

Ambiguity, Low Risk-Free Rates, and Consumption Inequality*

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Abstract

Macroeconomists failed to predict the Great Recession, suggesting that the existing macroeconomic models may have been misspecified. Bearing in mind this potential misspecification or “model uncertainty,” how do agents’ optimal decisions change? Furthermore, how large are the welfare costs of model misspecification? To shed light on these questions, we develop a tractable continuous-time general equilibrium model to show that a fear of model misspecification reduces both the equilibrium interest rate and the relative inequality of consumption to income, making the model’s predictions closer to the data. Our quantitative analysis shows that the welfare costs of model uncertainty are sizable.

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1. Introduction

In recent decades, both the real interest rate and the relative consumption inequality have declined in the United States. The real interest rate has declined significantly since 1980 (Figure 1), prompting intensive discussions among macroeconomists and policymakers. Understanding changes in the real interest rate is important because the real interest rate signals future growth prospects; a low interest rate may undermine financial stability (Fischer, 2016). Over the same period, relative consumption inequality, measured as the ratio of the standard deviation of the change in consumption to the standard deviation of the change in income, has also declined (Figure 1).¹ These similar trends suggest interest rates and consumption inequality may be related. Indeed, in standard models of incomplete consumption insurance, these variables are closely connected. When interest rates are low, wealthy households receive relatively little additional income compared with poor ones; as a result, lower interest rates may lead to lower consumption inequality. However, in general equilibrium, a low interest rate can be driven by high income risks and incomplete markets through increasing precautionary savings. In this case, the precautionary saving motive may lead poor households to spend relatively less than wealthy households, leading to relatively high consumption inequality.

In this paper we explain low real interest rates and consumption dispersion through the lens of model uncertainty, which is new in the literature. The basic idea of model uncertainty we pursue in this paper is based on Hansen and Sargent (1995, 2007), who first introduce a preference for robustness into economic models as a way to capture an agent's fear that their model is misspecified

¹In this paper, we use "inequality" and "dispersion" interchangeably to describe the cross-sectional distributions of changes in consumption and income. Specifically, we use household-level consumption data to construct our consumption dispersion, which is defined as the standard deviation of changes in household consumption in a given year. Income dispersion is defined in a similar way. Our measures use changes in consumption, following the early literature in which the consumption level is unstationary. Online Appendix A provides more details on how we construct the data.

(a form of ambiguity aversion).² In robustness models, agents have in mind a reference model that represents their best estimate of the model economy governing the dynamics of state variables. However, because they are concerned that this reference model may be misspecified in some way, they make their optimal decisions under the worst-case scenario.

Our approach is partially motivated by the increased concerns following the recent financial crisis about whether existing macroeconomic models are misspecified. Some recent papers argue that model uncertainty, the “unknown unknowns,” played an important role in the recent economic and financial crises (see Caballero and Krishnamurthy, 2008; Boyarchenko, 2012, as examples). At the very least, the failure of macroeconomists to predict the 2007–09 Great Recession suggests existing macroeconomic models are missing some critical ingredients. In addition, intuitively, when uncertainty about the model increases, an agent may spend more cautiously, which in turn may have implications for aggregate savings and equilibrium interest rates. As a result, we formally investigate the channel of model uncertainty.

This paper offers a unified framework to illustrate how model uncertainty can help simultaneously explain the low real interest rate and the decline in consumption inequality in the U.S. economy. We then use this framework to examine the welfare and policy implications of model uncertainty. Our analysis has made three main contributions. First, we derive closed-form solutions for a model featuring robustness, recursive exponential utility, uninsurable labor income, and consumption-portfolio choice. We characterise how the equilibrium interest rate is affected by robustness through its interactions with the elasticity of intertemporal substitution (EIS) and the coefficient of constant absolute risk aversion (CARA). An increase in the EIS affects the equilibrium interest rate through two distinct channels: (i) high EIS increases the relative importance of the impatience-induced dissaving effect (the direct channel) and (ii) reduces precautionary sav-

²See Hansen and Sargent (2007) for a textbook treatment on ambiguity and robustness. Although we could also capture concerns about model misspecification through recursive multiple-prior utility due to ambiguity aversion (Chen and Epstein, 2002), we follow Hansen and Sargent (2007) for technical simplicity.

ings by reducing the effect of robustness on the effective coefficient of absolute risk-uncertainty aversion (the indirect channel).³ We then show that a general equilibrium with a preference for robustness can be constructed in the vein of Huggett (1993) and Wang (2003).⁴ More model uncertainty increases precautionary savings and lowers interest rates. In addition, we show that the relative dispersion of consumption compared with income is determined only by the equilibrium interest rate and the persistence of the income process. In particular, the relative consumption dispersion decreases as the degree of robustness increases if the income process is stationary.

Second, our model succeeds quantitatively in explaining the declines in the equilibrium interest rate and the relative dispersion of consumption relative to income in the data. In the U.S. economy, the real risk-free interest rate averaged 1.87% from 1981 to 2010, and 1.37 percent from 1981 to 2015.⁵ To explain the observed real interest rate levels, an RE model without model uncertainty would require the coefficient of risk aversion parameter to be as high as 15 to 24 when the EIS takes reasonable values.⁶ However, when consumers take into account model uncertainty, the model can generate a low equilibrium interest rate with much lower values for the coefficient of risk aversion.⁷ Moreover, the reduction in the equilibrium interest rate following the 2007–09 financial crisis can be rationalised by possible increases in concerns about model misspecification during this period.

We slightly enrich our benchmark model by including a risky asset to better explain the relative

³The effective coefficient of absolute risk-uncertainty aversion ($\tilde{\gamma}$) is determined by the interaction between the true coefficient of absolute risk aversion (γ), the EIS (ψ), and the degree of RB (θ) via the following formula: $\tilde{\gamma} = \gamma + \theta/\psi$.

⁴Wang (2003) constructs a general equilibrium under rational expectations (RE) in the same Huggett-type model economy with CARA expected utility.

⁵Here the numbers are computed using the Consumer Price Index (CPI) to measure inflation. Using Personal consumption expenditures (PCE) leads to similar results. See Table 1 for different measures of the risk-free rates.

⁶We normalise the mean consumption level to be 1, so the coefficient of relative risk aversion is equal to the coefficient of absolute risk aversion. While estimates of the EIS fluctuate wildly, risk aversion coefficients of about 10 are generally not considered reasonable in the macroeconomic literature; finance is different, where very high risk aversion is seemingly accepted without comment.

⁷Barillas *et al.* (2009) show that most of the observed high market price of risk in the U.S. economy can be reinterpreted as a market price of model uncertainty and the risk-aversion parameter can thus be reinterpreted as measuring the representative agent's doubts about the model specification.

consumption dispersion. As in the benchmark model the relative consumption dispersion depends only on the equilibrium interest rate, it cannot match both the real interest rate and the relative consumption dispersion with the same parametrisation. We show that extending our model to include risky assets fixes this problem.⁸ In particular, we show that the presence of a risky asset affects equilibrium precautionary savings through two channels: (i) the risky asset can be used to hedge labor income risk, and (ii) the risky asset increases the amount of total risk that agents face if the net supply of the risky asset is positive. We find that, when the supply of risky asset increases, the relative dispersion of consumption increases while the risk-free rate declines. The main reason behind this result is that the positive net supply of the risky asset offers a higher expected return, increases current consumption, and makes consumption growth more dispersed relative to income growth. For plausibly calibrated parameter values of robustness, we find that the extended model can simultaneously generate the observed low risk free rate and high relative dispersion of consumption to income in the United States. Moreover, if we divide our sample period into two periods, we find that allowing for time-varying degree of robustness can help account for the observed decline in the relative inequality of consumption to income.⁹

Third, we evaluate the welfare costs associated with model uncertainty in two dimensions: the overall welfare costs and the marginal welfare costs. First, we assess the welfare gains associated with eliminating model uncertainty—what would an agent pay to find out the true stochastic process that governs his income? We find that these numbers are large—the cost can be as high as 15% of initial wealth (or 75% of initial income if initial wealth is 5 times larger than initial income).

⁸Related research has tried to understand how the supply of safe assets matters for low interest rates (Barro *et al.* 2017); our two-asset extension should be viewed as grappling with this question as well.

⁹The result that the relative dispersion of consumption to income decreases with income uncertainty through the general equilibrium interest rate channel provides an explanation for the empirical evidence documented in Blundell *et al.* (2008) that income and consumption inequality diverged over the studied sample period. It is worth noting that the authors use the variances of log consumption and log income to measure consumption and income inequality. Since our paper adopts the CARA-Gaussian setting, and the consumption process is nonstationary, we use the standard deviations of changes in consumption and income to measure the cross-sectional dispersions/inequality of changes in consumption and income, respectively.

