_1^e and M_r^e at the two nodes of a typical beam element

Formulae for shear force and bending moment at any section of beam

If one is interested in the shear force and bending moment at any section of a very long beam on viscoelastic Winkler foundation, it is unrealistic to directly use Eq. (17) because a very fine mesh must be adopted, which results in a dramatic increase in the number of elements. It is desirable to develop an efficient method for it. It is assumed that point A is any arbitrary section in the element between two adjacent nodes. Let us consider the segment between the left node of the element and point A as a free body, as shown in Fig. 2. There are several concentrated moving forces P_i ($i=1, 2, \dots, \bar{h}$) between the left node of the beam element and point A at time t . The shear force Q_A and bending moment M_A at point A are given by

$$Q_A = \int_0^{\xi_A} (\bar{m} \mathbf{N} \ddot{\mathbf{q}}^e + c_w \mathbf{N} \dot{\mathbf{q}}^e + k_w \mathbf{N} \mathbf{q}^e) d\xi - Q_1^e - \sum_{i=1}^{\bar{h}} P_i \tag{18}$$

$$M_A = Q_1^e \xi_A + \sum_{i=1}^{\bar{h}} P_i (\xi_A - \xi_i) - M_1^e - \int_0^{\xi_A} (\xi_A - \xi) (\bar{m} \mathbf{N} \ddot{\mathbf{q}}^e + c_w \mathbf{N} \dot{\mathbf{q}}^e + k_w \mathbf{N} \mathbf{q}^e) d\xi \tag{19}$$

where ξ_A denotes the distance between the left node of the beam element and point A. The positive directions of the shear force Q_A and bending moment M_A at point A are shown in Fig. 2. It should be noted that Q_1^e and M_1^e in Eqs. (18) and (19) have been obtained by solving Eq. (17).

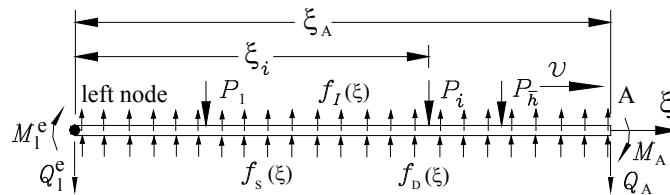


Figure 2: Free-body diagram of portion of a beam element on viscoelastic foundation, in which $f_I(\xi) = \bar{m} \mathbf{N} \ddot{\mathbf{q}}^e$, $f_S(\xi) = k_w \mathbf{N} \mathbf{q}^e$, and $f_D(\xi) = c_w \mathbf{N} \dot{\mathbf{q}}^e$

Eqs. (18) and (19) can be used to calculate the shear force and bending moment at any section within an element, including both nodes, whereas Eq. (17) can only be used to calculate the shear force and bending moment at the nodes.

Furthermore, Eqs. (14), (17)-(19) can be used to solve the “static” problem by setting $\ddot{\mathbf{q}} = \dot{\mathbf{q}} = \mathbf{0}$ and $\ddot{\mathbf{q}}^e = \dot{\mathbf{q}}^e = \mathbf{0}$.

4 Application of the proposed parameter and discussions

To demonstrate the use of the proposed dimensionless parameter “*s*” as a unified index to generate element mesh for a beam resting on elastic foundation with various values of flexural stiffness and elastic supporting coefficients, two types of loads, i.e., single static and moving concentrated forces *P* with 10,000 N, and three cases with the parameters *E*, *I* and *k_w* as shown in Tab. 1 are considered. The type of single static concentrated force acting on a very long beam of length *L* is shown in Fig. 3. Cases 1 and 2 in Tab. 1 have the same beam, but the foundation of Case 2 has substantially smaller stiffness. Cases 3 and 4 have the same beam with the least stiffness and foundation that is the least stiff. Case 5 is a beam with different parameters *I* and *k_w*. The parameter \bar{m} in Tab. 1 is only used to study the problem of moving concentrated force. To describe the moving concentrated force, the dimensionless speed parameter $\alpha = 0.2$ and the dimensionless damping parameter $\beta = 0.2$ are adopted. The dimensionless parameter “*s*” is varied from 0.2 to 2.0 with increments of 0.2, and the corresponding element length is $l = s \cdot L_c$, in which *L_c* can be obtained by Eq. (2). It should be noted that the element length varies with the parameter “*s*”. The static concentrated force is considered to act at different positions along the element length adjacent to the mid-point of the very long beam, while the moving concentrated force is considered to act at a position ranging from the left end to the right end of the beam.

