Quantum magnetotransport in massive Dirac materials

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Massive Dirac fermions break the chiral symmetry explicitly and also make the Berry curvature of the band structure non-Abelian. By utilizing the Green's function technique, we develop a microscopic theory to establish a set of quantum diffusive equations for massive Dirac materials in the presence of electric and magnetic fields. It is found that the longitudinal magnetoresistance is always negative and quadratic in the magnetic field, and decays quickly with the mass. The theory is applicable to the systems with non-Abelian Berry curvature and resolves the puzzles of anomalous magnetotransport properties measured in topological materials.

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I. INTRODUCTION

Symmetries and their corresponding conservation laws play an important role in understanding the fundamental nature of matter. A classical conservation law might turn out to be violated in its quantized version, i.e., the so-called quantum anomaly [1,2]. A well-known example is that the massless relativistic Dirac or Weyl fermions in three spatial dimensions possess the chiral anomaly [3,4]: the chiral charges at each node ($\chi = \pm 1$) will be nonconserved in the presence of the external gauge fields with nontrivial topology. The quantum fluctuation produces an additional term to the classical conservation law for the chiral charges:

$$\partial^{\mu} J^{\chi}_{\mu} = -\chi \frac{e^3}{4\pi^2\hbar^2} \mathbf{E}\cdot\mathbf{B}, \tag{1}$$

where J^{χ}_0 is the charge density and J^{χ}_i $(i = 1, 2, 3)$ is the electric current for the Weyl node χ. From this equation, with parallel electromagnetic fields $\mathbf{E}$ and $\mathbf{B}$, charge might flow between Weyl nodes and can not reach the equilibrium in the absence of internode scattering, generating the chirality imbalance. Nielsen and Ninomiya [5] proposed that the chiral anomaly of the Weyl fermions could be realized in the Weyl semimetals based on the picture of the Landau levels of the Weyl fermions in a finite magnetic field as shown in Fig. 1(left), and a negative magnetoresistance is regarded as a substantial signature of the effect. Son and Spivak have obtained such anomaly-related negative magnetoresistance effect through semiclassical Boltzmann transport theory [6]. It is assumed that the Hamilton dynamics equations of quasiparticle trajectories become modified due to the (Abelian) Berry curvature. After introducing the internode scattering time τ_a phenomenologically, a nonzero deviation of the chiral charge density $\delta\rho^5 = \delta\rho^+ - \delta\rho^- = \frac{e^2}{2\pi^2}\mathbf{E}\cdot\mathbf{B}\tau_a$ from its equilibrium value can be obtained in the steady-state condition, which results in an anomaly-related negative magnetoresistance $\sim B^2$. A lot of theoretical approaches have been developed for the anomaly-induced magnetoresistivity for massless Weyl fermions [6–11]. Recent advances in topological materials demonstrate a series of topological materials may host the chiral quasiparticles [12–20], which provide a practical route to detect the signatures of the purely quantum mechanical effect. The longitudinal negative magnetoresistance has been reported experimentally in a large class of topological materials, and the observation was regarded extensively as a smoking gun to confirm the chiral anomaly in condensed matter [21–30].

However, a puzzle arises as some topological materials with negative magnetoresistance are actually not Weyl or Dirac semimetals: for example, $ZrTe_5$ [30–35] actually has a tiny direct band gap, and Bi_2Se_3 is a typical topological insulator [36–38], in which the chiral anomaly might not exist according to the existing theories. It is known that mass term breaks the chiral symmetry explicitly, and also modifies the Nielsen-Ninomiya picture for chiral anomaly at a finite field. The conservation law for the axial charge is modified to be [39]

$$\partial_{\mu}\hat{J}^{a\mu}(x) = 2mv^2\hat{\mathfrak{n}}_P + \frac{e^3}{2\pi^2\hbar^2}\mathbf{E}\cdot\mathbf{B}, \tag{2}$$

where mv^2 and e are the fermion mass and charge, respectively, the axial or chiral current is defined as $\hat{J}^{a\mu} = \bar{\Psi}\gamma^{\mu}\gamma^5\Psi$, and $\hat{\mathfrak{n}}_P = \bar{\Psi}i\gamma^5\Psi$ is the pseudoscalar density, where $\bar{\Psi}$ and Ψ are fermionic field operators. γ^{μ} are the Dirac gamma matrices. The presence of term $\hat{\mathfrak{n}}_P$ in the right-hand side of Eq. (2) indicates that the axial charges are even not conserved at the classical level in the presence of the Dirac mass. Furthermore, the anomaly term arises as a consequence of the ultraviolet divergence of the "VVA" triangle diagrams which cannot be cured by a finite mass [39]. Another direct consequence of the chiral symmetry breaking is that the Berry curvature of the band structure becomes non-Abelian [40–43]. Actually, the two zeroth Landau levels are mixed together near the crossing point as shown in Fig. 1(right). And the charge tunneling process can still be realized through the smoothly connected region for nonzero Dirac mass, which is analogous

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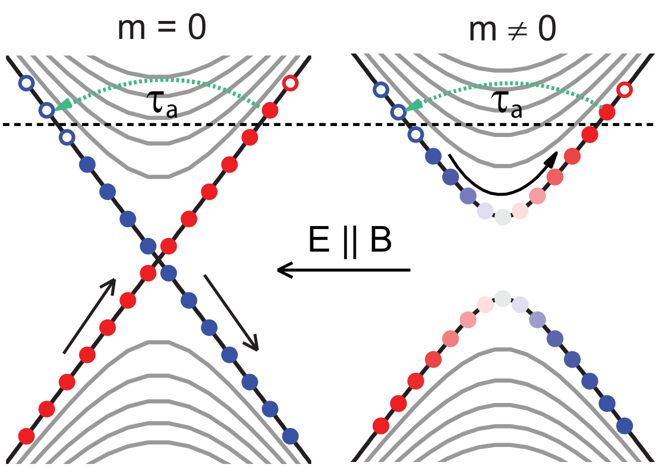

FIG. 1. Schematics for chiral anomaly-related magnetoresistance mechanism for massless (left panel) and massive (right panel) Dirac materials in the parallel electromagnetic field. The occupied and unoccupied states are shown as solid and open dots for the lowest Landau level, respectively, and the color of the dots indicates the averaged chirality $\langle\gamma^5\rangle$ as a function of the momentum. For massive case of $m \neq 0$, the states are mixed near $k_z = 0$. The populations for two chiralities are different due to the presence of parallel electric and magnetic field. The black solid line arrows demonstrate the charge transfer driven by the electric field. The exceeding right-hand electrons (red solid dots) are scattered back to left (blue open dots) as indicated by the dashed green line arrow, and the relaxation time is characterized by τ_a.

to the massless case that the chiral charge pumping is through the infinite Dirac sea [5]. Thus, it becomes an open issue whether the measured negative magnetoresistance could be still attributed to the chiral anomaly in the case of massive Dirac fermions. Some mechanisms have been proposed for topological and trivial states without invoking chiral anomaly [44–49]. Thus, it is desirable to develop a unified quantum magnetotransport theory for the topological materials with either Abelian or non-Abelian Berry curvature to clarify the puzzle.

The rest of this paper is organized as follows. In Sec. II we introduce the model Hamiltonian for massive Dirac fermion, give the Green's function in finite magnetic field, and analyze all the physical quantities and various types of disorder that may exist in this 4×4 Hamiltonian according to symmetry. Section III presents the derivation for the quantum diffusive equations in the real space in the framework of the linear response theory, which is one of the main findings in this paper. Section IV gives the solutions of the diffusive equations for the parallel electromagnetic fields in the diffusive regime, which contain the important results for the longitudinal magnetoconductivity and the axial density imbalance. Section V predicts an anomaly-related magnetic field correction to the polarization function from the obtained diffusive equations. Section VI derives the pseudoscalar density and the continuity equation for the massive Dirac fermion. Section VII discusses the physical origin of the anomalous coupling between the axial charge and vector current in Dirac materials in the presence of magnetic field. Finally, Sec. VIII is a conclusion.

TABLE I. Various types of physical quantities and disorder represented by fermionic bilinears ($i = 1, 2, 3$), their symmetries under time reversal ($\mathcal{T}$), parity ($\mathcal{I}$), and continuous chiral rotation ($\mathcal{C}$). The time-reversal symmetry $\mathcal{T}$ is generated by an antiunitary operator $\gamma^1\gamma^3\mathcal{K}$, where $\mathcal{K}$ is complex conjugation, such that $\mathcal{T}^2 = -1$. The parity operator is generated $\mathcal{I} = \gamma^0$. The continuous chiral symmetry ($\mathcal{C} = e^{i\theta\gamma^5}$) is generated by γ^5. Here, ✓ and × signify even and odd under a symmetry operation, respectively. And, we use the capital letters $\mathtt{A}, \mathtt{B}, \ldots$ for indices when the index runs through the entire hypercomplex system from 1 to 16.

Bilinear ($\gamma^{\mathtt{A}}$)	Physical quantity	$\mathcal{T}$	$\mathcal{I}$	$\mathcal{C}$	Disorder
$\bar{\Psi}\gamma^0\Psi$	Total charge (J^0)	✓	✓	✓	Δ
$\bar{\Psi}\gamma^0\gamma^5\Psi$	Axial charge (J^{a0})	✓	×	✓	Δ_a
$\bar{\Psi}\Psi$	Scalar mass ($\mathfrak{n}_\beta$)	✓	✓	×	Δ_m
$\bar{\Psi}i\gamma^5\Psi$	Pseudoscalar density ($\mathfrak{n}_P$)	×	×	×	Δ_P
$\bar{\Psi}\gamma^i\Psi$	Current (J^i)	×	×	✓	Δ_c
$\bar{\Psi}\gamma^i\gamma^5\Psi$	Axial current (J^{ai})	×	✓	✓	Δ_{ac}
$\bar{\Psi}i\gamma^0\gamma^i\Psi$	Electric polarization ($\mathfrak{p}_i$)	✓	×	×	Δ_p
$\bar{\Psi}\gamma^5\gamma^0\gamma^i\Psi$	Magnetization ($\mathfrak{m}_i$)	×	✓	×	Δ_M

II. MODEL HAMILTONIAN AND GREEN'S FUNCTION IN LANDAU-LEVEL REPRESENTATION

A. Model Hamiltonian

We start with the Hamiltonian for massive Dirac fermions

$$\mathscr{H}_0 = \int d^3\mathbf{x}\, \bar{\Psi}(\mathbf{x})(v\hat{\mathbf{p}} \cdot \boldsymbol{\gamma} + \varepsilon_F\gamma^0 + mv^2)\Psi(\mathbf{x}), \tag{3}$$

where m is the Dirac mass, v is the effective velocity, ε_F is the chemical potential, and $\hat{\mathbf{p}} = -i\hbar\nabla$ is the momentum operator. $\Psi(x)$ is the four-component Dirac spinor with the time-space position four-vector $x^\mu = (t, \mathbf{x})$, that the Greek indices (μ, ν, etc.) run over all the space-time indices (0,1,2,3). γ^μ are the Dirac gamma matrices in Weyl representation: $\gamma^0 = \tau^3 \otimes \sigma_0$ and $\gamma^i = i\tau^2 \otimes \sigma^i$ ($i = 1, 2, 3$) with τ^i and σ^i the Pauli matrices, acting on the orbital and spin degrees of freedom correspondingly. The chirality operator is $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \tau^1 \otimes \sigma^0$. In a uniform magnetic field, say along the z direction, the kinetic momentum operator $\hat{\mathbf{p}}$ is replaced by the canonical momentum operator in Eq. (3), $\boldsymbol{\pi} = -i\hbar(\nabla - ie\mathbf{A})$ with the gauge field chosen as $\mathbf{A} = (-Bx_2, 0, 0)$. In the case the model is solvable, and the energy dispersion becomes discrete to form the Landau levels [48]. The Green's functions for the free Dirac fermions at a magnetic field can be obtained analytically.