These costs increase as the degree of robustness increases but decrease as the EIS increases. Second, we provide formulas to evaluate, at the margin, the welfare costs/gains of changes in the degree of robustness and labor income volatility. We find that under our calibrated parameter values, a 10% increase in the degree of robustness leads to a welfare cost equivalent to a 1.23% reduction in initial income. This welfare cost is significantly larger when the income process is more volatile. To highlight the policy implications of our analysis, we show that a macro policy that reduces the income variance by 10% would lead to a welfare gain equivalent to roughly a 16% increase in initial income. The gain is much smaller absent any model uncertainty.

Literature review Our paper is related to several branches of literature. Our paper follows the Hansen and Sargent (2007) line of research and develops a tractable continuous-time general equilibrium model in which consumers have a recursive utility with an aversion to ambiguity and face uninsurable labor income risk.¹⁰

Our paper is also related to papers studying the determination of the equilibrium risk-free rate. Broadly speaking, three major factors determine the equilibrium real interest rate: aggregate savings, aggregate investment, and the relative demand for safe versus risky assets. Huggett (1993) and Aiyagari (1994) study the importance of uninsurable labour income risk in determining aggregate savings and equilibrium interest rates in a heterogeneous-agent framework. They find that labour income risk itself cannot explain the observed low real interest rate. Blanchard *et al.* (2014) and Summers (2014) argue that the low equilibrium real interest rates in the U.S. and other advanced economies were caused by increases in global savings. Their explanations for higher savings rely on demographic trends (such as an ageing population) or capital flows from emerging economies to advanced economies.

In addition, our paper is related to the relatively few papers that have studied consumption in-

¹⁰See Anderson *et al.* (2003); Maenhout (2004); Ju and Miao (2012); and Kasa and Lei (2018) for the applications of robustness in continuous-time models.

equality. Blundell *et al.* (2008) show that consumption and income inequality diverged during the 1980–2004 period, driven by the persistence of income shocks and the degree of consumption insurance. Krueger and Perri (2004) try to account for the observed difference between consumption and income inequality using a calibrated incomplete-markets model with limited commitment. All these papers have assumed the models that agents use to make decisions are correct. In this paper, we relax this assumption and show that a reasonable level of model uncertainty can help explain both the low real interest rate and consumption dispersion.

Finally, our paper is also related to the rapidly growing literature using continuous-time heterogeneous-agent models to address inequality issues, including Benhabib *et al.* (2011), Gabaix *et al.* (2016), and Kasa and Lei (2018). The key difference between their work and this paper is that we focus mainly on consumption inequality and equilibrium asset returns. Related to discrete-time rational expectations heterogeneous-agent models (such as Huggett, 1993; Aiyagari, 1994), we provide analytical solutions in a heterogeneous-agent model to help illustrate the key mechanisms through which model uncertainty influences our results.

This paper is organized as follows. Section 2 shows how to introduce robustness into a version of the Caballero-Huggett type model with incomplete markets. Section 3 presents the theoretical results regarding the general equilibrium implications of robustness for the interest rate and the joint dynamics of consumption and income. Section 4 presents our quantitative results showing how our robustness model can account for both a lower equilibrium interest rate and a lower relative consumption dispersion. Section 5 extends the model to include a risky asset to better explain the relative consumption dispersion. Section 6 examines the welfare implications of model uncertainty and shows that the welfare costs are sizable. Section 7 concludes and briefly discusses future research.

2. A Continuous-Time Heterogeneous-Agent Economy with Robustness

In this section, we lay out our continuous-time consumption-portfolio choice model with recursive utility and uninsurable labour income. To help explain the key structure of the model, we will introduce each of the key elements one by one, starting with specifications of the recursive utility preference, followed by labour income and investment opportunity set, and finally model uncertainty due to robustness and ambiguity.

2.1. Recursive Exponential Utility

Although the expected power utility model has many attractive features, that model implies that the elasticity of intertemporal substitution is the reciprocal of the coefficient of relative risk aversion. Conceptually, risk aversion (attitudes towards atemporal risks) and intertemporal substitution (attitudes towards shifts in consumption over time) capture two distinct aspects of decision-making and need not be so tightly connected.¹¹ By contrast, the class of recursive utility functions (Epstein and Zin, 1989; Duffie and Epstein, 1992) enable one to disentangle risk aversion from intertemporal substitution. In this paper, we assume that agents in our model economy have the Kreps-Porteus type preference with *recursive exponential utility* (REU): for every stochastic consumption stream, $\{c_t\}_{t=0}^{\infty}$, the utility stream, $\{f(U_t)\}_{t=0}^{\infty}$, is recursively defined by:¹²

$$f(U_t) = \left(1 - e^{-\delta\Delta t}\right) f(c_t) + e^{-\delta\Delta t} f(CE_t[U_{t+\Delta t}]), \quad (1)$$

¹¹Risk aversion describes the agent's reluctance to substitute consumption across different states of the world and is meaningful even in a static setting. By contrast, intertemporal substitution describes the agent's willingness to substitute consumption over time and is meaningful even in a deterministic setting.

¹²Skiadas (2009, ch.6) axiomatises and systematically characterises this type of recursive exponential utility (or transition-invariant recursive utility.) Skiadas (2009, ch.6) also compares this type of recursive utility with the scale-invariant Kreps-Porteus recursive utility (e.g., the Epstein-Zin-Weil parametric utility form). See also Angeletos and Calvet (2006) for an application of REU in a business cycles model.

where Δt is the time interval, $\delta > 0$ is the agent's subjective discount rate, $f(U_t) = -\psi \exp(-U_t/\psi)$, $f(c_t) = -\psi \exp(-c_t/\psi)$,

$$CE_t[U_{t+\Delta t}] = g^{-1}(E_t[g(U_{t+\Delta t})]), \quad (2)$$

is the certainty equivalent of U_{t+1} conditional on the period t information, and $g(U_{t+\Delta t}) = -\exp(-\gamma U_{t+\Delta t})/\gamma$. In (1), $\psi > 0$ governs the EIS, while $\gamma > 0$ governs the coefficient of CARA.¹³ A high value of ψ corresponds to a strong willingness to substitute consumption over time, and a high value of γ implies a low willingness to substitute consumption across states of nature. Note that if $\psi = 1/\gamma$, the functions f and g are the same and the recursive utility reduces to the standard time-separable expected utility function used in Caballero (1990) and Wang (2003).

2.2. Specifications of Investment Opportunity Set and Labor Income

We assume there is only one risk-free asset in the model economy and there are a continuum of consumers who face uninsurable labour income. The evolution of the financial wealth (w_t) of a typical consumer is:

$$dw_t = (rw_t + y_t - c_t) dt, \quad (3)$$

where r is the return to the risk-free asset and c_t and y_t are consumption and labour income at time t , respectively. Uninsurable labour income (y_t) follows an Ornstein-Uhlenbeck (OU) process:¹⁴

$$dy_t = \rho(\bar{y} - y_t) dt + \sigma_y dB_t, \quad (4)$$

where \bar{y} is the unconditional mean of y_t , σ_y is the unconditional volatility of the income change over an incremental unit of time, $\sigma_y^2 / (2\rho)$ is the unconditional variance of y_t , the persistence coefficient

¹³It is well-known that the CARA utility specification is tractable for deriving optimal policies and constructing general equilibrium in different settings. See Caballero (1990), Wang (2003), and Angeletos and Calvet (2006).

¹⁴In this paper, we abstract from income growth. It is worth noting that higher income growth generates higher risk-free rates. However, within our REU-OU framework, assuming constant income growth leads to time-varying risk-free rates, which greatly complicates our model. The detailed proof is available from the corresponding author by request.

ρ governs the speed of convergence or divergence from the steady state, and B_t is a standard Brownian motion on the real line \mathcal{R} . To present the model more compactly, we define a new state variable, s_t :

$$s_t \equiv w_t + h_t,$$

where h_t is human wealth at time t and is defined as the expected discounted present value of current and future labour income discounted at the risk-free interest rate r :

$$h_t \equiv E_t \left[\int_t^\infty \exp(-r(s-t)) y_s ds \right].$$

For the given the income process, (4), $h_t = y_t / (r + \rho) + \bar{y} / (r(r + \rho))$.¹⁵ Using s_t as the unique state variable, we can rewrite (3) as:

$$ds_t = (rs_t - c_t) dt + \sigma_s dB_t, \tag{5}$$

where $\sigma_s = \sigma_y / (r + \rho)$ is the unconditional variance of the innovation to s_t .¹⁶

The optimisation problem under rational expectations (RE) can thus be written as:

$$f(J_t) = \max_{c_t} \left\{ \left(1 - e^{-\delta\Delta t}\right) f(c_t) + e^{-\delta\Delta t} f(CE_t[J_{t+\Delta t}]) \right\}, \tag{6}$$

subject to (5). An educated guess is that $J_t = As_t + A_0$, where A and A_0 are undetermined coeffi-

¹⁵If $\rho > 0$, the income process is stationary and deviations of income from the steady state are temporary; if $\rho \leq 0$, income is non-stationary. We need to impose the restriction that $r > -\rho$ to guarantee the finiteness of human wealth.