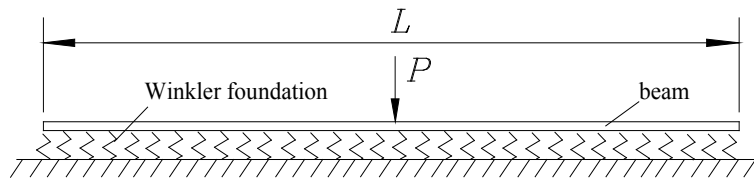


Figure 3: A very long uniform Euler beam with free end on Winkler foundation under a static concentrated force

Table 1: A beam on elastic foundation with different parameters

Cases	<i>E</i> (N/m ²)	<i>I</i> (m ⁴)	<i>k_w</i> (N/m ²)	\bar{m} (kg/m)
1	2.1×10 ¹¹	3.217×10 ⁻⁵	5.0×10 ⁷	60.64
2	2.1×10 ¹¹	3.217×10 ⁻⁵	5.0×10 ⁶	60.64
3	9.1×10 ⁹	7.326×10 ⁻⁵	1.0×10 ⁶	12.0
4	9.1×10 ⁹	7.326×10 ⁻⁵	4.0×10 ⁶	12.0
5	2.1×10 ¹¹	4.489×10 ⁻⁵	1.0×10 ⁷	74.414

An important feature of the very long beam on elastic foundation is that the vertical displacement and internal force decrease to zero over a distance of around $5L_c$ from the concentrated force. To ensure that the finite-length beam can be used to model an infinite-length beam on an elastic foundation, for each case in Tab. 1, the minimum distance between the loaded point and the beam end should be longer than $5L_c$. The length of the beam adopted in this study is not shorter than $42L_c$ so that the finite-length beam behaves effectively as an infinite-length one.

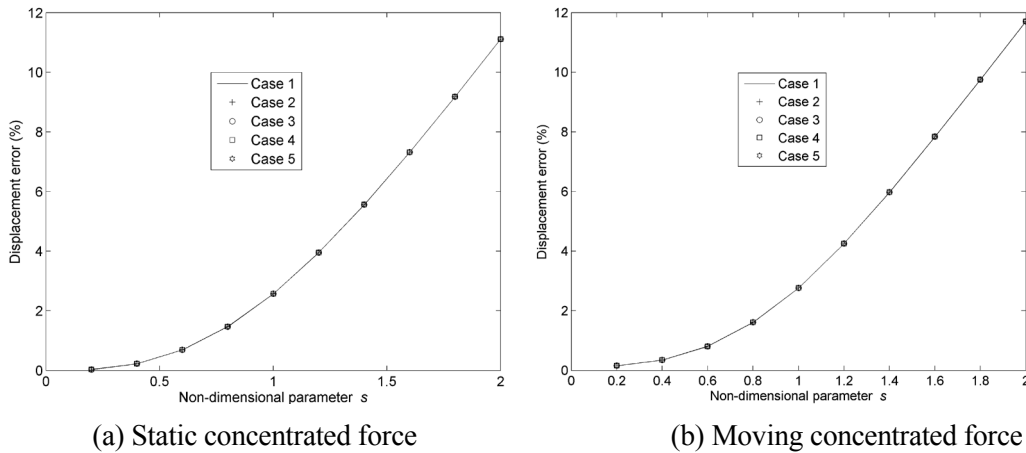


Figure 4: Displacement errors against dimensionless parameter “s” for Cases 1, 2 and 3