By using the five Dirac gamma matrices γ^μ ($\mu =$ 0, 1, 2, 3, and 5) and their descendants, we can define 16 physical quantities as shown in Table I. The enlarged (pseudospin $\otimes$ spin) gamma matrices will allow us to obtain a microscopic theory of diffusive transport for all the possible coupled physical observables in the presence of an external field.

B. Green's function in magnetic field

The Green' s function of free Dirac fermions in an external magnetic field reads as [50]

$$G(\mathbf{x}, \mathbf{x}'; i\omega_m) = \exp[i\Phi(\mathbf{x}_\perp, \mathbf{x}'_\perp)]\widetilde{G}(\mathbf{x} - \mathbf{x}'; i\omega_m), \tag{4}$$

where $\omega_m = (2m+1)\pi/\beta$ are the fermionic Matsubara frequencies and $\beta = 1/k_B T$ is the inverse temperature, $\Phi(\mathbf{x}_\perp, \mathbf{x}'_\perp) = e\int_{\mathbf{x}'_\perp}^{\mathbf{x}_\perp} d\mathbf{x}''_\perp \cdot \mathbf{A}(\mathbf{x}''_\perp) = -\frac{(x_1-x'_1)(x_2+x'_2)}{2\ell_B^2}$ is the Schwinger phase with $\mathbf{x}_\perp = (x_1, x_2)$, which depends on $x_2 + x'_2$ lacking translational invariance and reflecting the nonconservation of the direction of momentum evident in the circularity of motion in a magnetic field, also embodying the gauge dependence of the Green's function. Here we begin with the Fourier transform of $\widetilde{G}(\mathbf{x}-\mathbf{x}'; i\omega_m)$:

$$\widetilde{G}(\mathbf{k}, i\omega_m) = ie^{-k_\perp^2\ell_B^2}\sum_{n=0}^{\infty}\frac{(-1)^n D_n(\mathbf{k}, i\omega_m)}{(i\omega_m+\varepsilon_F)^2-\varepsilon_n^2(k^3)}, \quad (5)$$

where $\varepsilon_n(k^3) = \sqrt{m^2v^4 + 2nv^2\hbar|eB| + v^2(k^3)^2}$ are the energies of the relativistic Landau levels and the numerator of the nth Landau-level contribution is determined by

$$\begin{aligned} D_n(\mathbf{k}, i\omega_m) =& 2[(i\omega_m+\varepsilon_F)\gamma^0 - v\hbar k^3\gamma^3 + mv^2] \\ &\times \left[\mathcal{P}_- L_n\left(2k_\perp^2\ell_B^2\right) - \mathcal{P}_+ L_{n-1}\left(2k_\perp^2\ell_B^2\right)\right] \\ &- 4v\hbar\boldsymbol{\gamma}\cdot\mathbf{k}_\perp L_{n-1}^1\left(2k_\perp^2\ell_B^2\right) \end{aligned} \quad (6)$$

with $\mathbf{k}_\perp = (k^1, k^2)$, $\mathcal{P}_\pm = \frac{1}{2}[1 \mp i\gamma^1\gamma^2\mathrm{sgn}(eB)]$ being the spin projection operators, and $L_n^{(\alpha)}(z)$ are the generalized Laguerre polynomials. By definition, $L_{-1}^{(\alpha)} \equiv 0$. In the scalar product, the Dirac matrices appear with a lower index γ_i. In what follows, we also need the retarded and advanced Green' s functions that are obtained by analytic continuation from positive and negative discrete frequencies, respectively, $\widetilde{G}^R(\mathbf{k}, \omega + i0) = \widetilde{G}(\mathbf{k}, i\omega_m \to \omega + i0)$ and $\widetilde{G}^A(\mathbf{k}, \omega - i0) = \widetilde{G}(\mathbf{k}, i\omega_m \to \omega - i0)$. When the impurity scattering induced self-energy is included, they acquire the form

$$\widetilde{G}^{R/A}(\mathbf{k}, \omega) = ie^{-k_\perp^2\ell_B^2}\sum_{n=0}^{\infty}\frac{(-1)^n D_n[\mathbf{k}, \omega+\varepsilon_F \pm i\hbar/2\tau]}{[\omega+\varepsilon_F \pm i\hbar/2\tau] - \varepsilon_n^2(k^3)}, \quad (7)$$

where τ is the relaxation time which will be defined later. Up to the linear order in B, the translation-invariant part of the free-fermion propagator in the momentum representation [Eq. (7)] has the following structure:

$$\widetilde{G}^{R/A}(\mathbf{k}, \omega) = \widetilde{G}_0^{R/A}(\mathbf{k}, \omega) + \widetilde{G}_1^{R/A}(\mathbf{k}, \omega) + \cdots, \quad (8)$$

where $\widetilde{G}_0^{(R,A)}(\mathbf{k}, \omega)$ is the fermion propagator in the absence of magnetic field and $\widetilde{G}_1^{(R,A)}(\mathbf{k}, \omega)$ is the linear part in the magnetic field. It can be derived by means of a generalized Schwinger parametrization [50]

$$\widetilde{G}_0^{R/A}(\mathbf{k}, \omega) = i\frac{\left(\omega+\varepsilon_F \pm \frac{i\hbar}{2\tau}\right)\gamma^0 - v\hbar\mathbf{k}\cdot\boldsymbol{\gamma} + mv^2}{\left(\omega+\varepsilon_F \pm \frac{i\hbar}{2\tau}\right)^2 - (v^2\hbar^2|\mathbf{k}|^2 + m^2v^4)} \quad (9)$$

and

$$\begin{aligned} \widetilde{G}_1^{R/A}(\mathbf{k}, \omega) =& -\gamma^1\gamma^2 ev^2\hbar B \\ &\times \frac{\left(\omega+\varepsilon_F \pm \frac{i\hbar}{2\tau}\right)\gamma^0 - v\hbar k^3\gamma^3 + mv^2}{\left[\left(\omega+\varepsilon_F \pm \frac{i\hbar}{2\tau}\right)^2 - (v^2\hbar^2|\mathbf{k}|^2 + m^2v^4)\right]^2}. \end{aligned} \quad (10)$$

C. Disorder effects

Various types of disorder are described by an appropriate choice of the 4 × 4 gamma matrices (last column in Table I) due to the Dirac structure of the Hamiltonian. All the disorder potential $V_{\mathtt{F}}(\mathbf{x})$ are quenched, the random variable behaving as white noises,

$$\langle V_{\mathtt{A}}(\mathbf{x})V_{\mathtt{B}}(\mathbf{x}')\rangle_{\mathrm{dis}} = \delta_{\mathtt{AB}}\Delta_{\mathtt{A}}\delta^3(\mathbf{x}-\mathbf{x}'), \quad (11)$$

where $\langle\ldots\rangle_{\mathrm{dis}}$ denotes disorder averaging. After configuration average, the impurity vertex acquires tensor form $\sum_{\mathtt{F}}\Delta_{\mathtt{F}}\gamma^{\mathtt{F}} \otimes \gamma^{\mathtt{F}}$. Our formalism allows us to give a generic description for various type of disorders with symmetry consideration. For example, the constraint of time-reversal symmetry allows only six bilinears: $\{\bar{\Psi}\gamma^0\Psi, \bar{\Psi}\Psi, \bar{\Psi}\gamma^0\gamma^5\Psi, i\bar{\Psi}\gamma^0\gamma^j\Psi\ (j = 1, 2, 3)\}$ which can be used to describe the nonmagnetic static disorder in Dirac materials. For simplicity, we only concentrate on impurities with time reversal and parity invariance

$$\mathscr{H}_{\mathrm{dis}} = \int d^3\mathbf{x}[V(\mathbf{x})\bar{\Psi}(\mathbf{x})\gamma^0\Psi(\mathbf{x}) + V_m(\mathbf{x})\bar{\Psi}(\mathbf{x})\Psi(\mathbf{x})], \quad (12)$$

which corresponds to the random chemical potential and mass, respectively. After introducing the impurities in the system, we can calculate the self-energy in the Born approximation. The real part of the self-energy gives a shift of the zero of energies and will not be considered either. The imaginary part of the self-energy can be evaluated as

$$\begin{aligned} \mathrm{Im}\,\Sigma^R =& -\frac{\pi\rho}{2}(\Delta+\Delta_m)(\gamma^0+\eta\mathbf{1}) \\ &\times\left[1+\frac{2}{k_F\ell_B}\sum_{p=1}\frac{e^{-p\lambda_D}}{\sqrt{p}}\cos\left(\pi p k_F^2\ell_B^2 - \frac{\pi}{4}\right)\right], \end{aligned} \quad (13)$$

where $\rho = k_F^2/(2\pi^2\hbar v_F)$ is the density of states at zero magnetic field with Fermi wave vector $k_F = \sqrt{\varepsilon_F^2 - m^2v^4}/\hbar v$ and Fermi velocity $v_F = (\hbar v^2 k_F)/\varepsilon_F$. The orbital polarization is defined by

$$\eta \equiv \langle\gamma^0\rangle = \frac{mv^2}{\varepsilon_F}, \quad (14)$$

where $\langle\ldots\rangle = \frac{1}{2}\sum_s\langle s\mathbf{k}|\ldots|s\mathbf{k}\rangle$ denotes the sum of the expectation values for the degenerate states in the conduction band. The Dingle factor is $\lambda_D = (\Delta+\Delta_m)(1+\eta^2)\frac{\varepsilon_F^2 k_F}{v^4\hbar^4}\ell_B^2$. When in the weak magnetic field regime $k_F\ell_B \gg 1$, the quantum oscillation of the relaxation time can be neglected. In this regime, the self-energy is approximately independent of the magnetic field. Thus, the total relaxation time (or quasiparticle lifetime) is given by

$$\tau = \frac{\hbar}{\pi\rho(\Delta+\Delta_m)(1+\eta^2)} \quad (15)$$

and the random mass induced relaxation time is

$$\tau_m = \frac{\hbar}{\pi\rho\Delta_m(1+\eta^2)}. \quad (16)$$

Further assume the random chemical potential dominates the elastic scattering processes $\Delta \gg \Delta_m$. The impurities are considered to distribute randomly in the sample which can be modeled as a random potential. In the Dirac material, one also needs to introduce small quadratic momentum terms ($C\hbar^2k^2$) in the Hamiltonian to incorporate the nonlinearity of the band dispersion. Here, we use the mass-type disorder (Δ_m) to simulate the small pseudospin mixing caused by quadratic momentum terms which can be estimated as $\Delta_m \sim \Delta(\frac{C\hbar k_F}{2v_F})^2$. Also, the anomaly-induced contribution dominates the magnetoconductivity only when $\Delta \gg \Delta_m$ [51].