¹⁶In the next subsection, we will introduce robustness directly into this “reduced” precautionary savings model. It is not difficult to show that the reduced univariate model and the original multivariate model are observationally equivalent in the sense that they lead to the same consumption and saving decisions, because the financial wealth part of total wealth is deterministic between periods. The detailed proof is available from the corresponding author by request.

cients. The J function at time $t + \Delta t$ can thus be written as:

$$J(s_{t+\Delta t}) = As_{t+\Delta t} + A_0 \approx As_t + A(rs_t - c_t)\Delta t + A\sigma_s\Delta B_t + A_0,$$

where $\Delta s_t \equiv s_{t+\Delta t} - s_t$ and $\Delta s_t \approx (rs_t - c_t)\Delta t + \sigma_s\Delta B_t$ where $\Delta B_t = \sqrt{\Delta t}\epsilon$ and ϵ is a standard normal innovation. The Hamilton-Jacobi-Bellman (HJB) equation is then:

$$\delta f(J_t) = \sup_{c_t} \{\delta f(c_t) + \mathcal{D}f(s_t)\}, \quad (7)$$

where

$$\mathcal{D}f(s_t) = f'(J_t) \left(A(rs_t - c_t) - \frac{1}{2}\gamma A^2\sigma_s^2 \right), \quad (8)$$

and the transversality condition, $\lim_{t \rightarrow \infty} \{E|\exp(-\delta t)f_t|\} = 0$, holds at the optimum.

2.3. Incorporating Fear of Model Uncertainty

To introduce aversion to model uncertainty into our model (and thus generate a demand for robust decision rules), we follow the continuous-time methodology proposed by Anderson *et al.* (2003) and adopted in Maenhout (2004). Consumers take Equation (5) as the approximating model. The corresponding set of distorting models can thus be obtained by adding an *endogenous* distortions $b(s_t)$ to (5):

$$ds_t = (rs_t - c_t)dt + \sigma_s(\sigma_s b(s_t)dt + dB_t). \quad (9)$$

Consumers accept (5) as the best approximating (or reference) model, but are still concerned that the model is misspecified. They therefore want to consider a range of models (the distorted models (9)) surrounding the approximating model when computing the continuation payoff. A preference for robustness (ambiguity aversion) manifests by having the agent guard against distorting models that are reasonably close to the approximating model. The drift adjustment $b(s_t)$ is chosen to

minimise the sum of (i) the expected continuation payoff adjusted to reflect the additional drift component in (9) and (ii) an entropy penalty:

$$\inf_{b_t} \left[\mathcal{D}f + f' (J_t) Ab (s_t) \sigma_s^2 + \frac{1}{\theta_t} \frac{(b (s_t) \sigma_s)^2}{2} \right], \quad (10)$$

where the first two terms are the expected continuation payoff when the state variable follows (9), $(b (s_t) \sigma_s)^2 / 2$ is the relative entropy or the expected log likelihood ratio between the distorted model and the approximating model, and $1/\theta_t$ is the weight on the entropy penalty term.¹⁷ θ_t is fixed and state independent in Anderson *et al.* (2003), whereas it is state-dependent in Maenhout (2004). The role of the state-dependent counterpart to θ_t in Maenhout (2004) is to assure the homotheticity or scale invariance of the decision problem under a CRRA utility function. Note that the evil agent's minimisation problem, (10), is invariant to the scale of total resources s_t under the state-dependent specification for $\theta_t (s_t)$, which we use as well so that the demand for robustness does not disappear as the value of total wealth increases.

We can then obtain the HJB equation for the RB model:

$$\delta f (J_t) = \sup_{c_t \in \mathcal{C}} \inf_{b_t} \left\{ \delta f (c_t) + \mathcal{D}f (s_t) + f' (J_t) Ab (s_t) \sigma_s^2 + \frac{1}{\theta (s_t)} \frac{(b (s_t) \sigma_s)^2}{2} \right\}. \quad (11)$$

Note that here following Hansen and Sargent (2011) and Kasa and Lei (2018), we scale the robustness parameter (θ_t) by the sampling interval (Δt), effectively making the consumer have stronger preference for robustness as the sampling interval shrinks.¹⁸ Solving first for the infimisation part

¹⁷The last term in (10) is due to the consumer's preference for robustness. Note that the $\theta_t = 0$ case corresponds to the standard RE case. This robustness specification is called the *multiplier (or penalty) robust control problem*. It is worth noting that this multiplier preference of RB expresses *ambiguity* with a multiplier that penalises deviations from the approximating model as measured by relative entropy, and they express *ambiguity aversion* with the minimization operator.

¹⁸See also Hansen and Miao (2018) for a discussion on distinguishing ambiguity from risk in the continuous time limit.

of (11) yields:

$$b^*(s_t) = -\theta(s_t) f'_s(s_t),$$

where $\theta(s_t) = -\vartheta/f(s_t) > 0$ and $f'_s(s_t) = f'(J_t) A$ (see Online Appendix B for the derivation). Here we can also define “ $1/f(s_t)$ ” in the $\theta(s_t)$ specification as a *normalisation* factor that is introduced to convert relative entropy (the distance between the approximating model and a given distorted model) into units of utility so that it is consistent with the units of the expected future value function evaluated with the distorted model. It is worth noting that this state-dependent robustness parameter follows a geometric Brownian motion in general equilibrium. (See Section 3.2 for the details.) This resulting process is similar to the AR(1) ambiguity shocks proposed in Bhandari *et al.* (2019).¹⁹

Since $\theta(s_t) > 0$, the preference for robustness adds a negative drift term to the state transition equation because $J_s > 0$. Substituting for b^* in (11) gives:

$$\delta f(J_t) = \sup_{c_t \in \mathcal{C}} \left\{ \delta f(c_t) + f'(J_t) A \left(rs_t - c_t - \frac{1}{2} \gamma A \sigma_s^2 + \frac{\vartheta}{f(J_t)} A f'(J_t) \sigma_s^2 \right) - \frac{\vartheta}{2f(J_t)} A^2 (f'(J_t))^2 \sigma_s^2 \right\}. \quad (12)$$

2.4. The Robust Consumption Function and Precautionary Saving

We can now solve (12) and obtain the consumption rule under robustness. The following proposition summarises the solution.

Proposition 1. *Under robustness, the consumption function and the saving function are*

$$c_t^* = rs_t + \Psi(r) - \Gamma(\vartheta, r), \quad (13)$$

¹⁹They identify AR(1) ambiguity shocks using U.S. survey data, and find that in the data, the ambiguity shocks are an important source of variation in labour market variables.

and

$$d_t^* = x_t + \Gamma - \Psi, \quad (14)$$

respectively, where $x_t \equiv \rho (y_t - \bar{y}) / (r + \rho)$ is the demand for savings “for a rainy day”,

$$\Psi(r) \equiv \psi \left(\frac{\delta}{r} - 1 \right) \quad (15)$$

captures the saving demand of relative patience,

$$\Gamma(\vartheta, r) \equiv \frac{1}{2} r \tilde{\gamma} \sigma_s^2 \quad (16)$$

is the demand for precautionary savings due to the interaction of income uncertainty, intertemporal substitution, and risk and uncertainty aversion, and

$$\tilde{\gamma} \equiv \gamma + \frac{\vartheta}{\psi} \quad (17)$$

is the effective coefficient of absolute risk-uncertainty aversion. The corresponding value function is

$$f(s_t) = -\frac{\delta \psi}{r} \exp \left(- \left(\frac{\delta}{r} - 1 - \frac{1}{2} \frac{r}{\psi} \tilde{\gamma} \sigma_s^2 \right) - \frac{r}{\psi} s_t \right). \quad (18)$$

Finally, the worst possible distortion is

$$b^* = -r \frac{\vartheta}{\psi}. \quad (19)$$

Proof. See Online Appendix B. ■

From (15), it is clear that if the consumer is impatient relative to the market ($\delta > r$), the higher the EIS, the stronger the demand for consumption. If $\delta > r$, households want consumption to fall

over time, and a higher EIS implies that consumption will be allowed to fall faster for a given value of δ/r ; as a result, consumption must initially be high. Following the literature on precautionary savings, we measure the demand for precautionary saving as the amount of saving induced by the combination of uninsurable labour income risk and risk aversion. Expression (16) shows that the precautionary savings demand now depends on the effective coefficient of risk-uncertainty aversion $\tilde{\gamma}$, which is a function of the EIS (ψ), the CARA (γ), and the degree of robustness (ϑ). Specifically, it increases with γ and ϑ , whereas it decreases with ψ . From (16), one can also see that the precautionary saving demand is larger for a more volatile income innovation (higher σ_y) and a larger persistence coefficient (lower ρ).²⁰ Holding other parameters constant, we can see from (13) to (16) that intertemporal substitution and risk aversion have opposing effects on consumption and saving decisions if $\delta > r$ (which will be the case in general equilibrium).²¹

Another interesting question here is the relative importance of RB (ϑ) and CARA (γ) in determining the precautionary savings demand, holding other parameters constant. We can use the elasticities of precautionary saving as a measure of their importance.