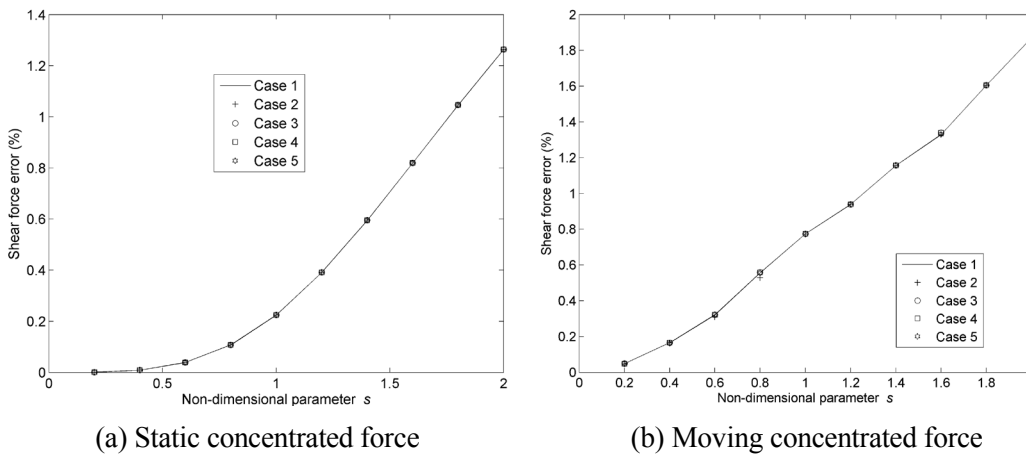


Figure 5: Shear force errors against dimensionless parameter “s” for Cases 1, 2 and 3

Generally, the maximum values of the vertical displacement, shear force and bending moment of beam are of concern. For the concentrated force P , these maxima occur underneath the load. For each case in Tab. 1, the errors between the FE and analytical solutions for the maximum vertical displacement, shear force and bending moment of the

beam are investigated with respect to the dimensionless parameter “*s*” and the element length. The percentage error of a response is defined in terms of the absolute error as

$$\text{Percentage error of } R = \left| \frac{R_{FE} - R_{An}}{R_{An}} \right| \times 100\% \tag{19}$$

where *R* denotes the response, i.e., vertical displacement, shear force and bending moment, and *R_{FE}* and *R_{An}* denote the FE and analytical solutions of the response, respectively. The FE solutions are obtained by using Eqs. (15), (18) and (19), while the analytical solutions are calculated by using Eqs. (3)-(5) for the static concentrated force and Eqs. (7)-(9) for the moving concentrated force.

The maximum errors of the vertical displacement, shear force and bending moment of beam underneath the static and moving concentrated forces *P* against the dimensionless parameter “*s*” and element length are plotted in Figs. 4-9. Fig. 4 shows that, whether for the static or moving load, the curves for the maximum errors of the vertical displacement against the dimensionless parameter “*s*” for Cases 1-3 are virtually identical to one another. The same holds for the curves for the maximum errors of the shear force and bending moment against the dimensionless parameter “*s*” as shown in Figs. 5 and 6. However, both Figs. 7(a) and 7(b) show that the curves for the maximum errors of the vertical displacement against the element length for Cases 1-3 under the static or moving concentrated force differ from one another. Similar phenomena can also be observed for the shear force and bending moment, as shown in Figs. 8 and 9, respectively. In addition, Figs. 4-6 show that the errors between the FE and analytical solutions for the maximum vertical displacement, shear force and bending moment of the beam increase with the dimensionless parameter “*s*”.