III. QUANTUM DIFFUSIVE EQUATIONS IN THE REAL SPACE

With the help of Table I, we can introduce the 16-dimensional vectors for all the possible physical observables in terms of the Dirac matrices,

$$\hat{\mathcal{S}}_{\mathtt{A}}(x) = d_{\mathtt{A}}\bar{\Psi}(x)\gamma^{\mathtt{A}}\Psi(x), \tag{17}$$

where $d_{\mathtt{A}} = e$ for the density operators (the first four quantities in Table I) and $d_{\mathtt{A}} = ev$ for the current operators. To investigate the response to the external potential $\mathcal{A}_{\mathtt{A}}(x)$, we consider the generic external perturbation

$$\mathscr{H}_1(t) = \sum_{\mathtt{A}} \int d^3\mathbf{x}\, \hat{\mathcal{S}}_{\mathtt{A}}(x)\mathcal{A}_{\mathtt{A}}(x) \tag{18}$$

with $\mathcal{A}_{\mathtt{A}}$ are also 16-dimensional vectors [52] . The observables can be evaluated within the framework of the linear response theory [53]

$$\mathcal{S}_{\mathtt{A}} \approx \mathcal{S}^{(0)}_{\mathtt{A}} + \mathcal{S}^{(1)}_{\mathtt{A}} + \mathcal{O}\big(\mathcal{A}^2_{\mathtt{A}}\big). \tag{19}$$

$\mathcal{S}^{(0)} = -d_{\mathtt{A}}\mathrm{Tr}[\gamma^{\mathtt{A}}G(x,x)]$ is the zeroth-order term in $\mathscr{H}_1$ and $G(x,x')$ is the fermion propagator for $\mathscr{H}_0$:

$$\mathcal{S}^{(1)}_{\mathtt{A}}(x) = \int d^4x'\,\chi^R_{\mathtt{AB}}(x,x')\mathcal{A}_{\mathtt{B}}(x') \tag{20}$$

is the first-order response to $\mathscr{H}_1$, with $\chi^R_{\mathtt{AB}}(x,x')$ is the retarded response function which can be evaluated by analytical continuation $[\chi^R_{\mathtt{AB}}(\Omega) = \chi_{\mathtt{AB}}(i\Omega_m \to \Omega + i0)]$ of the imaginary-time expression

$$\chi_{\mathtt{AB}}(\mathbf{x},\mathbf{x}';i\Omega_m) = \int_0^\beta d\tau\, e^{i\Omega_m(\tau-\tau')}\langle T_\tau\hat{\mathcal{S}}_{\mathtt{A}}(\mathbf{x},\tau)\hat{\mathcal{S}}_{\mathtt{B}}(\mathbf{x}',\tau')\rangle, \tag{21}$$

where $\hat{\mathcal{S}}_{\mathtt{B}}$ is the observable coupled with the external field and $\hat{\mathcal{S}}_{\mathtt{A}}$ is the goal observable to calculate. Both $\hat{\mathcal{S}}_{\mathtt{A}}$ and $\hat{\mathcal{S}}_{\mathtt{B}}$ can be chosen as the one of 16 physical quantities in Table I. Substituting the generalized current operator (17) into Eq. (21), and using the Fourier series expansion of imaginary-time Green's function $G(\mathbf{x},\mathbf{x}',\tau-\tau') = \frac{1}{\beta}\sum_n e^{-i\omega_n(\tau-\tau')}G(\mathbf{x},\mathbf{x}';i\omega_n)$, where the imaginary-time Green's function $G(\mathbf{x},\mathbf{x}',\tau-\tau') = -\langle T_\tau\psi(\mathbf{x},\tau)\bar{\psi}(\mathbf{x}',\tau')\rangle$ defined in Eq. (4), we obtain

$$\chi_{\mathtt{AB}}(\mathbf{x},\mathbf{x}';i\Omega_m) = -\frac{d_{\mathtt{A}}d_{\mathtt{B}}}{\beta}\sum_n \mathrm{Tr}[\gamma^{\mathtt{A}}G(\mathbf{x},\mathbf{x}',i\omega_n)\gamma^{\mathtt{B}} \times G(\mathbf{x}',\mathbf{x},i\omega_n - i\Omega_m)], \tag{22}$$

where Tr includes also the summation over the flavor index. Note that in the presence of a magnetic field, the average Green's function is no longer translation invariant and the real-space formalism is more appropriate. However, when considering the combination of the type $G(\mathbf{x},\mathbf{x}')G(\mathbf{x}',\mathbf{x})$, the Schwinger phase exactly cancels out and translational invariance still holds for diffuson. In order to introduce the disorder effect, we express the translation-invariant part of the Green's function as a frequency integral over its respective spectral function

$$\widetilde{G}(\mathbf{k},i\omega_n) = \int_{-\infty}^{\infty} d\omega\frac{\mathscr{A}(\mathbf{k},\omega)}{i\omega_n - \omega}, \tag{23}$$

where the spectral function is given by the discontinuity relation $\mathscr{A}(\mathbf{k},\omega) = \frac{1}{2\pi i}[\widetilde{G}^A(\mathbf{k},\omega) - \widetilde{G}^R(\mathbf{k},\omega)]$.

The sum over Matsubara frequencies in Eq. (22) is easily evaluated when the fermion Green' s function is written, using the spectral representation (23),

$$\chi_{\mathtt{AB}}(\mathbf{x},\mathbf{x}';i\Omega_m) = d_{\mathtt{A}}d_{\mathtt{B}}\int_{-\infty}^{\infty}d\omega\int_{-\infty}^{\infty}d\omega'\int\frac{d^3\mathbf{k}\,d^3\mathbf{k}'}{(2\pi)^6}e^{i(\mathbf{k}-\mathbf{k}')\cdot(\mathbf{x}-\mathbf{x}')}\frac{n_F(\omega')-n_F(\omega)}{\omega-\omega'-i\Omega_m}\mathrm{Tr}[\gamma^{\mathtt{A}}\mathscr{A}(\mathbf{k},\omega)\gamma^{\mathtt{B}}\mathscr{A}(\mathbf{k}',\omega')]. \tag{24}$$

Here, n_F is the Fermi distribution function $n_F(\omega) = 1/[\exp(\beta\hbar\omega)+1]$. We then perform the analytical continuation $i\Omega_m \to \Omega + i0$ and integral over ω:

$$\begin{aligned}\chi_{\mathtt{AB}}(\mathbf{x},\mathbf{x}';\Omega+i0) = \frac{d_{\mathtt{A}}d_{\mathtt{B}}}{2\pi i}\bigg\{&\int_{-\infty}^{\infty}d\omega[n_F(\omega+\Omega)-n_F(\omega)]\mathrm{Tr}[\gamma^{\mathtt{A}}G^R(\mathbf{x},\mathbf{x}';\omega+\Omega)\gamma^{\mathtt{B}}G^A(\mathbf{x}',\mathbf{x};\omega)]\\ &+\int_{-\infty}^{\infty}d\omega\, n_F(\omega)\mathrm{Tr}[\gamma^{\mathtt{A}}G^R(\mathbf{x},\mathbf{x}';\omega+\Omega)\gamma^{\mathtt{B}}G^R(\mathbf{x}',\mathbf{x};\omega)]\\ &-\int_{-\infty}^{\infty}n_F(\omega+\Omega)\mathrm{Tr}[\gamma^{\mathtt{A}}G^A(\mathbf{x},\mathbf{x}';\omega+\Omega)\gamma^{\mathtt{B}}G^A(\mathbf{x}',\mathbf{x};\omega)]\bigg\}.\end{aligned} \tag{25}$$

We are only interested in the response function in a low-frequency limit $\Omega \to 0$ that $n_F(\omega) - n_F(\omega+\Omega) \approx -\Omega\partial_\omega n_F(\omega)$ at zero temperature $\partial_\omega n_F(\omega) = -\delta(\omega-\varepsilon)$. The Ward identity expresses the conservation of the symmetry currents which follows from the global symmetry of the system. It puts direct constraint between the vertex function and the self-energy function arising from impurities' scattering [52]. In order to satisfy the Ward identity, the vertex

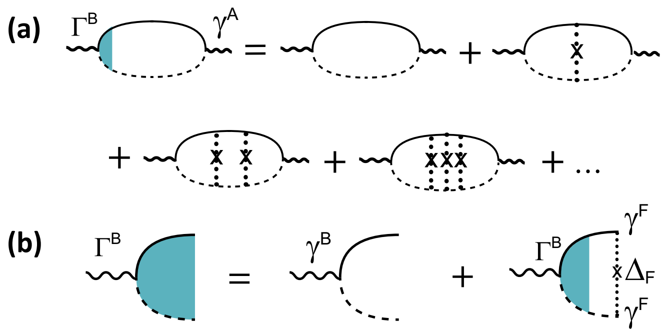

FIG. 2. (a) Diagrammatic representation of the disorder-averaged correlation function. (b) The iterative equation for the vertex correlation of the generalized current operator. The solid (dashed) lines represent the retarded (advanced) Green's function, the dotted lines denote the disorder correlations.

renormalization due to the multiscattering needs to be considered which appears perturbatively as a series of impurity line ladder diagrams, in which only combinations of retarded and advanced Green's functions, having poles on opposite sides of the real axis, will contribute. To evaluate the vertex corrections, we can first neglect the *RR* and *AA* contributions

$$\chi_{\mathrm{AB}}(\mathbf{x},\mathbf{x}';\Omega+i0) = -\frac{d_{\mathrm{A}}d_{\mathrm{B}}}{2\pi i}\Omega\,\mathrm{Tr}[\gamma^{\mathrm{A}}G^{R}(\mathbf{x},\mathbf{x}';\Omega)\gamma^{\mathrm{B}}G^{A}(\mathbf{x}',\mathbf{x};0)]. \quad (26)$$

Taking the impurity vertex corrections into consideration, the calculation of the response function (21) reduces to the evaluation of a series of ladderlike diagrams, depicted as in Fig. 2. Analytically, these diagrams are given by the expression

$$\chi_{\mathrm{AB}}(\mathbf{x},\mathbf{x}';\Omega+i0) = \frac{id_{\mathrm{A}}d_{\mathrm{B}}}{2\pi}\Omega\int d\mathbf{x}_1\mathrm{Tr}[\gamma^{\mathrm{A}}G^{R}(\mathbf{x},\mathbf{x}_1;\Omega) \times \Gamma^{\mathrm{B}}(\mathbf{x}_1,\mathbf{x}';\Omega)G^{A}(\mathbf{x}_1,\mathbf{x};0)], \quad (27)$$

where the bare vertex γ^{B} has been replaced by the dressed vertex $\Gamma^{\mathrm{B}}(\mathbf{x}_1,\mathbf{x}';\Omega)$ which satisfies the following Bethe-Salpeter equation:

$$\Gamma^{\mathrm{B}}(\mathbf{x}_1,\mathbf{x}';\Omega) = \gamma^{\mathrm{B}}\delta(\mathbf{x}_1-\mathbf{x}') + \int d\mathbf{x}_2\sum_{\mathrm{F}}\Delta_{\mathrm{F}}\gamma^{\mathrm{F}}G^{R}(\mathbf{x}_1,\mathbf{x}_2;\epsilon+\Omega) \times \Gamma^{\mathrm{B}}(\mathbf{x}_2,\mathbf{x}';\omega)G^{A}(\mathbf{x}_2,\mathbf{x}_1;\Omega)\gamma^{\mathrm{F}}. \quad (28)$$