Proposition 2. *The relative sensitivity of precautionary saving to risk aversion (γ), intertemporal substitution (ψ), and robustness (ϑ) can be measured by*

$$\mu_{\gamma\vartheta} \equiv \frac{e_\gamma}{e_\vartheta} = \frac{\gamma}{\vartheta/\psi'} \quad (20)$$

$$\mu_{\psi\vartheta} \equiv \frac{e_\psi}{e_\vartheta} = -1, \quad (21)$$

respectively, where $e_\gamma \equiv \frac{\partial \Gamma/\Gamma}{\partial \gamma/\gamma}$, $e_\psi \equiv \frac{\partial \Gamma/\Gamma}{\partial \psi/\psi}$, and $e_\vartheta \equiv \frac{\partial \Gamma/\Gamma}{\partial \vartheta/\vartheta}$ are the elasticities of the precautionary saving demand to CARA, EIS, and RB, respectively.

²⁰As argued in Caballero (1990) and Wang (2003), a more persistent income shock takes a longer time to wear off and thus induces a stronger precautionary saving demand by a prudent forward-looking consumer.

²¹As a side note, incomplete markets generally imply that aggregate dynamics depend on the wealth distribution, this ‘‘curse of dimensionality’’ is circumvented by our CARA-Gaussian specification since savings functions are linear.

Proof. The proof is straightforward. ■

The interpretation of (20) is that the precautionary savings demand is more sensitive to the actual coefficient of (absolute) risk aversion (γ) than it is to RB (ϑ) if the actual CARA is greater than RB amplified by the inverse of the EIS, i.e., $\gamma > \vartheta/\psi$. Of course, it is not exactly clear how to interpret a proportional change in either parameter since they do not have units, but we report this result to show that risk aversion does not clearly dominate the motives of the agents in the model.

Hansen *et al.* (1999) show that the discount factor and the degree of RB are observationally equivalent in the sense that they lead to the same consumption and saving decisions in a discrete-time LQ Gaussian representative-agent permanent income model. The reason for this result is that introducing a concern about model misspecification increases savings in the same way as increasing the agent's degree of patience, so that the discount factor can be changed to offset the effect of a change in RB on consumption and investment.²² In contrast, in our continuous-time CARA-Gaussian model, we have a more general observational equivalence result between δ , γ , and ϑ .

Proposition 3. *Let*

$$\gamma^{fi} = \gamma + \frac{\vartheta}{\psi}, \quad (22)$$

where γ^{fi} is the coefficient of absolute risk aversion in the RE model. Then consumption and savings are identical in the RE and RB models, holding other parameter values constant. Furthermore, let $\delta = r$ in the RB model, and

$$\delta^{fi} = r - \frac{1}{2}\vartheta \left(\frac{r}{\psi}\right)^2 \sigma_s^2, \quad (23)$$

where δ^{fi} is the discount rate in the RE model. Then consumption and savings are identical in the RE and

²²As shown in Hansen *et al.* (1999), the two models have different implications for asset prices because continuation valuations would alter as one alters the values of the discount factor and the RB parameter within the observational equivalence set.

RB models, ceteris paribus.

Proof. Using (13) and (16), the proof is straightforward. ■

Expression (22) means that a consumer with a preference for robustness (ϑ) and recursive utility with EIS (ψ) and CARA (γ) is observationally equivalent to a consumer with full-information and recursive utility with EIS (ψ) and CARA ($\gamma + \vartheta/\psi$). In contrast, within a Merton model with recursive utility, Maenhout (2004) showed that an agent with a preference for robustness and Epstein-Zin recursive utility with EIS (ψ) and CRRA (γ) is observationally equivalent to an agent with full information and recursive utility with EIS (ψ) and CARA ($\gamma + \vartheta$). In other words, in Maenhout's model, the effective coefficient of relative risk aversion ($\gamma + \vartheta$) does not depend on the EIS (ψ).

3. General Equilibrium Implications of RB

3.1. Definition of the General Equilibrium

As in Huggett (1993) and Wang (2003), we assume that the economy is populated by a continuum of *ex ante* identical, but *ex post* heterogeneous agents, with each agent having the saving function, (16). In addition, we also assume that the risk-free asset in our model economy is a pure-consumption loan and is in zero net supply.²³ The key insights can be also obtained in a CARA-Gaussian production economy with incomplete markets (as in Angeletos and Calvet, 2006) using a neoclassical production function with capital and bonds as saving instruments. We consider the simpler Huggett-type endowment economy for two reasons. First, in the endowment economy, we can directly compare the model's predictions on the dynamics of individual consumption and income with its empirical counterpart, and do not need to infer the idiosyncratic productiv-

²³We can easily generalise to fixed positive net savings, as in a Lucas-style tree model. Nothing would change.

ity shock process. Second, the endowment economy allows us to solve the models explicitly, and thus helps us identify distinct channels via which RB interacts with risk aversion, discounting, and intertemporal substitution and affects the consumption-saving behaviour.

In the model economy, the initial cross-sectional distribution of income is assumed to be its stationary distribution $\Phi(\cdot)$. By the law of large numbers (LLN), provided that the spaces of agents and the probability space are constructed appropriately, aggregate income and the cross-sectional distribution of permanent income $\Phi(\cdot)$ will be constant over time.

Proposition 4. *The total savings demand “for a rainy day” in the precautionary savings model with RB equals zero for any positive interest rate. That is, $F_t(r) = \int_{y_t} x_t(r) d\Phi(y_t) = 0$, for $r > 0$.*

Proof. Given that labour income is a stationary process, the LLN can be directly applied. The proof is the same as that in Wang (2003). ■

This proposition states that the total savings “for a rainy day” is zero, at any positive interest rate; with a constant income distribution and linear decision rules, agents in the stationary wealth distribution follow the “American dream and American nightmare” path, where any rise in income today is eventually offset by a future decline. Therefore, from (14), for $r > 0$, the expression for total savings under RB in the economy at time t can be written as:

$$D(\vartheta, r) \equiv \Gamma(\vartheta, r) - \Psi(r), \quad (24)$$

where $\Psi(r)$ and $\Gamma(\vartheta, r)$ are defined in (15) and (16), respectively. We can now define a general equilibrium.

Definition 1. Given (24), a general equilibrium under RB is defined by an interest rate r^* satisfying

$$D(\vartheta, r^*) = 0. \quad (25)$$

3.2. Theoretical Results

The following proposition summarises the equilibrium dynamics of consumption and wealth:

Proposition 5. Suppose the true economy is governed by the approximating model, where $\theta_t = -\vartheta/f(s_t)$ and $f(s_t)$ is provided in (18). In equilibrium, each consumer's optimal consumption is described by the permanent income hypothesis (PIH):

$$c_t^* = r^* s_t. \quad (26)$$

Furthermore, the evolution equations of wealth and consumption are:

$$dw_t^* = x_t dt, \quad (27)$$

$$dc_t^* = \frac{r^*}{r^* + \rho} \sigma_y dB_t, \quad (28)$$

respectively, where $x_t = \rho(y_t - \bar{y}) / (r^* + \rho)$. Finally, in equilibrium, θ_t follows a geometric Brownian motion

$$\frac{d\theta_t}{\theta_t} = \frac{1}{2} (r^* \sigma_s)^2 dt + (r^* \sigma_s) dB_t. \quad (29)$$

If $\rho > \delta$, the equilibrium is unique.