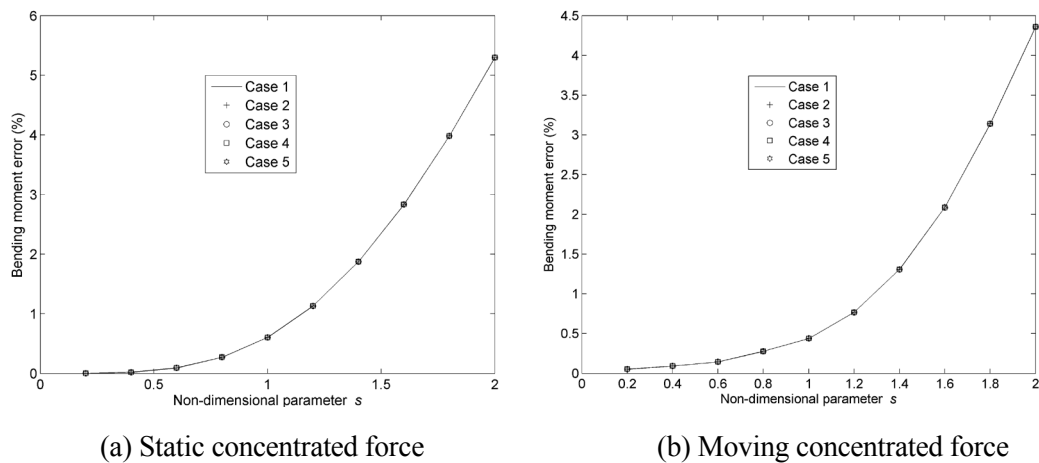
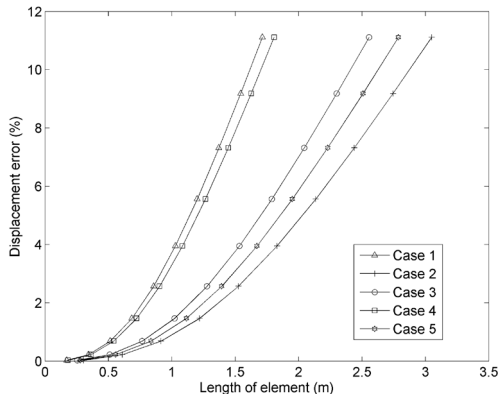
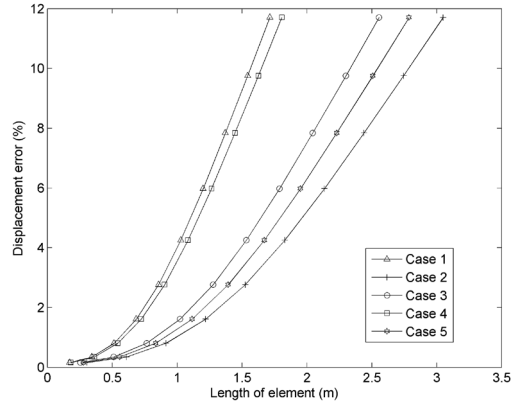


Figure 6: Bending moment errors against dimensionless parameter “*s*” for Cases 1, 2 and 3