The dressed vertex $\Gamma^{\mathrm{B}}(\mathbf{x}_2,\mathbf{x}';\omega)$ is obtained from iteration of the elementary vertex γ^{B} with the further propagation of the electron in-between two scattering events characterized by its retarded and advanced propagator which are also 4×4 matrices of the form (4). By projecting Eq. (28) onto the elements of the Clifford algebra,

$$\widetilde{\Gamma}^{\mathrm{BC}}(\mathbf{x},\mathbf{x}';\Omega) = \tfrac{1}{4}\mathrm{Tr}[\Gamma^{\mathrm{B}}(\mathbf{x},\mathbf{x}';\Omega)\gamma^{\mathrm{C}}], \quad (29)$$

thus the notation $\widetilde{\Gamma}^{\mathrm{BC}}$ with a wide tilde denotes each projection component $\Gamma^{\mathrm{B}}(\mathbf{x},\mathbf{x}';\Omega) = \sum_{\mathrm{C}}\widetilde{\Gamma}^{\mathrm{BC}}(\mathbf{x},\mathbf{x}';\Omega)\gamma^{\mathrm{C}}$. We then recast the Bethe-Salpeter equation (28) into the form

$$\widetilde{\Gamma}^{\mathrm{BC}}(\mathbf{x},\mathbf{x}';\Omega) = \delta_{\mathrm{BC}}\delta(\mathbf{x}-\mathbf{x}') + \sum_{\mathrm{D}}\int d\mathbf{x}_1\widetilde{\Gamma}^{\mathrm{BD}}(\mathbf{x}_1,\mathbf{x}';\Omega)\frac{1}{4}\sum_{\mathrm{F}}\Delta_{\mathrm{F}} \times \mathrm{Tr}[\gamma^{\mathrm{F}}G^{R}(\mathbf{x},\mathbf{x}_1;\epsilon+\Omega)\gamma^{\mathrm{D}}G^{A}(\mathbf{x}_1,\mathbf{x};\epsilon)\gamma^{\mathrm{F}}\gamma^{\mathrm{C}}]. \quad (30)$$

For any two matrices γ^{F} and γ^{C}, one always has $\gamma^{\mathrm{F}}\gamma^{\mathrm{C}} = \kappa_{\mathrm{FC}}\gamma^{\mathrm{C}}\gamma^{\mathrm{F}}$ with $\kappa_{\mathrm{FC}} = \pm 1$ depending on whether they are commuting or anticommuting. With this definition, Eq. (30) is rewritten as

$$\widetilde{\Gamma}^{\mathrm{BC}}(\mathbf{x},\mathbf{x}';\Omega) = \delta_{\mathrm{BC}}\delta(\mathbf{x}-\mathbf{x}') + \sum_{\mathrm{D,F}}\int d\mathbf{x}_1\widetilde{\Gamma}^{\mathrm{BD}}(\mathbf{x}_1,\mathbf{x}';\Omega) \times \mathcal{M}^{\mathrm{DC}}(\mathbf{x},\mathbf{x}_1,\Omega)\Delta_{\mathrm{F}}\kappa_{\mathrm{FC}}, \quad (31)$$

with

$$\mathcal{M}^{\mathrm{DC}}(\mathbf{x},\mathbf{x}',\Omega) = \tfrac{1}{4}\mathrm{Tr}[\gamma^{\mathrm{C}}G^{R}(\mathbf{x},\mathbf{x}';\epsilon+\Omega)\gamma^{\mathrm{D}}G^{A}(\mathbf{x}',\mathbf{x};\epsilon)]. \quad (32)$$

Rewrite Eq. (32) in matrix form:

$$\widetilde{\Gamma}(\mathbf{x},\mathbf{x}';\Omega) = \delta(\mathbf{x}-\mathbf{x}') + \int d\mathbf{x}_1\widetilde{\Gamma}(\mathbf{x}_1,\mathbf{x}';\Omega)\mathcal{M}(\mathbf{x},\mathbf{x}_1,\Omega)\mathcal{W}, \quad (33)$$

where the impurity-related diagonal matrix $\mathcal{W}$ with its elements is defined as

$$\mathcal{W}_{\mathrm{CC}} = \sum_{\mathrm{F}}\kappa_{\mathrm{FC}}\Delta_{\mathrm{F}}. \quad (34)$$

In the diffusive regime, the spatial variations of $\widetilde{\Gamma}(\mathbf{x},\mathbf{x}';\Omega)$ are small on the scale of mean-free path $\ell_e = v_F\tau$ and the integral equation (33) simplifies. We expand $\widetilde{\Gamma}(\mathbf{x}_1,\mathbf{x}';\Omega)$ about $\mathbf{x}_1 = \mathbf{x}$:

$$\widetilde{\Gamma}(\mathbf{x}_1,\mathbf{x}';\Omega) \approx \mathcal{T}(\mathbf{x}_1,\mathbf{x})\widetilde{\Gamma}(\mathbf{x},\mathbf{x}';\Omega) \quad (35)$$

with

$$\mathcal{T}(\mathbf{x}_1,\mathbf{x}) = 1 + (\mathbf{x}_1-\mathbf{x})\cdot\nabla_{\mathbf{x}} + \tfrac{1}{6}(\mathbf{x}_1-\mathbf{x})^2\nabla_{\mathbf{x}}^2.$$

The diffusion equation for $\widetilde{\Gamma}(\mathbf{x},\mathbf{x}';\Omega)$ takes the form

$$\mathscr{D}_{\mathbf{x}}^{-1}\widetilde{\Gamma}^{T}(\mathbf{x},\mathbf{x}';\omega) = \mathcal{W}^{-1}\delta(\mathbf{x}-\mathbf{x}'), \quad (36)$$

with the inverse diffusion operator $\mathscr{D}_{\mathbf{x}}^{-1}$ defined as

$$\mathscr{D}_{\mathbf{x}}^{-1} = \mathcal{W}^{-1} - \int d\mathbf{x}_1\mathcal{M}^{T}(\mathbf{x},\mathbf{x}_1;0)\mathcal{T}(\mathbf{x}_1,\mathbf{x}) - \Omega\int d\mathbf{x}_1\partial_{\Omega}\mathcal{M}^{T}(\mathbf{x},\mathbf{x}_1;\Omega)|_{\Omega=0}, \quad (37)$$

where we have expanded $\mathcal{M}^{T}$ near $\Omega = 0$ under low-frequency approximation because, as mentioned above, we are only interested in the diffusive regime, which corresponds to low frequencies and long wavelengths. Substituting Eqs. (29) and (32) into Eq. (27), we derive the relation which connects the generalized response function with the renormalized vertex:

$$\chi(\mathbf{x}-\mathbf{x}',\Omega)=4d_{\mathtt{A}}d_{\mathtt{B}}\frac{i\Omega}{2\pi}\int d\mathbf{x}_1\widetilde{\Gamma}(\mathbf{x}_1,\mathbf{x}';\Omega)\mathcal{M}(\mathbf{x},\mathbf{x}_1,\Omega). \tag{38}$$

The coordinate $\mathbf{x}_1$ can be further integrated out by using Eq. (33), and the response functions can be expressed in terms of the dressed vertex functions

$$\chi(\mathbf{x}-\mathbf{x}',\Omega)=d_{\mathtt{A}}d_{\mathtt{B}}4\frac{i\Omega}{2\pi}[\widetilde{\Gamma}(\mathbf{x},\mathbf{x}';\Omega)-\delta(\mathbf{x}-\mathbf{x}')]\mathcal{W}^{-1}. \tag{39}$$

From the linear response theory, the first-order term in $\mathscr{H}_1$ is known to be

$$\mathcal{S}^{(1)}_{\mathtt{A}}(\mathbf{x},t)=\int_{-\infty}^{\infty}\frac{d\Omega}{2\pi}e^{-i\Omega t}\int d\mathbf{x}'\chi^R_{\mathtt{AB}}(\mathbf{x},\mathbf{x}';\Omega)A_{\mathtt{B}}(\mathbf{x}',\Omega). \tag{40}$$

By introducing the 16-dimensional vectors, the physical responses $\mathcal{S}$ and the generalized external field $\mathcal{A}$ with the elements as $\mathcal{S}^{(1)}_{\mathtt{A}}$ and $A_{\mathtt{A}}$, respectively, the above set of equations can be recast as the following matrix form:

$$\mathcal{S}(\mathbf{x},\Omega)=\int d\mathbf{x}'\chi(\mathbf{x},\mathbf{x}';\Omega)\mathcal{A}(\mathbf{x}',\Omega). \tag{41}$$

Then, applying the inverse diffusion operator to the both sides of Eq. (41) and using the relation (38), we arrive at a set of quantum diffusive equations

$$\mathscr{D}^{-1}_{\mathbf{x}}\mathcal{S}^{(1)}(\mathbf{x},\omega)=-\frac{2}{\pi}\left[\mathcal{W}^{-1}-\mathscr{D}^{-1}_{\mathbf{x}}\right]\mathcal{W}^{-1}i\omega\mathcal{A}(\mathbf{x},\omega), \tag{42}$$

which describes the coupled dynamics of all the physical quantities. These equations provide a unified description of transport phenomena and (pseudo)spin density relaxation at finite magnetic field for Dirac materials. Due to the coupling in $\mathscr{D}^{-1}$ between different physical quantities, we can use two different ways to induce the current. The first choice is to apply the electric field and investigate the current-current response. One can also induce the current indirectly by exciting a coupled observable through $\mathscr{D}^{-1}$. For instance, as shown below, the axial charge density is coupled to the current (directed along the magnetic field) in the presence of the magnetic field. Therefore, for massless Dirac materials, the axial charge injection which will generate the chemical potential difference between the right- and left-hand fermion and finally induces a net electric current, this is the chiral magnetic effect [54].