Proof. See Online Appendix C. ■

The intuition behind this proposition is similar to that in Wang (2003). With an individual's constant total precautionary savings demand $\Gamma(\vartheta, r)$, for any $r > 0$, the equilibrium interest rate

r^* must be at a level with the property that individual's dissavings demand due to impatience is exactly balanced by their total precautionary-savings demand, $\Gamma(\vartheta, r^*) = \Psi(r^*)$. We can see from (25) that EIS affects the equilibrium interest rate via two channels: (i) the precautionary saving channel and (ii) the impatience-induced dissaving channel. As EIS decreases, it increases the precautionary saving demand via increasing the effective coefficient of risk-uncertainty aversion and also reduces the impatience-induced dissaving effect; both channels drive down the equilibrium interest rate. It is also clear from (25) that a high value of ψ would amplify the relative importance of the dissaving effect $\Psi(r)$ for the equilibrium interest rate. The intuition behind this result is simple. When ψ is higher, consumption growth responds less to changes in the interest rate. In order to clear the market, the consumer must be offered a higher equilibrium risk free rate in order to be induced to save more and make his consumption tomorrow even more in excess of what it is today (less smoothing).

From the equilibrium condition:

$$\frac{1}{2}r^* \left(\gamma + \frac{\vartheta}{\psi} \right) \frac{\sigma^2}{(r^* + \rho)^2} - \psi \left(\frac{\delta}{r^*} - 1 \right) = 0, \quad (30)$$

it is straightforward to show that

$$\frac{dr^*}{d\vartheta} = -\frac{r^* \sigma_s^2}{\psi} \left(\tilde{\gamma} \sigma_s^2 \frac{\rho - r^*}{\rho + r^*} + \frac{2\delta\psi}{r^{*2}} \right)^{-1}; \quad (31)$$

if $\rho > \delta > r^*$, then this derivative is negative, so that r^* is decreasing in the degree of RB, ϑ . In addition, it is straightforward to see that:

$$\frac{dr^*}{d\gamma} < 0 \text{ and } \frac{dr^*}{d\psi} > 0.$$

That is, the equilibrium interest rate decreases with the degree of risk aversion and increases with the degree of intertemporal substitution. From (27) and (28), we can conclude that although both the CARA model and the linear-quadratic (LQ) model lead to the PIH in general equilibrium, both risk aversion and intertemporal substitution play roles in affecting the dynamics of consumption and wealth in the CARA model via the equilibrium interest rate channel. Figure 2 shows that the aggregate saving function $D(\vartheta, r)$ is increasing with the interest rate for different values of ϑ when $\gamma = 3$, $\psi = 0.5$, $\delta = 0.036$, $\sigma_y = 0.182$, and $\rho = 0.083$, and there exists a unique interest rate r^* for every given ϑ such that $D(\vartheta, r^*) = 0$.²⁴

Note that mathematically, the cross-sectional dispersion of consumption (relative to income) can be measured by the relative volatility of consumption to income, as our model satisfies a mixing condition in the steady state. The following result is then immediate.

Proposition 6. *The relative dispersion of changes in consumption to changes in income is:*

$$\mu \equiv \frac{\text{sd}(dc_t^*)}{\text{sd}(dy_t)} = \frac{r^*}{r^* + \rho'} \quad (32)$$

where $\text{sd}(\cdot)$ denotes standard deviation.

Figure 2 also shows how RB (ϑ) affects the equilibrium interest rate (r^*). It is clear from the figure that the stronger the preference for robustness, the lower the equilibrium interest rate. From (32), we can see that RB will affect the dispersion of consumption by reducing the equilibrium interest rate. The following proposition summarizes the results about how the persistence coefficient of income changes the effect of RB on μ .

²⁴In Section 4.1, we will discuss the choice of these preference parameters and provide more details about how to estimate the income process using the U.S. panel data. The main result here is robust to the choices of these parameter values.

Proposition 7. *Using (32), we have:*

$$\frac{\partial \mu}{\partial \theta} = \frac{\rho}{(r^* + \rho)^2} \frac{\partial r^*}{\partial \theta} < 0$$

because $\rho > 0$ and $\partial r^ / \partial \theta < 0$.*

Proof. The proof is straightforward. ■

4. Quantitative Analysis

In this section, we first describe how we estimate the income process and calibrate the robustness parameter. We then present quantitative results on how RB affects the equilibrium interest rate and relative dispersion of consumption to income.

4.1. Estimation of the Income Process

To implement the quantitative analysis, we need to first estimate ρ and σ_y in the income process specification (4). We use micro data from the the Panel Study of Income Dynamics (PSID). Following Blundell *et al.* (2008), we define the household income as total household income (including wage, financial, and transfer income of head, wife, and all others in household) minus financial income (defined as the sum of annual dividend income, interest income, rental income, trust fund income, and income from royalties for the head of the household only) minus the tax liability of non-financial income. This tax liability is defined as the total tax liability multiplied by the non-financial share of total income. Tax liabilities after 1992 are not reported in the PSID, and so we estimate them using the TAXSIM program from the National Bureau of Economic Research (NBER). We report details on sample selection in Online Appendix A.

To eliminate possible heteroskedasticity in the income measures, we regress each on a series of

demographic variables to remove variation caused by differences in age and education. We next subtract these fitted values from each measure to create a panel of income residuals. We then use this panel to estimate the household income process as a stationary AR(1) process with Gaussian innovations:

$$y_t = \phi_0 + \phi_1 y_{t-1} + \sigma \varepsilon_t, \quad t \geq 1, \quad |\phi_1| < 1, \quad (33)$$

where $\varepsilon_t \sim N(0, 1)$, $\phi_0 = (1 - \phi_1) \bar{y}$, \bar{y} is the mean of y_t , and the initial level of labour income y_0 are given. Once we have estimates of ϕ_1 and σ , we can recover the drift and diffusion coefficients in the Ornstein-Uhlenbeck process specified in (4) by rewriting (33) in the time interval $[t, t + \Delta t]$ as

$$y_{t+\Delta t} = \phi_0 + \phi_1 y_t + \sigma \sqrt{\Delta t} \varepsilon_{t+\Delta t}, \quad (34)$$

where $\phi_0 = \kappa (1 - \exp(-\rho \Delta t)) / (\rho \Delta t)$, $\phi_1 = \exp(-\rho \Delta t)$, $\sigma = \sigma_y \sqrt{(1 - \exp(-2\rho \Delta t)) / (2\rho \Delta t)}$, and $\varepsilon_{t+\Delta t}$ is the time- $(t + \Delta t)$ standard normal distributed innovation to income.²⁵ As the time interval, Δt , converges to 0, (34) reduces to the OU process, (4). The estimation results and the recovered persistence and volatility coefficients in (4) are reported in Table 2.

4.2. Calibration of the Robustness Parameter

We adopt the calibration procedure outlined in Anderson *et al.* (2003) to calibrate the value of the RB parameter (ϑ) that governs the degree of robustness. Specifically, we calibrate ϑ by using the method of detection error probabilities (DEP) that is based on a statistical theory of model selection. We can then infer what values of ϑ imply reasonable fears of model misspecification for empirically-plausible approximating models. The model detection error probability denoted by $p(\vartheta)$ is a measure of how far the distorted model can deviate from the approximating model without being discarded; low values for this probability mean that agents are unwilling to discard

²⁵Note that here we use the fact that $\Delta B_t = \varepsilon_t \sqrt{\Delta t}$, where ΔB_t represents the increment of a Wiener process.

many models, implying that the cloud of models surrounding the approximating model is large. Let model P denote the approximating model, (5), and model Q be the distorted model, (9). See Online Appendix D for the detailed description of the calibration procedure.

In Online Appendix D, we show that the DEP, $p(\vartheta)$, can be written as:

$$p(\vartheta) = \Pr\left(x < \frac{\bar{b}}{2}\sqrt{N}\right), \quad (35)$$

where $\bar{b} \equiv b^*\sigma_s = -\vartheta r^*\sigma_s/\psi$, N is the sample size, and x follows a standard normal distribution. From the expressions of \bar{b} and $p(\vartheta)$, it is clear that the value of p is decreasing with the value of ϑ . Under the observational equivalence condition between the multiplier and constraint robustness formulations, (35) can be rewritten as $p(\vartheta) = \Pr\left(x < -\sqrt{2\eta}\sqrt{N}\right)$, where η is the upper bound on the distance between the two models and measures the consumer's tolerance for model misspecification.