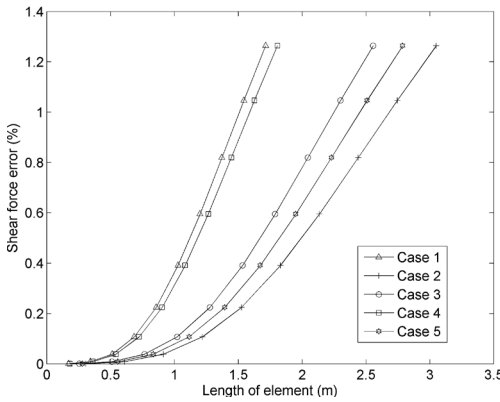


(a) Static concentrated force

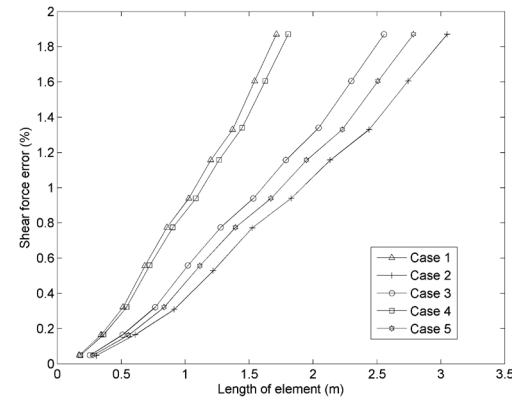


(b) Moving concentrated force

Figure 7: Displacement errors against element length for Cases 1, 2 and 3

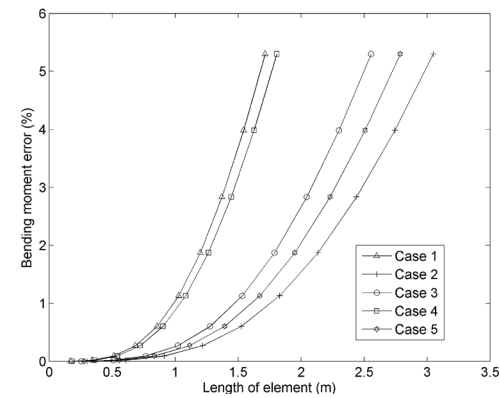


(a) Static concentrated force

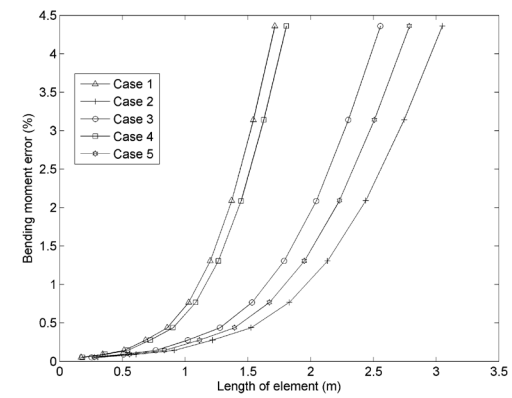


(b) Moving concentrated force

Figure 8: Shear force errors against element length for Cases 1, 2 and 3



(a) Static concentrated force



(b) Moving concentrated force

Figure 9: Bending moment errors against element length for Cases 1, 2 and 3

This example illustrates that, even if the values of the flexural stiffness of the beam and the elastic supporting coefficient of the elastic foundation are different, the same parameter “*s*” can ensure the same accuracy of the FE solution, but the accuracy may be different for the same element length. Therefore, the proposed dimensionless parameter “*s*” can indeed be used as a unified index to generate element mesh for a beam resting on elastic foundation, whereas the use of the same element length as a criterion may be misleading.

This example also demonstrates how the proposed parameter “*s*” can guide the choice of element length. Tab. 2 provides the values of the dimensionless parameter “*s*” for calculating the vertical displacement, bending moment and shear force with various allowable errors under static concentrated force. The procedures for obtaining the element length to achieve specific accuracy are described as follows.

- (a) Assume an allowable error of the vertical displacement, shear force and bending moment.
- (b) Find the value of the dimensionless parameter “*s*” for the vertical displacement, shear force and bending moment in the same column of the allowable error in Tab. 2.
- (c) The element length *l* is obtained as the product of *s* and *L_c* by using Eq. (1).

Table 2: The values of dimensionless parameter “*s*” for calculating vertical displacement, shear force and bending moment with given error under static concentrated force

Allowable error	2.5%	1.0%	0.5%	0.25%	0.1%
<i>s</i> for vertical displacement	0.9887	0.6903	0.5338	0.4157	0.3010
<i>s</i> for bending moment	1.5351	1.1569	0.9480	0.7823	0.6109
<i>s</i> for shear force	2.7870	1.7590	1.3100	1.0344	0.7831

(Note: The corresponding element length is $l = s \cdot L_c$ and the dimensionless parameter “*s*” is the maximum value to satisfy the allowable error)

Table 3: The values of dimensionless parameter “*s*” for calculating vertical displacement with various values of α and β for allowable error 5% under moving concentrated force

α	β						
	0.1	0.15	0.2	0.25	0.3	0.35	0.4
0.1	1.328	1.325	1.325	1.322	1.319	1.317	1.314
0.2	1.289	1.291	1.291	1.286	1.278	1.270	1.259
0.3	1.256	1.295	1.284	1.269	1.247	1.219	1.188
0.4	1.247	1.288	1.259	1.225	1.177	1.119	1.047

(Note: The corresponding element length is $l = s \cdot L_c$ and the dimensionless parameter “*s*” in the Table is the maximum value to satisfy the allowable error)

Tabs. 3-8 provide the values of the dimensionless parameter “*s*” for calculating the vertical displacement, bending moment and shear force with various values of α and β for allowable errors of 5% and 2.5% under a moving concentrated force. The procedures for obtaining the element length with allowable errors of 5% and 2.5% are described as follows.