IV. LONGITUDINAL MAGNETORESISTANCE

A general solution of the quantum diffusive equations is quite complicated. As an application to explore the longitudinal magnetoresistance, we focus on the linear response for the electric field also along the z direction. The perturbation part of the Hamiltonian is $\mathscr{H}_1(t)=\int d^3\mathbf{x}A_3(t)\bar{\Psi}(x)\gamma^3\Psi(x)$. In this case, the diffusion operator $\mathscr{D}^{-1}$ can be reduced into a block-diagonal form: among all the 16 physical quantities, only 4 observables we are interested in are coupled together in the quantum diffusive regime. Thus, we extract the following coupled 4×4 submatrix of $\mathscr{D}^{-1}$ which is spanned by $(J^0,J^{a0},\mathfrak{n}_\beta,J^3)$:

$$\frac{\hbar}{\pi\rho\tau^2}\mathscr{D}^{-1}_{4\times4}=\begin{pmatrix} i\omega+\frac{\Lambda_0}{\tau}-\mathcal{D}\partial_z^2 & \Upsilon v\partial_z & \eta\left(i\omega-\frac{1}{\tau}-\mathcal{D}\partial_z^2\right) & \frac{1}{3}(1-\eta^2)v\partial_z \\ \Upsilon v\partial_z & (1-\eta^2)\left(i\omega+\frac{\Lambda_{a0}}{\tau}-\mathcal{D}\partial_z^2\right) & \frac{\eta}{2}\Upsilon v\partial_z & -\frac{1}{\tau}\Upsilon \\ \eta(i\omega-\frac{1}{\tau}-\mathcal{D}\partial_z^2) & \frac{\eta}{2}\Upsilon v\partial_z & \eta^2\left(i\omega+\frac{\Lambda_\beta}{\tau}-\mathcal{D}\partial_z^2\right) & \frac{1}{3}\eta(1-\eta^2)v\partial_z \\ \frac{1}{3}(1-\eta^2)v\partial_z & -\frac{1}{\tau}\Upsilon & \frac{1}{3}\eta(1-\eta^2)v\partial_z & \frac{1}{3}(1-\eta^2)\left(i\omega+\frac{\Lambda_3}{\tau}-\mathcal{D}\partial_z^2\right)\end{pmatrix}. \tag{43}$$

The coefficient $\Upsilon\equiv\frac{1}{2}(\frac{\hbar v}{\varepsilon_F\ell_B})^2$ with the magnetic length $\ell_B=\sqrt{\hbar/eB}$ and the dimensionless diffusion channel relaxation rates are

$$\Lambda_0=\eta^2, \tag{44}$$

$$\Lambda_\beta=\frac{1}{\eta^2}, \tag{45}$$

$$\Lambda_3=2\frac{(\Delta+2\Delta_m)+\eta^2(2\Delta+\Delta_m)}{(\Delta-\Delta_m)(1-\eta^2)}. \tag{46}$$

Most importantly, the axial relaxation rate is

$$\Lambda_{a0}=2\frac{\Delta_m+\Delta\eta^2}{(\Delta-\Delta_m)(1-\eta^2)}, \tag{47}$$

from the chiral symmetry breaking. By substituting (43) into (42) and using the explicit form of $\mathcal{W}=\Delta\mathbf{1}_4+\Delta_m\mathrm{diag}(1,-1,1,-1)$ for this 4×4 case, we arrive at the coupled charge-current dynamics equations in the presence of electromagnetic field. By transforming into frequency-momentum space, the coupled charge-current equations can be solved in the diffusive regime ($\omega\tau,\mathcal{D}\tau q_z^2\ll 1$):

(i) The chirality imbalance induced by the parallel electric and magnetic field is

$$\delta J^{a0}=\frac{\varepsilon_F}{v\hbar k_F}\frac{i\omega}{i\omega+\mathcal{D}^*q_z^2}\frac{e^3}{\hbar^2}\frac{EB\tau_a}{2\pi^2}. \tag{48}$$

(ii) The current along the direction of the magnetic field is

$$\delta J^3=\frac{i\omega\sigma_D E}{i\omega+\mathcal{D}^*q_z^2}\left(1+\frac{3}{4}\frac{\tau_a}{\tau^*}\frac{1}{k_F^4\ell_B^4}\frac{i\omega}{i\omega+q_z^2\mathcal{D}^*}\right). \tag{49}$$

(iii) The density fluctuations induced by external field are

$$\delta J^0 = \frac{iq_z\sigma_D E}{i\omega + \mathcal{D}^* q_z^2}\left(1 + \frac{3}{4}\frac{\tau_a}{\tau^*}\frac{1}{k_F^4\ell_B^4}\frac{i\omega}{q_z^2\mathcal{D}^* + i\omega}\right), \tag{50}$$

which can be used to determine the density-density response function.

(iv) The scalar mass which has nonzero polarization (η) in the absence of external electric field acquires a variation due to an applied electric field

$$\delta \mathfrak{n}_\beta = \eta \delta J^0. \tag{51}$$

We have introduced the renormalized diffusion coefficient $\mathcal{D}^* = \frac{3}{2}\frac{(1+\eta^2)(\Delta+\Delta_m)}{(\Delta+2\Delta_m)+\eta^2(2\Delta+\Delta_m)}\mathcal{D}$ with the classical diffusion constant $\mathcal{D} = v_F^2\tau/3$, and the Drude conductivity as

$$\sigma_D = 2e^2\rho\mathcal{D}^*. \tag{52}$$

Most importantly, the axial relaxation time which describes the attenuation time of the axial charge in the disordered medium is defined as

$$\tau_a = \frac{1}{\Lambda_{a0}}\frac{\mathcal{D}^*}{\mathcal{D}}\tau^*, \tag{53}$$

with $\tau^* = \hbar/[\pi\rho(\Delta + \Delta_m)]$. The ratio of two relaxation times depends only on two parameters η and Δ_m/Δ:

$$\frac{\tau_a}{\tau^*} = \frac{3}{4}\frac{(1-\eta^4)\left(1 - \frac{\Delta_m^2}{\Delta^2}\right)}{\left(\eta^2 + \frac{\Delta_m}{\Delta}\right)\left[1 + 2\eta^2 + (2+\eta^2)\frac{\Delta_m}{\Delta}\right]}. \tag{54}$$

For symmetry with chiral symmetry nearly preserved ($\eta, \Delta_m/\Delta \sim 0$), we have $\tau_a/\tau^* \approx \frac{3}{4}(\eta^2 + \Delta_m/\Delta)^{-1}$. The solution in Eq. (49) gives the dynamic longitudinal conductivity $\sigma_{zz}(\omega, \mathbf{q}, B) = \delta J^3(\omega, \mathbf{q}, B)/E(\omega, \mathbf{q})$ in a finite magnetic field. In the "slow limit," $\lim_{q\to 0}\lim_{\omega\to 0}(\omega/q_z^2\mathcal{D}^*) \to 0$ [55], such that the perturbing potential is nearly constant on the timescale $1/q_z^2\mathcal{D}^*$. Consequently, in the thermodynamic equilibrium no current will be generated: $\lim_{q\to 0}\lim_{\omega\to 0}\sigma_{zz}(\omega, \mathbf{q}, B) = 0$, which is also a requirement of the gauge invariance that a purely longitudinal and static vector potential cannot induce any physical current. In the "rapid limit," $\lim_{\omega\to 0}\lim_{q\to 0}(q_z^2\mathcal{D}^*/\omega) \to 0$ [55]. In this case, we obtain the remarkable result $\lim_{\omega\to 0}\lim_{q\to 0}\sigma_{zz}(\omega, \mathbf{q}, B) = \sigma_D + \sigma_{CA}(\eta, B)$, with the anomaly-induced magnetoconductivity as

$$\sigma_{CA}(\eta, B)/\sigma_D = \frac{3}{4}\frac{\tau_a}{\tau^*}\frac{1}{k_F^4\ell_B^4} = \frac{3}{16}\frac{\tau_a}{\tau^*}\left(\frac{B}{B_F}\right)^2. \tag{55}$$

The magnetoconductivity is positive and quadratic in B and $B_F = \frac{\hbar}{2e}k_F^2$. In the massless case of $\eta = 0$, only the impurities which break the chiral symmetry (the impurity matrix anticommutes with the chiral symmetry operator) can cause the scattering between different chiralities (nodes) control the axial relaxation time, and we may reproduce the previous results from the semiclassical theory $\lim_{\eta\to 0}\sigma_{CA}(\eta, B) = \frac{3}{4\pi^2 2\rho}\left(\frac{eB}{\hbar}\right)^2\frac{\tau_a}{\tau}\frac{\hbar v}{\varepsilon_F^2}\sigma_D$ [6]. In a massive case of $\eta \neq 0$, the chiral symmetry is broken explicitly due to the Dirac mass m, and the eigenstates near the Fermi level mix the chiralities. The disorder with the chiral symmetry (e.g., the chemical potential randomness) can cause the backscattering between opposite helicity, giving an axial relaxation time proportional to the inverse of η^2. As shown in Eq. (55), when the carrier density of the system is fixed, the relative magnetoconductivity $\sigma_{CA}(\eta, B)/\sigma_D$ is only determined by two parameters, the relative impurity strength Δ_m/Δ and the orbital polarization η. As shown in Fig. 3(a), when the relative impurity strength Δ_m/Δ is fixed, the relative anomaly-induced magnetoconductivity $\sigma_{CA}(\eta)/\sigma_{CA}(\eta = 0)$ is suppressed as the parameter η grows. When Δ_m/Δ goes to zero, the mass becomes dominant in chiral symmetry breaking, and the anomaly-induced magnetoconductivity quenches to zero more quickly as η grows. We also plot the absolute value of σ_{CA} by using the realistic parameters according to Ref. [30]. As shown in Fig. 3(b), σ_{CA} is strongly suppressed as Δ_m/Δ and η grows. Experimental observation of the anomaly-induced magnetoconductivity requires a long axial current relaxation time τ_a, which stems from the near conservation of chiral charge, and a lower carrier density. A finite mass cannot forbid such an effect, but only suppress its contribution. From Eq. (48), in the "rapid limit," the chirality imbalance $\delta J^{a0} = \frac{e^3}{\hbar^2}\frac{\varepsilon_F}{v\hbar k_F}\frac{EB\tau_a}{2\pi^2}$ is also self-consistently obtained, further confirming our calculations. The chiral anomaly-induced magnetoconductivity is rooted in the current vertex renormalization from the axial charge density in the presence of parallel electromagnetic field. Therefore, we could not obtain such an anomaly correction only in the Drude approximation by considering the bubble diagram.

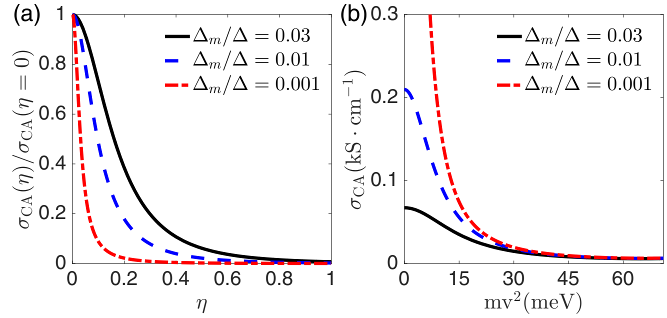

FIG. 3. (a) The universal behavior of the relative anomaly magnetoconductivity correction for different Δ_m/Δ as a function of the parameter $\eta = mv^2/\varepsilon_F$. (b) The anomaly-related positive magnetoconductivities at $B = 1$ T are plotted as a function of Dirac mass mv^2 for several different $\Delta_m/\Delta$with the chemical potential $\varepsilon_F = 90$ meV and Ohm resistance at zero field $\mathcal{R} = \sigma^{-1} = 1.2$ mΩ cm. The other parameter is chosen as $\hbar v = 6 \times 10^{-5}$ meV cm.