We first explore the relationship between the DEP (p) and the value of the RB parameter, ϑ . A general finding is a negative relationship between these two variables. The upper panels of Figure 3 illustrates how p varies with the value of ϑ for different values of EIS (ψ) and CARA (γ).²⁶ We can see from the figures that the stronger the preference for robustness (higher ϑ), the less the DEP (p) is. Following the literature on precautionary savings (e.g., Caballero, 1990; Wang, 2003), we set $\gamma = 3$. The magnitude of the EIS (ψ) is an open and unresolved question, as the literature has found a very wide range of values. Vissing-Jorgensen and Attanasio (2003) estimate the EIS to be well in excess of one, while Campbell (2003) estimates a value well below one (and possibly zero). Guvenen (2006) finds that stockholders have a higher EIS (around 1.0) than non-stockholders

²⁶Based on the estimation results, we set $\bar{y} = 1$, $\sigma_y = 0.182$, and $\rho = 0.083$. The implied CRRA in our CARA utility specification can be written as either γc or γy . Given that the value of the CRRA is very stable and \bar{b} can be expressed as $r(\vartheta/\psi)\sigma_y/(r+\rho)$, proportional changes in the mean and standard deviation of y do not change our calibration results because their effects on γ and σ_y cancel each other out.

(around 0.1). Havránek (2015) surveys the vast literature and suggests that a range around 0.3-0.4 is appropriate after correcting for selective reporting bias, while Crump *et al.* (2015) find that the EIS is precisely and robustly estimated to be around 0.8 in the general population using the newly released FRBNY Survey of Consumer Expectations (SCE). For example, let $\psi = 0.5$, then $p = 0.403$ and $r^* = 2.83$ percent when $\vartheta = 1$, while $p = 0.163$ and $r^* = 1.99$ percent when $\vartheta = 5$. Both values of p are reasonable as argued in Anderson *et al.* (2003), Maenhout (2004), and Hansen and Sargent (2007, ch.7). In other words, a value of ϑ below 5 is reasonable in this case in which $\gamma = 3$ and $\psi = 0.5$. Furthermore, from the two upper panels of Figure 3, we can also see that the DEP increases with both ψ and γ for given values of ϑ , and the impact of a change in ψ is much larger than that of a change in γ . The intuition is that a change in the EIS has two channels to affect the values of \bar{b} and p : (i) the direct channel and (ii) the indirect channel via affecting the general interest rate channel (r^*), and the direct channel dominates the indirect channel. In contrast, a change in CARA only affects the values of \bar{b} and p via the indirect equilibrium interest rate channel, which is relatively weak. Furthermore, using (20), in this case, we have $\mu_{\gamma\vartheta} = 1.5$ and 0.3 when we set $\vartheta = 1$ and 5 , respectively. That is, the relative importance of risk aversion to RB in determining the precautionary savings demand decreases with the value of ϑ , holding other parameters constant.

The two lower panels of Figure 3 illustrate how DEP (p) varies with ϑ for different values of σ_y and ρ if $\psi = 0.5$ and $\gamma = 3$.²⁷ It also shows that the higher the value of ϑ , the less the DEP (p). In addition, to calibrate the same value of p , smaller values of σ_y (less volatile labor income processes) or higher values of ρ (less persistent income processes) lead to higher values of ϑ . The intuition behind this result is that σ_s and ϑ have opposite effects on \bar{b} and then p (see (35)).

As emphasised in Hansen and Sargent (2007), in the robustness model, p is a measure of the amount of model uncertainty, whereas ϑ is a measure of the agent's aversion to model uncertainty.

²⁷Since $\sigma_s = \sigma_y / (r + \rho)$, both changes in the persistence coefficient (ρ) and changes in volatility coefficient (σ_y) will change the value of σ_s .

If we keep p constant when recalibrating ϑ for different values of γ , ρ , or σ_y , the amount of model uncertainty is held constant—that is, the set of distorted models with which we surround the approximating model does not change. In contrast, if we keep ϑ constant, p will change accordingly if the values of γ , ρ , or σ_y change; in this case, the amount of model uncertainty is “elastic” and will change accordingly as the agent’s aversion to uncertainty changes.

4.3. Effects of RB on the Equilibrium Interest Rate and Consumption Dispersion

The equilibrium interest rate and relative dispersion of consumption to income are jointly determined by the degree of robustness, risk aversion, intertemporal substitution, and the income process. To better see how RB affects the equilibrium interest rate and the relative dispersion, we present two quantitative exercises here. The first exercise fixes the parameters of the income process at the estimated values and allows the risk aversion and intertemporal substitution parameters to change, while the second exercise fixes the risk aversion and intertemporal substitution parameters and allows the key income process parameter to vary.

Figure 4 shows that the equilibrium interest rate and the equilibrium relative consumption dispersion decrease with the calibrated value of ϑ for different values of ψ and γ when $\sigma_y = 0.182$ and $\rho = 0.083$. For example, if ϑ is increased from 1 to 5 (p decreases from 0.403 to 0.163), r^* falls from 2.63% to 1.98% and μ falls from 0.241 to 0.193, given $\psi = 0.5$ and $\gamma = 3$.²⁸ In addition, the figure also shows that the interest rate and the relative dispersion decrease with γ and increase with ψ for different values of ϑ .

Our model has the potential to explain the observed low real interest rate in the U.S. economy; see Laubach and Williams (2015) or Hall (2017) for evidence on low real rates. One of our theoretical results shows that a stronger aversion to model uncertainty lowers the equilibrium real interest rate. In the US, the average real risk-free interest rate has been about 1.87% between 1981 and 2010

²⁸In the RE case, $r^* = 2.92\%$ and $\mu = 0.26$.

if we use CPI to measure inflation, and about 1.96% if we use PCE to measure inflation.²⁹ Therefore, depending on what inflation index is used, the risk-free rate is between 1.87% and 1.96%. In our following discussion, we set the risk-free rate to be 1.91%, which is the average of the two real interest rates under CPI and PCE. Using the equilibrium condition, we find that the RE model without RB requires the coefficient of risk aversion parameter to be 23 to match this rate if $\psi = 0.8$, and requires the coefficient to be 14.5 if $\psi = 0.5$.³⁰

In contrast, when consumers take into account model uncertainty, the model can generate an equilibrium interest rate of 1.91% with much lower values of the coefficient of risk aversion.³¹ Figure 5 shows the relationship between γ and ϑ for interest rates equal to 1.91% for different values of ψ . For example, if $\psi = 0.5$, the RB model with $\gamma = 4.5$ and $\vartheta = 5$ leads to the same interest rate as in the RE model with $\gamma = 14.5$. Note that $\gamma = 4.5$ is much lower than the risk aversion levels used in most macro-asset pricing models. Using the same calibration procedure discussed in Section 4.1, we find that the corresponding DEP is $p = 0.171$. In other words, agents tolerate a 17.1% probability that they cannot distinguish the distorted model from the approximating model. We have summarised these results in Table 3. As argued in Hansen and Sargent (2007) and in Section 4.2, this value is viewed as reasonable in the literature.

The explanation that agents have become more concerned about model misspecification after the 2007-2009 financial crisis does not seem unreasonable given the long and deep recession that generated skepticism (at least in the popular press) about whether the standard macro models fully

²⁹Following Campbell (2003), we calculate the average of the real 3-month Treasury yields. Here we choose the 1981 – 2010 period because it is more consistent with our sample period of the panel data in estimating the joint consumption and income process. When we consider an extended period from 1981 to 2015, the real interest rate is 1.37% when using CPI and is 1.75% when using PCE. Hall (2017) finds that real rates (computed using TIPS) have been consistently falling for several decades, so we are overstating the current rate.

³⁰Note that since we set the mean income level to be 1, the coefficient of CRRA evaluated at this level is equal to the coefficient of CARA.

³¹This result is comparable to that obtained in Barillas *et al.* (2009). They find that most of the observed high market price of uncertainty in the U.S. can be reinterpreted as a market price of model uncertainty rather than the traditional market price of risk.

capture the key features of the economy. To provide a numerical example, under our calibrated parameter values, $\psi = 0.5$, and $\gamma = 4.5$, an increase in model uncertainty reflected by a reduction in the DEP from $p = 0.297$ to $p = 0.171$ (an increase in ϑ from 2.5 to 5) leads to a reduction in the equilibrium interest rate from 2.21% to 1.91%.

To examine how RB affects the relative dispersion of consumption to income ($\mu = \text{sd}(dc_t^*) / \text{sd}(dy_t)$), we follow Luo *et al.* (2017) and construct a panel data set that contains both consumption and income at the household level. Figure 1 shows the relative dispersion of consumption to income between 1980 and 2010.³² From the figure, the average empirical value of the relative dispersion (μ) is 0.377 for the 1980-1996 period and 0.326 for the 1980-2010 period. The minimum and maximum values of the empirical relative dispersion from 1980 to 2010 are 0.195 (year 2006) and 0.55 (year 1982), respectively. From the expression for the equilibrium relative dispersion (32), we can see that when the real interest rate is low, it is impossible for the model to generate sufficiently high relative dispersion of consumption to income without using an implausible value for ρ . For example, when $r^* = 1.91\%$, we obtain $\mu = 0.19$, which is well below the average value $\mu = 0.326$. To get $\mu = 0.326$, we would need $\rho = 0.039$, a value that can be rejected given our estimated value of $\rho = 0.082$. Because this moment matters for the welfare calculations which are the focus of the paper, in the next section, we will resolve the disparity between data and model using a risky asset in positive net supply.