- (a) Determine the values of the dimensionless speed parameter α and the dimensionless damping parameter β .
- (b) Find the value of the dimensionless parameter “ s ” for the vertical displacement, shear force and bending moment in Tabs. 3-5 with allowable error of 5% or in Tabs. 6-8 with allowable error of 2.5% according to the values of α and β .
- (c) The element length l is obtained as the product of s and L_c by using Eq. (1).

Table 4: The values of dimensionless parameter “ s ” for calculating shear force with various values of α and β for allowable error 5% under moving concentrated force

α	β						
	0.1	0.15	0.2	0.25	0.3	0.35	0.4
0.1	3.469	3.441	3.413	3.381	3.350	3.319	3.288
0.2	3.499	3.456	3.391	3.325	3.257	3.191	3.125
0.3	3.425	3.438	3.353	3.150	2.944	2.744	2.557
0.4	1.507	1.566	1.605	1.805	1.844	1.908	1.863

Table 5: The values of dimensionless parameter “ s ” for calculating bending moment with various values of α and β for allowable error 5% under moving concentrated force

α	β						
	0.1	0.15	0.2	0.25	0.3	0.35	0.4
0.1	1.988	1.991	1.991	1.988	1.988	1.988	1.988
0.2	2.075	2.091	2.097	2.097	2.097	2.094	2.091
0.3	2.394	2.375	2.338	2.319	2.288	2.263	2.241
0.4	1.908	2.000	2.200	2.225	2.244	2.253	2.259

Table 6: The values of dimensionless parameter “ s ” for calculating vertical displacement with various values of α and β for allowable error 2.5% under moving concentrated force

α	β						
	0.1	0.15	0.2	0.25	0.3	0.35	0.4
0.1	0.981	0.980	0.978	0.976	0.9727	0.970	0.966
0.2	0.965	0.966	0.960	0.950	0.937	0.921	0.902
0.3	0.949	0.957	0.940	0.912	0.877	0.833	0.777
0.4	0.913	0.939	0.902	0.841	0.754	0.635	0.434

Table 7: The values of dimensionless parameter “ s ” for calculating shear force with various values of α and β for allowable error 2.5% under moving concentrated force

α	β						
	0.1	0.15	0.2	0.25	0.3	0.35	0.4
0.1	2.796	2.775	2.740	2.667	2.595	2.526	2.459
0.2	2.761	2.670	2.536	2.367	2.223	2.105	2.009
0.3	1.199	1.304	1.370	1.491	1.505	1.532	1.499
0.4	0.904	0.912	0.968	0.976	1.148	1.163	1.162

Table 8: The values of dimensionless parameter “ s ” for calculating bending moment with various values of α and β for allowable error 2.5% under moving concentrated force

α	β						
	0.1	0.15	0.2	0.25	0.3	0.35	0.4
0.1	1.564	1.566	1.567	1.568	1.568	1.569	1.569
0.2	1.680	1.684	1.682	1.677	1.673	1.669	1.666
0.3	1.615	1.763	1.779	1.776	1.770	1.763	1.759
0.4	1.274	1.400	1.450	1.557	1.602	1.596	1.513

5 Conclusions

In this paper, a unified dimensionless parameter is proposed to guide the mesh generation for cubic FEs for the analysis of very long beams resting on elastic foundation with various values of flexural stiffness and elastic supporting coefficients. The main conclusions that can be drawn from this investigation are as follows.