V. ANOMALY-INDUCED MAGNETIC FIELD CORRECTION TO POLARIZATION FUNCTION

The conservation of total charge also makes an anomaly-induced correction to the dynamical polarization function or the density-density response function $\chi_{00}(\omega, \mathbf{q}, B)$. The gauge invariance poses some constraints on the elements of the response function: $\chi_{00} = -\frac{q_z}{\omega}\chi_{03}$ with $\chi_{03} = \frac{\delta J^0}{A_3}$, we yield the following compact form for the polarization function from Eq. (50):

$$\chi_{00}(\omega, \mathbf{q}, B) = 2\rho\frac{q_z^2\widetilde{D}(B)}{q_z^2\widetilde{D}(B) + i\omega}, \tag{56}$$

where $\widetilde{D}(B)=\mathcal{D}^*(1+\frac{3}{4}\frac{\tau_a}{\tau^*}\frac{1}{k_F^4\ell_B^4})$ is the field-dependent diffusion constant. This factor $(1/[q_z^2\widetilde{D}(B)+i\omega])$ is known as the "diffusion pole," which emerges from the repeated elastic scattering (the ladder diagram), and also reflects the conservation of total charge. Some many-body effects are directly associated with this diffusion pole. For example, when the electron-electron interaction cannot be neglected, each electron will be influenced by the electronic density fluctuation from other electrons described by χ_{00}. As a consequence, the spectral and transport properties are modified by the interaction effect. One way to detect the effect is to measure the tunnel conductance, which directly reflects the variation of the density of states $\delta\rho$ due to the Coulomb interaction. The reduction of the tunnel conductance is given by $\delta G_t(V)/G_t=\delta\rho(V)/\rho\propto(\sqrt{|eV|/\widetilde{D}(B)}-C)/\widetilde{D}(B)$ where V is the voltage difference between two leads and C is a constant independent of the bias [56]. Since the change is maximal around the Fermi energy ε_F, the tunneling spectrum will display a downward cusp at the Fermi level, i.e., the so-called zero-bias anomaly. Due to the magnetic field dependence of the diffusion constant, we can expect the zero bias downward cusp should be weakened under the magnetic field. Furthermore, this interaction correction in conductivity shows a strong dependence on the configuration of the electric and magnetic field in sharp contrast with the contribution from weak localization, providing a fruitful way to distinguish the two effects.

VI. CALCULATION OF THE PSEUDOSCALAR DENSITY $\langle\mathfrak{n}_P\rangle$ AND DERIVATION OF CONTINUITY EQUATION

Within the framework of semiclassical Boltzmann theory, the chiral anomaly-induced negative magnetoresistance for a massless semimetal can be derived through the Hamiltonian dynamic equations which are modified by the (Abelian) Berry curvature. This is a nontrivial extension even in the case of small mass. The Berry curvature becomes non-Abelian in the case of massive Dirac fermions, and the semiclassical theory is usually limited to the system with the Abelian Berry curvature. Also, the conservation law of the axial charge cannot be applied to the semiclassical Boltzmann theory explicitly if we have no response result of the pseudoscalar density to the external fields.

In the linear response theory, the pseudoscalar density $\mathfrak{n}_P$ is associated with the electric field

$$\mathfrak{n}_P(\mathbf{q},\Omega)=2imv^2\chi_{\mathfrak{n}_P,J_3}(\mathbf{q},\Omega)A_3(\Omega). \tag{57}$$

From the previous calculations, we find that the retarded-advanced part of $\chi_{\mathfrak{n}_P,J_3}$ vanishes. From Eq. (25), we only need to evaluate the RR and AA parts of contributions,

$$\chi_{\mathfrak{n}_P,J_3}(\mathbf{q},\Omega)=\Pi^{RR}_{\mathfrak{n}_P,J_3}(\mathbf{q};\Omega)-\Pi^{AA}_{\mathfrak{n}_P,J_3}(\mathbf{q};\Omega), \tag{58}$$

where we define the RR part of response function as

$$\Pi^{RR}_{\mathtt{AB}}(\mathbf{x},\mathbf{x}';\Omega)=\frac{e^2v}{2\pi i}\int_{-\infty}^{\infty}d\omega\, n_F(\omega)\times\mathrm{Tr}[G^R(\mathbf{x},\mathbf{x}';\omega+\Omega)\gamma^{\mathtt{A}}G^R(\mathbf{x}',\mathbf{x};\omega)\gamma^{\mathtt{B}}] \tag{59}$$

and transform into momentum space,

$$\Pi^{RR}_{\mathtt{AB}}(\mathbf{q},\Omega)=\frac{e^2v}{2\pi i}\int_{-\infty}^{\infty}d\omega\, n_F(\omega)\int\frac{d^3\mathbf{k}}{(2\pi)^3}\times\mathrm{Tr}[\widetilde{G}^R(\mathbf{k}+\mathbf{q},\omega+\Omega)\gamma^{\mathtt{A}}\widetilde{G}^R(\mathbf{k};\omega)\gamma^{\mathtt{B}}]. \tag{60}$$

$\Pi^{AA}_{\mathtt{AB}}$ can be obtained by replacing the retarded Green's functions by the advanced ones. We are only interested in small $|\mathbf{q}|$ and Ω limits, and also the electric field with no spatially variation, then we first set $\mathbf{q}=0$. Here, we directly use the magnetic field expansion of $\widetilde{G}$ in Eq. (8), and truncate to the linear order in B:

$$\begin{aligned}\Pi^{RR}_{\mathfrak{n}_P,J_3}(0,\Omega)\approx{}&\frac{e^2v}{2\pi i}\int_{-\infty}^{\infty}d\omega\, n_F(\omega)\int\frac{d^3\mathbf{k}}{(2\pi)^3}\\&\times\big\{\mathrm{Tr}\big[\gamma^5\widetilde{G}^R_0(\mathbf{k},\omega+\Omega)\gamma^3\widetilde{G}^R_0(\mathbf{k};\omega)\big]\\&+\mathrm{Tr}\big[\gamma^5\widetilde{G}^R_1(\mathbf{k},\omega+\Omega)\gamma^3\widetilde{G}^R_0(\mathbf{k};\omega)\big]\\&+\mathrm{Tr}\big[\gamma^5\widetilde{G}^R_0(\mathbf{k},\omega+\Omega)\gamma^3\widetilde{G}^R_1(\mathbf{k};\omega)\big]\big\}.\end{aligned} \tag{61}$$

The trace of γ^5 timing a product of two or three γ^μ matrices gives zero, thus, the first term in the absence of magnetic field vanishes, or in other words the pseudoscalar density has no response only in the presence of the electric field. By substituting the explicit expression for the Green's function which is linear in magnetic field [Eq. (10)] into Eq. (61), we find the expressions for $\Pi^{RR}_{\mathfrak{n}_P,J_3}(0,\Omega)$ and $\Pi^{AA}_{\mathfrak{n}_P,J_3}(0,\Omega)$. Since the integrand is already first order in Ω we can neglect the Ω dependence of n_F in Π^{AA}. Using the trace identity for the Dirac gamma matrices $\mathrm{Tr}[\gamma^5\gamma^1\gamma^2\gamma^0\gamma^3]=-4i$, and performing the integration over frequency and momentum, we finally obtain

$$\chi_{\mathfrak{n}_P,J_3}(0,\Omega)=-\frac{e^3}{4\pi^2\hbar^2}\frac{\Omega B}{mv^2}\left(\frac{\varepsilon_F}{v\hbar k_F}-1\right). \tag{62}$$

By substituting Eq. (62) into (57), we finally obtain the remark result for anomaly-induced pseudoscalar condensate [57,58] in the presence of electric and magnetic field:

$$\mathfrak{n}_P(\mathbf{0},\Omega)=\frac{e^3BE}{2\pi^2\hbar^2}\left(\frac{\varepsilon_F}{v\hbar k_F}-1\right). \tag{63}$$

It is emphasized that Eq. (2) is exact as an operator relation, thus, in order to examine the properties of measurable physical quantities we need to calculate the expectation value of the corresponding operator. The pseudoscalar density $\hat{\mathfrak{n}}_P$ modifies directly the continuity equation for the axial charge and current density in Eq. (2). In the presence of the electric and magnetic field, it is found that the expectation value of the pseudoscalar density has the form $\langle\hat{\mathfrak{n}}_P\rangle=\frac{1}{2mv^2}(\frac{\varepsilon_F}{\hbar vk_F}-1)\frac{e^3}{2\pi^2\hbar^2}\mathbf{E}\cdot\mathbf{B}$, which vanishes when $m=0$ [57,58]. Thus, the anomaly equation is reduced to

$$\partial_\mu J^{a\mu}=\frac{\varepsilon_F}{\hbar vk_F}\frac{e^3}{2\pi^2\hbar^2}\mathbf{E}\cdot\mathbf{B}. \tag{64}$$

As the chemical potential $\varepsilon_F=\sqrt{(v\hbar k_F)^2+m^2v^4}$, the prefactor $\frac{\varepsilon_F}{\hbar vk_F}=\sqrt{1+(\frac{mv}{\hbar k_F})^2}$ is always larger than 1 for a finite mass. Assume the electric field is caused by a spatially varying chemical potential ε_F, $\mathbf{E} = -\nabla\frac{\varepsilon_F}{e}$. By integrating the above anomaly equation for massive Dirac fermion and further assuming a spatially varying chemical potential μ is modeled by a electric potential $\phi = \mu/e$, we directly obtain the dissipationless axial current $\mathbf{J}^a = -e\mathbf{B}v\hbar k_F\frac{e^2}{2\pi^2\hbar^2}$ for massive Dirac fermion which coincides the previous results for the massless Dirac fermions [59], the so-called chiral separation effect. As the name suggests, the axial current will induce a chiral charge separation (i.e., a nonzero chiral chemical potential μ_5), thus triggers the chiral magnetic effect and the longitudinal NMR in the presence of parallel electromagnetic field even with finite Dirac mass. After introducing the relaxation mechanism, a finite chirality imbalance can be generated which plays a major role in accessing the chiral anomaly. However, in many studies the chirality imbalance is introduced by hand. Thus, in order to get a full understanding of the anomaly-related magnetotransport, one should perform a first-principles and self-consistent derivation including the dynamics and impurity effects.