5. Extension to an RU-RB Model with Multiple Financial Assets

5.1. Model Specification

In this section, we follow Viceira (2001) and Maenhout (2004), and assume that consumers can assess two financial assets: one risk-free asset and one risky asset. Our aim here is to resolve the

³²See Online Appendix A for more details on how we constructed the panel.

anomaly from the benchmark model regarding the relative dispersion of consumption to income at low interest rates. Specifically, the consumer can purchase both a risk-free asset with a constant interest rate r and a risky asset (the market portfolio) with a risky return r_t^e . The instantaneous return dr_t^e of the risky market portfolio over dt is given by:

$$dr_t^e = (r + \pi) dt + \sigma_e dB_{e,t}, \quad (36)$$

where π is the market risk premium, σ_e is the standard deviation of the market return, and $B_{e,t}$ is a standard Brownian motion. Let ρ_{ye} be the contemporaneous correlation between the labor income process and the return of the risky asset. If $\rho_{ye} = 0$, then the labor income risk is purely idiosyncratic, so the risky asset does not provide a hedge against declines in labor income. The agent's financial wealth evolution is then given by:

$$dw_t = (rw_t + y_t - c_t) dt + \alpha_t (\pi dt + \sigma_e dB_{e,t}), \quad (37)$$

where α_t denotes the amount of wealth that the investor allocates to the market portfolio at time t .

As in the benchmark model, we define a new state variable, s_t : $s_t \equiv w_t + h_t$, where h_t is human wealth at time t and is defined as the expected present value of current and future labour income discounted at the risk-free interest rate r : $h_t \equiv E_t [\int_t^\infty \exp(-r(s-t)) y_s ds]$. Following the same state-space-reduction approach used in the benchmark model, the budget constraint can be written as:

$$ds_t = (rs_t - c_t + \pi\alpha_t) dt + \sigma dB_t, \quad (38)$$

where $\sigma dB_t = \sigma_e \alpha_t dB_{e,t} + \sigma_s dB_{y,t}$, $\sigma_s = \sigma_y / (r + \rho)$, and

$$\sigma = \sqrt{\sigma_e^2 \alpha_t^2 + \sigma_s^2 + 2\rho_{ye} \sigma_s \sigma_e \alpha_t} \quad (39)$$

is the unconditional variance of the innovation to s_t .

5.2. Consumption and Portfolio Rules Under RB

To introduce robustness into the above recursive utility model, we follow the same procedure as in Section 2.3 and write the distorting model by adding an endogenous distortion $b(s_t)$ to the law of motion of the state variable s_t , (38):

$$ds_t = (rs_t - c_t + \pi\alpha_t) dt + \sigma(\sigma b(s_t) dt + dB_t). \quad (40)$$

Following the same procedure we used in solving the benchmark model, we can also solve the multiple-asset case explicitly. The following proposition summarises the solution to the dynamic problem.

Proposition 8. *Under robustness, the consumption function, the portfolio rule, and the saving function are*

$$c_t^* = r \left(s_t - \frac{\pi\rho_{ye}\sigma_s\sigma_e}{r\sigma_e^2} \right) + \Psi(r) - \Gamma(\vartheta, r) + \Pi(\vartheta, r), \quad (41)$$

$$\alpha^* = \frac{\pi}{r\tilde{\gamma}\sigma_e^2} - \frac{\rho_{ye}\sigma_s\sigma_e}{\sigma_e^2}, \quad (42)$$

and

$$d_t^* = x_t + \Gamma - \Psi + \Pi, \quad (43)$$

respectively, where $x_t \equiv \rho(y_t - \bar{y}) / (r + \rho)$ is the demand for savings “for a rainy day,”

$$\Gamma(\vartheta, r) \equiv \frac{1}{2}r\tilde{\gamma} \left(1 - \rho_{ye}^2 \right) \sigma_s^2 \quad (44)$$

is the demand for precautionary savings due to the interactions of income uncertainty, intertemporal substi-

tution, and risk and uncertainty aversion;

$$\Psi(r) \equiv \psi \left(\frac{\delta}{r} - 1 \right) \quad (45)$$

captures the saving demand of relative patience;

$$\Pi(\vartheta, r) \equiv \frac{\pi^2}{2r\tilde{\gamma}\sigma_e^2} \quad (46)$$

is the additional saving demand due to the higher expected return of the risky asset; and $\tilde{\gamma} \equiv \gamma + \vartheta/\psi$ is the effective coefficient of absolute risk aversion. Finally, the worst possible distortion is $b^* = -r(\vartheta/\psi)$.

Proof. See Online Appendix E. ■

Expression (41) shows that the presence of the risky asset in the agent's investment opportunity has two effects on current consumption. First, it reduces the risk-adjusted certainty equivalent human wealth by $\pi\rho_{ye}\sigma_s\sigma_e / (r\sigma_e^2)$ because the agent faces more risk when holding the risky asset. Second, it increases current consumption because it offers a higher expected return. In general equilibrium, the second effect dominates the first effect. Furthermore, from the definition of individual saving, we can see that the presence of $\pi\alpha^*$ term has the potential to increase saving because it offers a higher expected return. Combining these two effects, it is clear from (43) that the net effect of the risky asset on current saving is governed by $\Pi > 0$ defined in (46). In addition, since the risky asset can be used to hedge labour income risk (provided the correlation is not zero), it will reduce the precautionary saving demand arising from income uncertainty by a factor $1 - \rho_{ye}^2 \in (0, 1)$. It is thus clear from (43) that there are four saving motives in the model with a risky asset. The first three saving motives— x_t , Γ , and Ψ —are the same as that mentioned in our benchmark model. The fourth term captures the additional saving demand due to the higher expected return of the risky

asset and obviously does not appear in the benchmark model.

Since the effective coefficient of absolute risk aversion depends on both the EIS and the degree of RB, it is clear from (42) that even if the consumer only has a constant investment opportunity set, the optimal share invested in the risky asset not only depends on risk aversion, but also depends on intertemporal substitution if $\vartheta > 0$.³³

5.3. General Equilibrium Implications

We first consider the equilibrium in the market for the risky asset. Assuming that the net supply of the risky asset is $\bar{\alpha} \geq 0$, the equilibrium condition in the market for the risky asset is:

$$\bar{\alpha} = \frac{\pi}{r\tilde{\gamma}\sigma_e^2} - \frac{\rho_{ye}\sigma_s\sigma_e}{\sigma_e^2} \quad (47)$$

for a given risk free rate, r .

Using the individual saving function (43) and following the same aggregation procedure used in the previous section, we have the following result on savings:

Proposition 9. *The total demand of savings “for a rainy day” equals zero for any positive interest rate.*

That is, $F_t(r) = \int_{y_t} x_t(r) d\Phi(y_t) = 0$, for $r > 0$.

Proof. The proof uses the LLN and is the same as that in Wang (2003). ■

Using this result, from (43), after aggregating across all consumers, the expression for total savings can be written as:

$$D^{total}(\vartheta, r) \equiv \Gamma(\vartheta, r) - \Psi(r) + \Pi(\vartheta, r), \quad (48)$$

³³A constant investment opportunity set means a constant interest rate, a constant expected return on risky assets, and a constant volatility of the returns on risky assets.

where $\Gamma(\vartheta, r)$, $\Psi(r)$, and $\Pi(\vartheta, r)$ are given in (44), (45), and (46), respectively. To compare $D(\vartheta, r)$ with the aggregate saving function obtained in the benchmark model, we rewrite $D(\vartheta, r)$ as follows:

$$D^{total}(\vartheta, r) = \tilde{\Gamma}(\vartheta, r) - \Psi(r) + \tilde{\Pi}(\vartheta, r), \quad (49)$$

where $\tilde{\Gamma}(\vartheta, r) = r\tilde{\gamma}\sigma_s^2/2$, $\tilde{\Pi}(\vartheta, r) = \bar{\alpha}r\tilde{\gamma}\rho_{ye}\sigma_s\sigma_e + \bar{\alpha}^2r\tilde{\gamma}\sigma_e^2/2$, and $\bar{\alpha}$ is determined by (47). Comparing the two aggregate saving functions, $\tilde{\Pi}(\vartheta, r)$ is an additional term due to the positive net supply of the risky asset in this model. As in the benchmark model, we still assume that the net supply of the risk-free asset is zero in equilibrium, i.e., an equilibrium interest rate r^* satisfies:

$$D(\vartheta, r^*) \equiv \Gamma(\vartheta, r^*) - \Psi(r^*) = 0, \quad (50)$$

where $D(\vartheta, r^*)$ denotes the amount of saving in the risk-free asset. The following proposition proves that an equilibrium exists and that the PIH is satisfied.