- (a) Whether for static or moving load, for the same value of the proposed parameter “ s ”, the accuracies of cubic FE solutions of beams with different values of flexural stiffness and elastic supporting coefficient are the same. However, for the same value of element length, the accuracies of the solutions may not be the same. Therefore, the proposed dimensionless parameter “ s ” can indeed be used as a unified index to generate FE mesh for the beam, whereas the use of the element length as a criterion may be misleading.
- (b) For various values of allowable errors of the vertical displacement, shear force and bending moment of beams on elastic foundation under static or moving concentrated force, the corresponding values of the dimensionless parameter “ s ” are provided to guide the FE mesh generation.
- (c) The errors between the FE and analytical solutions for the maximum vertical displacement, shear force and bending moment of the beam increase with the dimensionless parameter “ s ”. For the same allowable errors of the vertical displacement, shear force and bending moment of the beam, the values of their respective dimensionless parameter “ s ” can be different.

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References

Borák, L.; Marcián, P. (2014): Beams on elastic foundation using modified Betti's theorem. *International Journal of Mechanical Sciences*, vol. 88, pp. 17-24.

Bowles, J. E. (1977): *Foundation Analysis and Design*, 2nd ed. New York: McGraw-Hill Co., Inc.

Correa, R. T.; da Costa, A. P.; Simoes, F. M. F. (2018): Finite element modeling of a rail resting on a Winkler-Coulomb foundation and subjected to a moving concentrated load. *International Journal of Mechanical Sciences*, vol. 140, pp. 432-445.

Dimitrova, Z. (2019): Semi-analytical solution for a problem of a uniformly moving oscillator on an infinite beam on a two-parameter visco-elastic foundation. *Journal of Sound and Vibration*, vol. 438, pp. 257-290.

Dodagoudar, G. R.; Rao, B. N.; Sunitha, N. V. (2015): A meshfree method for beams on elastic foundation. *International Journal of Geotechnical Engineering*, vol. 9, no. 3, pp. 298-306.

Esveld, C. (2001): *Modern Railway Track (2nd ed)*. MRT-Productions.

Feng, Z. H.; Cook, R. D. (1983): Beam elements on two-parameter elastic foundations. *Journal of Engineering Mechanics*, vol. 109, no. 6, pp. 1390-1402.

Froio, D.; Rizzi, E. (2016): Analytical solution for the elastic bending of beams lying on a variable Winkler support. *Acta Mechanica*, vol. 227, no. 4, pp. 1157-1179.

Froio, D.; Rizzi, E. (2017): Analytical solution for the elastic bending of beams lying on a linearly variable Winkler support. *International Journal of Mechanical Sciences*, vol. 128, pp. 680-694.

Froio, D.; Rizzi, E.; Simoes, F. M. F.; Da Costa, A. P. (2018): Universal analytical solution of the steady-state response of an infinite beam on a Pasternak elastic foundation under moving load. *International Journal of Solids Structures*, vol. 132, pp. 245-263.

Fryba, L. (1999): *Vibration of Solids and Structures Under Moving Loads (3rd ed)*. Academia Prague.

Hetenyi, M. (1961): *Beams on Elastic Foundation*. University of Michigan Press, Ann Arbor, MI.

Lou, P. (2008): Finite-element formulae for calculating the sectional forces of a Bernoulli-Euler beam on continuously viscoelastic foundation subjected to moving concentrated loads. *Shock and Vibration*, vol. 15, no. 2, pp. 147-162.

Paz, M. (1997): *Structural Dynamics: Theory and Computation (4th ed)*. Chapman & Hall, New York.

Rieker, J. R.; Lin, Y. H.; Trethewey, M. W. (1996): Discretization considerations in moving load finite element beam models. *Finite Elements in Analysis and Design*, vol. 21, no. 3, pp. 129-144.

Sunitha, N. V.; Dodagoudar, G. R.; Rao, B. N. (2007): Element free Galerkin method for beams on elastic foundation. *Journal of Structural Engineering (Madras)*, vol. 34, no. 5, pp. 327-334.

Timoshenko, S. (1956): *Strength of Materials Part II Advance Theory and Problems*. Van Nostrand Reinhold Company Ltd., London.