VII. ANOMALOUS COUPLING BETWEEN THE AXIAL CHARGE AND VECTOR CURRENT

As shown in the previous section, the coupling between the axial charge and the vector current along the magnetic field, which originates from the chiral anomaly, leads to the anomalous negative magnetoresistance. We will show that the coupling only comes from the lowest Landau level (LLL). Let us first evaluate the vector-axial current coupling components in Landau-level representation:

$$\mathcal{Q}^{a\mu,\nu} \equiv \int d\mathbf{x}'\mathcal{M}^{a\mu,\nu}(\mathbf{x},\mathbf{x}';0) = \int \frac{d^3\mathbf{k}}{(2\pi)^3}\frac{1}{4}\mathrm{Tr}[\gamma^5\gamma^\mu\widetilde{G}^R(\mathbf{k},\varepsilon_F)\gamma^\nu\widetilde{G}^A(\mathbf{k},\varepsilon_F)]. \tag{65}$$

By substituting the translation-invariant part of the fermion propagator (7), we arrive at

$$\mathcal{Q}^{a\mu,\nu} = \int\frac{dk^3}{2\pi}\sum_{n,n'=0}^{\infty}(-1)^{n+n'}\int\frac{d^2\mathbf{k}_\perp}{(2\pi)^2}e^{-k_\perp^2\ell_B^2}\frac{1}{4}\frac{\mathrm{Tr}[\gamma^5\gamma^\mu D_{n'}(\varepsilon_F+i\hbar/2\tau,\mathbf{k})\gamma^\nu D_n(\varepsilon_F-i\hbar/2\tau,\mathbf{k})]}{[(\varepsilon_F+i\hbar/2\tau)^2-\varepsilon_{n'}^2(k^3)][(\varepsilon_F-i\hbar/2\tau)^2-\varepsilon_n^2(k^3)]}. \tag{66}$$

We first perform the integration over $\mathbf{k}_\perp$ by using the orthogonality relation for the generalized Laguerre polynomials (A5). The odd terms in k^3 vanish after integration over k^3 have been dropped, and also the trace of terms involving $\mathbf{k}_\perp\cdot\boldsymbol{\gamma}$ equals zero. We only need to evaluate the following two traces:

$$\tfrac{1}{4}\mathrm{Tr}[\gamma^5\gamma^\mu(\varepsilon_F\gamma^0+mv^2)\mathcal{P}_s\gamma^\nu(\varepsilon_F\gamma^0+mv^2)\mathcal{P}_{s'}] = (s+s')\{\eta^{\mu0}\eta^{3\nu}(-\varepsilon_F^2+m^2v^4)+\eta^{\mu3}\eta^{\nu0}(-\varepsilon_F^2-m^2v^4)\} \tag{67}$$

and

$$\tfrac{1}{4}\mathrm{Tr}[\gamma^5\gamma^\mu\gamma^3\mathcal{P}_s\gamma^\nu\gamma^3\mathcal{P}_{s'}] = -(s+s')(\eta^{\mu0}\eta^{\nu3}+\eta^{\mu3}\eta^{\nu0}) \tag{68}$$

with $\eta^{\mu\nu}$ the Minkowski metric with signature $(+,-,-,-)$. The spin projection operator $\mathcal{P}_s$ projects out the spin component of the state on the direction of the magnetic field. As seen from Eqs. (67) and (68), the opposite spin part has a vanishing trace and only the same spin part contributes to the off-diagonal coupling. Furthermore, LLs with $n \geqslant 1$ are degenerate with the opposite spin state, leading to an exact cancellation of the contributions from $s = \pm$. The LL of $n = 0$, on the other hand, has no spin degeneracy. Thus, even in a small field that the Fermi energy crosses many LLs, only the LLL contributes to the Eq. (65). Then, we obtain

$$\mathcal{Q}^{a\mu,\nu} = \eta^{\mu0}\eta^{3\nu}\frac{eB}{4\pi\hbar}\frac{k_F\tau}{\hbar\varepsilon_F}+\eta^{\mu3}\eta^{\nu0}\frac{eB}{4\pi\hbar}\frac{\varepsilon_F\tau}{v^2\hbar^3k_F}. \tag{69}$$

It is also instructive to investigate this coupling term from the weak magnetic field expansion at the first beginning. By making use of the explicit form of $G_1^{R/A}$ in Eq. (10) and truncating to the linear order in B, we straightforwardly derive the contribution of the coupling between the axial and vector currents which coincides with the results obtained by using the expansion over the Landau levels, where it comes only from the LLL states. However, in the language of weak magnetic field without the Landau quantization, the anomaly-related anomalous coupling comes from the Fermi surface. The above calculations imply that a semiclassical approach for the anomaly-related topological contribution can be viewed as a dual description as the fully quantum theory.

VIII. SUMMARY

In this section, we summarize the main results in this work.

Quantum diffusive equations. Even a small magnetic field is not a weak perturbation, and many subtleties may be missed out in calculations by using the perturbative basis of the plane-wave function. In a magnetic field, electrons perform the cyclotron motion and exhibit the Landau quantization. The quantized Landau levels provide an accurate description of the wave-function properties, and serve as a more proper basis for perturbative calculations. In this work, we develop a fully microscopic theory of the quantum diffusive magnetotransport for massive Dirac materials by means of the Green's function in the Landau-level representation in a fully quantum mechanical manner. By using the diagrammatic perturbation theory, we derive the coupled diffusive equation for all the 16 relevant physical quantities of 4×4 massive Dirac equation

$$\mathscr{D}_{\mathbf{x}}^{-1}\mathcal{S}^{(1)}(\mathbf{x},\omega) = -\frac{2}{\pi}[\mathcal{W}^{-1}-\mathscr{D}_{\mathbf{x}}^{-1}]\mathcal{W}^{-1}i\omega\mathcal{A}(\mathbf{x},\omega), \tag{70}$$

which is one of our main findings.

Chiral anomaly-induced positive longitudal magnetoconductivity. The coupled dynamic equations are solved analytically in small magnetic field and the diffusive regime. We obtain the quadratic positive magnetoconductivity contribution from chiral anomaly

$$\frac{\sigma_{CA}(\eta,B)}{\sigma_D}=\frac{3}{16}\frac{\tau_a}{\tau^*}\left(\frac{B}{B_F}\right)^2, \tag{71}$$

which recovers the previous result in the massless limit. Thus, we may conclude that the anomaly-related magnetoconductivity will dominate the magnetotransport in a wide class of Dirac materials with small η, e.g., the topological insulators, not a transport characteristic unique to semimetals. As required by the Ward identities, the series of ladder diagrams need to be considered in the presence of impurity scatterings to implement the charge conservation laws, although it appears as a higher-order perturbation. Our calculations also demonstrate that the anomaly correction is rooted in the current vertex renormalization from the axial charge density in the presence of parallel electromagnetic field. The vertex corrections are important and can never be dismissed in studying the transport properties of Dirac materials.

Anomaly-induced magnetic field correction to polarization function. Another important finding is that the polarization function acquires a magnetic field correction due to chiral anomaly,

$$\chi_{00}(\omega,\mathbf{q},B)=2\rho\frac{q_z^2\widetilde{D}(B)}{q_z^2\widetilde{D}(B)+i\omega}, \tag{72}$$

which is closely related to many-body effects. We hope this new effect originated from chiral anomaly can be detected experimentally in the future.

Axial continuity equation for massive Dirac fermion. Last but not least, the expectation value of the pseudoscalar density is calculated within the linear response theory

$$\mathfrak{n}_P(\mathbf{0},\Omega)=\frac{e^3BE}{2\pi^2\hbar^2}\left(\frac{\varepsilon_F}{v\hbar k_F}-1\right). \tag{73}$$

With the help of this result, we reach at the axial charge continuity equation for massive Dirac fermion

$$\partial_\mu J^{a\mu}=\frac{\varepsilon_F}{\hbar v k_F}\frac{e^3}{2\pi^2\hbar^2}\mathbf{E}\cdot\mathbf{B}, \tag{74}$$

which can be directly applied to the semiclassical Boltzmann theory and thereby the magnetotransport properties can be calculated.

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APPENDIX: EVALUATION OF THE DIFFUSION OPERATOR

In order to evaluate the explicit form for $\mathscr{D}_{\mathbf{x}}^{-1}$ in the diffusive regime, it is convenient to use the diffusion kernel for Dirac fermions

$$P_0(\mathbf{x},\mathbf{x}';\Omega)=G^R(\mathbf{x},\mathbf{x}';\epsilon+\Omega)\otimes G^{AT}(\mathbf{x}',\mathbf{x};\epsilon). \tag{A1}$$

The superscript "T" means taking the transpose of the advanced Green's function. Then, we can directly get all the matrix element of $\mathcal{M}$ through the index contraction

$$\mathcal{M}^{\mathsf{AB}}(\mathbf{x},\mathbf{x}',\Omega)=\sum_{\mu,\nu,\lambda,\xi}\frac{1}{4}(\gamma^{\mathsf{B}})_{\lambda\xi}[P_0(\mathbf{x},\mathbf{x}';\omega)]_{4(\xi-1)+\lambda,4(\mu-1)+\nu}(\gamma^{\mathsf{A}})_{\mu\nu}. \tag{A2}$$

As a consequence of the scattering trajectories traversed in the same direction, the diffusion kernel depends only on the translation-invariant part of the Green's function. It is more convenient to work in momentum space. Using the Fourier transforms of the Green's function, Eq. (A1) becomes

$$P_0(\mathbf{x},\mathbf{x}',\Omega)=\int\frac{d^3\mathbf{k}\,d^3\mathbf{k}'}{(2\pi)^6}e^{i(\mathbf{k}-\mathbf{k}')\cdot(\mathbf{x}-\mathbf{x}')}\widetilde{G}^R(\epsilon+\Omega,\mathbf{k})\otimes\widetilde{G}^{AT}(\epsilon,\mathbf{k}'). \tag{A3}$$

We only present the crucial steps in computing $\int d\mathbf{x}'P_0(\mathbf{x},\mathbf{x}',0)$, and give directly the results for $\int d\mathbf{x}'(x_3-x_3')P_0(\mathbf{x},\mathbf{x}',0)$, $\int\partial_\omega P_0(\mathbf{x},\mathbf{x}';\omega)|_{\omega=0}d\mathbf{x}'$, and $\frac{1}{6}\int d\mathbf{x}'(\mathbf{x}-\mathbf{x}')^2P_0(\mathbf{x},\mathbf{x}',0)$. Thus, the average over $\mathbf{x}'$ after setting $\omega=0$ in Eq. (A3) yields

$$\int d\mathbf{x}'P_0(\mathbf{x},\mathbf{x}',0)=\int\frac{d^3\mathbf{k}}{(2\pi)^3}\widetilde{G}^R(\epsilon,\mathbf{k})\otimes\widetilde{G}^{AT}(\epsilon,\mathbf{k}). \tag{A4}$$

Substitute Eq. (7) into (A4) and perform the integration over $\boldsymbol{k}_\perp$ by using

$$\int_0^\infty x^\alpha\exp(-x)L_n^{(\alpha)}(x)L_m^{(\alpha)}(x)dx=\frac{\Gamma(n+\alpha+1)}{n!}\delta_{n,m}, \tag{A5}$$

that the Kronecker' s delta symbols appear due to the orthogonality relation for the Laguerre polynomials:

$$\int d\mathbf{x}'P_0(\mathbf{x},\mathbf{x}',0)=\int\frac{dk^3}{2\pi}\frac{1}{4|e\hbar v^2B|^2}\sum_{n,n'=0}^{\infty}(-1)^{n+n'}\frac{1}{(n+\chi)(n'+\chi^*)}\left\{\frac{eB}{8\pi\hbar}\Big[4(\varepsilon\gamma^0-v\hbar k^3\gamma^3+mv^2)\mathcal{P}_-\otimes(\varepsilon\gamma^0+v\hbar k^3\gamma^3\right.$$
$$+mv^2)\mathcal{P}_-\frac{\Gamma(n+1)}{n!}\delta_{nn'}+4(\varepsilon\gamma^0-v\hbar k^3\gamma^3+mv^2)\mathcal{P}_+\otimes(\varepsilon\gamma^0+v\hbar k^3\gamma^3+mv^2)\mathcal{P}_+\frac{\Gamma(n)}{(n-1)!}\delta_{nn'}$$

$$- 4(\varepsilon\gamma^0 - v\hbar k^3\gamma^3 + mv^2)\mathcal{P}_- \otimes (\varepsilon\gamma^0 + v\hbar k^3\gamma^3 + mv^2)\mathcal{P}_+ \frac{\Gamma(n+1)}{n!}\delta_{nn'-1}$$
$$- 4(\varepsilon\gamma^0 - v\hbar k^3\gamma^3 + mv^2)\mathcal{P}_+ \otimes (\varepsilon\gamma^0 + v\hbar k^3\gamma^3 + mv^2)\mathcal{P}_- \frac{\Gamma(n)}{(n-1)!}\delta_{n-1,n'}\Big]$$
$$- \left[16\frac{1}{2}(\gamma^1\otimes\gamma^1 - \gamma^2\otimes\gamma^2)\frac{e^2v^2B^2}{16\pi}\frac{\Gamma(n+1)}{(n-1)!}\delta_{n,n'}\right]\Bigg\}, \tag{A6}$$

where we have introduced the shorthand notation

$$\chi = -\frac{\left(\varepsilon + \frac{i\hbar}{2\tau}\right)^2 - (v\hbar k^3)^2 - m^2v^4}{2v^2\hbar|eB|}. \tag{A7}$$

Then, one of the two summations over the Landau index n in the retarded and advanced Green's functions can be easily done:

$$\int d\mathbf{x}'P_0(\mathbf{x},\mathbf{x}',0)$$
$$= \int \frac{dk^3}{2\pi}\frac{eB}{2\pi\hbar}\frac{1}{4e^2\hbar^2v^4B^2}\sum_{n=0}^{\infty}\Bigg\{\frac{(\varepsilon\gamma^0 - v\hbar k^3\gamma^3 + mv^2)\mathcal{P}_- \otimes (\varepsilon\gamma^0 + v\hbar k^3\gamma^3 + mv^2)\mathcal{P}_-}{(n+\chi)(n+\chi^*)}$$
$$+ \frac{(\varepsilon\gamma^0 - v\hbar k^3\gamma^3 + mv^2)\mathcal{P}_+ \otimes (\varepsilon\gamma^0 + v\hbar k^3\gamma^3 + mv^2)\mathcal{P}_+}{(n+\chi+1)(n+\chi^*+1)} + \frac{(\varepsilon\gamma^0 - v\hbar k^3\gamma^3 + mv^2)\mathcal{P}_- \otimes (\varepsilon\gamma^0 + v\hbar k^3\gamma^3 + mv^2)\mathcal{P}_+}{(n+\chi)(n+\chi^*+1)}$$
$$+ \frac{(\varepsilon\gamma^0 - v\hbar k^3\gamma^3 + mv^2)\mathcal{P}_+ \otimes (\varepsilon\gamma^0 + v\hbar k^3\gamma^3 + mv^2)\mathcal{P}_-}{(n+\chi+1)(n+\chi^*)} - \frac{ne\hbar v^2B(\gamma^1\otimes\gamma^1 - \gamma^2\otimes\gamma^2)}{(n+\chi)(n+\chi^*)}\Bigg\}. \tag{A8}$$

It can be recast by using the the polygamma functions

$$\int d\mathbf{x}'P_0(\mathbf{x},\mathbf{x}',0) = \int \frac{dk^3}{2\pi}\frac{|eB|}{2\pi\hbar}\frac{1}{4e^2\hbar^2v^4B^2}\Bigg\{\frac{\psi(\chi)-\psi(\chi^*)}{\chi-\chi^*}(\varepsilon\gamma^0 - v\hbar k^3\gamma^3 + mv^2)\mathcal{P}_- \otimes (\varepsilon\gamma^0 + v\hbar k^3\gamma^3 + mv^2)\mathcal{P}_-$$
$$+ \frac{\psi(\chi)+\frac{1}{\chi}-\psi(\chi^*)-\frac{1}{\chi^*}}{\chi-\chi^*}(\varepsilon\gamma^0 - v\hbar k^3\gamma^3 + mv^2)\mathcal{P}_+ \otimes (\varepsilon\gamma^0 + v\hbar k^3\gamma^3 + mv^2)\mathcal{P}_+$$
$$+ \frac{\psi(\chi)+\frac{1}{\chi}-\psi(\chi^*)}{\chi+1-\chi^*}(\varepsilon\gamma^0 - v\hbar k^3\gamma^3 + mv^2)\mathcal{P}_- \otimes (\varepsilon\gamma^0 + v\hbar k^3\gamma^3 + mv^2)\mathcal{P}_+$$
$$+ \frac{\psi(\chi)-\psi(\chi^*)-\frac{1}{\chi^*}}{\chi-1-\chi^*}(\varepsilon\gamma^0 - v\hbar k^3\gamma^3 + mv^2)\mathcal{P}_+ \otimes (\varepsilon\gamma^0 + v\hbar k^3\gamma^3 + mv^2)\mathcal{P}_-$$
$$- \frac{e\hbar v^2B}{\chi-\chi^*}(\gamma^1\otimes\gamma^1 - \gamma^2\otimes\gamma^2)\Bigg\}, \tag{A9}$$

where $\psi^{(n)}$ is the polygamma function of order n. With the help of the asymptotic expansion of $\psi^{(n)}$ for large arguments, i.e., a small magnetic field,

$$\psi(z) \sim \log z - \frac{1}{2z} - \sum_{n=1}^{\infty}\frac{B_{2n}}{2nz^{2n}} \tag{A10}$$

and

$$\psi^{(m)}(z) \sim (-1)^{m+1}\sum_{k=0}^{\infty}\frac{(k+m-1)!}{k!}\frac{B_k}{z^{k+m}} \tag{A11}$$

for $m \geqslant 1$, where B_k is the kth Bernoulli number, the above expression allows us to study the magnetic dependence order by order. Retaining to the linear terms in B and performing the integration over k^3, we arrive at the final expression

$$\int d\mathbf{x}'P_0(\mathbf{x},\mathbf{x}',0) = \frac{1}{4\pi}\frac{k_F}{\varepsilon v^2\hbar^2}\frac{\tau}{\hbar}\left[\varepsilon^2(\gamma^0+\eta\mathbf{1}_4)\otimes(\gamma^0+\eta\mathbf{1}_4) + \sum_{i=1,2,3}\frac{1}{3}v^2\hbar^2k_F^2\gamma^i\otimes(\gamma^i)^T\right]$$
$$+ \frac{|eB|}{8\pi\hbar}\Bigg\{ik_F\frac{\tau^2}{\hbar^2}[(\gamma^0+\eta\mathbf{1}_4)\otimes(\gamma^0+\eta\mathbf{1}_4)i\gamma^1\gamma^2 - (\gamma^0+\eta\mathbf{1}_4)i\gamma^1\gamma^2\otimes(\gamma^0+\eta\mathbf{1}_4)]$$

$$+\frac{1}{3}\frac{iv^2\hbar^2k_F^3}{\varepsilon^2}\frac{\tau^2}{\hbar^2}(-\gamma^3\otimes i\gamma^3\gamma^1\gamma^2+i\gamma^3\gamma^1\gamma^2\otimes\gamma^3)+\frac{1}{2}\frac{\varepsilon}{v^2\hbar^2k_F}\frac{\tau}{\hbar}[(1+\eta\gamma^0)\otimes(\gamma^0+\eta\mathbf{1}_4)i\gamma^1\gamma^2$$
$$+(1+\eta\gamma^0)i\gamma^1\gamma^2\otimes(\gamma^0+\eta\mathbf{1}_4)]+\frac{1}{2}\frac{k_F}{\varepsilon}\frac{\tau}{\hbar}(\gamma^3\otimes i\gamma^3\gamma^1\gamma^2+i\gamma^3\gamma^1\gamma^2\otimes\gamma^3)\bigg\}+\mathcal{O}(B^2). \tag{A12}$$

Following the similar procedure, we can find the results for the term linear in the gradient

$$\int d\mathbf{x}'(x_3-x_3')P_0(\mathbf{x},\mathbf{x}',0)=-\frac{v\hbar k_F^3}{12\pi\varepsilon}\frac{\tau^2}{\hbar^2}[\gamma^3\otimes(\gamma^0+\eta\mathbf{1}_4)-(\gamma^0+\eta\mathbf{1}_4)\otimes\gamma^3]$$
$$+\frac{i|eB|}{16\pi\hbar}\bigg\{-i\frac{4}{3}\frac{v^3\hbar^3k_F^3}{\varepsilon^2}\frac{\tau^3}{\hbar^3}[\gamma^3\otimes(\gamma^0+\eta\mathbf{1}_4)\gamma^1\gamma^2-\gamma^3\gamma^1\gamma^2\otimes(\gamma^0+\eta\mathbf{1}_4)]$$
$$+i\frac{v\hbar k_F}{\varepsilon}\frac{\tau^2}{\hbar^2}[\gamma^3\otimes(\gamma^0+\eta\mathbf{1}_4)\gamma^1\gamma^2+\gamma^3\gamma^1\gamma^2\otimes(\gamma^0+\eta\mathbf{1}_4)]-[\gamma^3\Leftrightarrow(\gamma^0+\eta\mathbf{1}_4)]\bigg\}+\mathcal{O}(B^2); \tag{A13}$$

the notation $[\gamma^3\Leftrightarrow(\gamma^0+\eta\mathbf{1}_4)]$ is an instruction to interchange the two matrices in the previous expression to generate a second term and finally the quadratic term is

$$\frac{1}{6}\int d\mathbf{x}'(\mathbf{x}-\mathbf{x}')^2P_0(\mathbf{x},\mathbf{x}',0)=\mathcal{D}\tau\frac{\varepsilon k_F}{4\pi v^2\hbar^2}\frac{\tau}{\hbar}\left[(\gamma^0+\eta\mathbf{1}_4)\otimes(\gamma^0+\eta\mathbf{1}_4)+\sum_{i=1,2,3}\frac{1}{3}(1-\eta^2)\gamma^i\otimes\left(\gamma^i\right)^T\right], \tag{A14}$$

and the term linear in ω,

$$\int d\mathbf{x}'\partial_\omega P_0(\mathbf{x},\mathbf{x}',\omega)|_{\omega=0}=-\frac{i\varepsilon k_F}{4\pi v^2\hbar^2}\frac{\tau^2}{\hbar^2}\left[(\gamma^0+\eta\mathbf{1}_4)\otimes(\gamma^0+\eta\mathbf{1}_4)+\sum_{i=1,2,3}\frac{1}{3}(1-\eta^2)\gamma^i\otimes(\gamma^i)^T\right]. \tag{A15}$$

By using the results (A12)–(A15), we can obtain the explicit expression for the 16×16 matrix form of $\mathscr{D}_{\mathbf{x}}^{-1}$.

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