Proposition 10. *Suppose the true economy is governed by the approximating model. In equilibrium, each consumer's optimal consumption-portfolio rules are described by:*

$$c_t^* = r^* s_t, \quad (51)$$

and

$$\alpha^* = \bar{\alpha}, \quad (52)$$

respectively. Furthermore, in equilibrium, the evolution equation of s_t is:

$$ds_t = \left(\frac{\pi^{*2}}{r^*\tilde{\gamma}\sigma_e^2} \right) dt + \sigma dB_t, \quad (53)$$

and

$$\pi^* = r^* \tilde{\gamma} \sigma_e (\rho_{ye} \sigma_s + \bar{\alpha} \sigma_e). \quad (54)$$

If $\rho > \delta$ and $\rho_{ye} \geq 0$, the equilibrium is unique.

Proof. See Online Appendix E. ■

Figure 6 shows that the aggregate saving function $D(\vartheta, r)$ is increasing with the interest rate for different values of ρ_{ye} .³⁴ It also clearly shows that there exists a unique interest rate r^* for every given ρ_{ye} such that $D(\vartheta, r^*) = 0$, and a higher correlation between the equity return and labour income leads to a higher equilibrium interest rate given ϑ . The intuition behind this result is that the presence of the risky asset helps hedge labour income risk, leading to less precautionary savings. However, we can see from the figure that ρ_{ye} does not have significant effects on the equilibrium interest rate. The following result is an immediate implication on how the presence of the risky asset affects the relative dispersion of consumption growth to income growth under RB.

Proposition 11. *The relative dispersion of changes in consumption to changes in income is*

$$\mu \equiv \frac{\text{sd}(dc_t^*)}{\text{sd}(dy_t)} = r^* \sqrt{\left(\frac{1}{r^* + \rho}\right)^2 + \left(\frac{\sigma_e}{\sigma_y}\right)^2 \bar{\alpha}^2 + 2 \frac{\rho_{ye}}{r^* + \rho} \frac{\sigma_e}{\sigma_y} \bar{\alpha}}. \quad (55)$$

Comparing (32) with (55), it is clear that the positive net supply of the risky asset will be helpful at increasing the relative dispersion of consumption to income while keeping the real interest rate at a low level. To quantitatively examine the effects of RB on the relative dispersion of consumption growth to income growth, we first use the observed risk premium of 7.2 percent to calibrate the value of $\bar{\alpha}$ using (47). Estimating the correlation between individual labor income and the equity return is complicated by the lack of panel data on household portfolio choice, and we find

³⁴Here, we set $\gamma = 4.5$, $\psi = 0.5$, $\vartheta = 2.9$, and $\delta = 0.03$. The parameters in the income process are the same as before.

several estimates in the literature: Viceira (2001) adopts $\rho_{ye} = 0.35$ when simulating his life-cycle consumption-portfolio choice model. Davis and Willen (2000) estimate that the correlation is between 0.1 and 0.3 for college-educated men, and is 0.25 or more for college-educated women. Here, we follow Viceira (2001) and set $\rho_{ye} = 0.35$. In our model, if $\gamma = 4.5$, $\psi = 0.5$, $\vartheta = 2.9$, $\delta = 0.03$, and $\rho_{ye} = 0.35$, the corresponding DEP (p) is 0.19, the equilibrium interest rate (r^*) is 1.91%, and the relative dispersion (μ) is 0.31, which equals the empirical counterpart for the sample from 1980 to 1996. We summarise these results in Table 3.

5.4. Explaining the Decline in the Relative Consumption Dispersion

To test the model's predictions on the effects of robustness on the dynamic relative consumption dispersion, we quantitatively examine how well a calibrated version of our extended model can explain the decline in the relative consumption dispersion from 1980 to 2010 (see Figure 1). To do this, we divide our sample into two periods, 1980-1995 and 1996-2010. We calibrate our model to the first period by choosing the robustness parameter (ϑ) and the aggregate asset supply parameter ($\bar{\alpha}$) to match the observed real interest rate (r) and the relative consumption dispersion (μ_{cy}). To focus on the effects of robustness, we fix the other parameters at the same values used in the previous section. Then, we let the robustness parameter vary to match the real interest rate in the second period. In other words, this is the amount of model uncertainty the model needs to explain the decline in the real interest rate from the first period to the second period. Finally, we check how much this change in the amount of model uncertainty can explain the decline in the relative consumption dispersion from the first period to the second period.

The results in Table 4 show the model does a good job of explaining the decline in the relative consumption dispersion. The first row shows that the model with $\bar{\alpha} = 8.8$ and $\vartheta = 0.57$ matches the average real interest rate of 3.1% and the average relative consumption dispersion of 0.39 in

the first period, 1980-1995. To generate a real interest rate of 1.5% in the second period, 1996-2010, we need ϑ to increase from 0.57 to 19, which corresponds to a decrease in the DEP from 0.46 to 0.06 (third column in the table). Remember a larger DEP means that either it is more difficult to distinguish the approximating model and the distorted model, or there is less model uncertainty. The calibrated results therefore suggest that both the degree of robustness and the amount of model uncertainty increased significantly from the first period to the second period. With this increase in the degree of robustness, the model predicts a decrease in the relative consumption dispersion from 0.39 to 0.21, which nearly matches the observed decrease in the data. This accurate out-of-sample prediction provides additional evidence that incorporating model uncertainty due to ambiguity and robustness can help the model to better explain the data.

6. The Welfare Cost of Model Uncertainty

The uncertainty about model specifications due to a preference for robustness or ambiguity aversion generates welfare losses. We measure the welfare cost of model uncertainty in a standard and intuitive way—the amount of wealth or income an average consumer is willing to pay to remove or reduce such uncertainty. In particular, we provide two approaches to calculate the welfare cost of model uncertainty from different angles. The first approach is based on Lucas' (1987) elimination-of-risk method that tells us how much the consumers would pay to fully resolve all model uncertainty. The second approach is based on Barro's (2009) local welfare analysis which allows us to answer questions such as “how much consumers would pay to reduce partial uncertainty, such as 10% of model uncertainty, in order to keep the level of lifetime utility unchanged.” We evaluate welfare costs based on our extended model that better accounts for data on the joint behavior of consumption, income, and asset returns than the benchmark model, consistent with Barro's (2009) argument that welfare and policy analyses of the impacts of consumption uncertainty should be carried out within models that can at least roughly capture the stylized facts in the asset markets.

However, our qualitative points remain valid in the benchmark model without the risky asset.

It is worth noting that although our welfare analysis under robustness is related to that discussed in the endowment economies proposed in Barillas *et al.* (2009) and Ellison and Sargent (2015), our paper is the first to examine the welfare costs of model uncertainty if consumers are allowed to choose optimal consumption-portfolio rules.

6.1. Total Welfare Gains from Eliminating Model Uncertainty

In this section, we follow Lucas's elimination-of-risk method (see Lucas, 1987; Obstfeld, 1994) to quantify the welfare cost of RB in the general equilibrium. It is worth noting that in Lucas (1987), the welfare cost of aggregate fluctuations is expressed in terms of percentage of consumption the representative agent in the *endowment* economy is willing to give up at all dates to switch to the deterministic world. However, such a welfare measure may not be very informative when the consumers can choose optimal consumption–saving–portfolio plans because the marginal propensity of consumption out of total wealth is now endogenous and affects consumption growth. We thus follow Obstfeld (1994) and compute the welfare gains of removing uncertainty as an equivalent variation: what percentage of initial wealth a typical consumer is willing to give up to be as well off in the RE economy as he is in the RB economy.³⁵

Specifically, we define:

$$\tilde{f}(s_0(1 - \Delta)) = f(s_0), \quad (56)$$

where

$$\tilde{f}(s_0(1 - \Delta)) = -\frac{\delta}{\tilde{\alpha}_1} \exp(-\tilde{\alpha}_0 - \tilde{\alpha}_1 s_0(1 - \Delta)) \text{ and } f(s_0) = -\frac{\delta}{\alpha_1} \exp(-\alpha_0 - \alpha_1 s_0) \quad (57)$$

³⁵Epaulard and Pommeret (2003) also use this approach to examine the welfare cost of volatility in a representative-agent model with recursive utility. In their model, the total welfare cost of volatility is defined as the percentage of capital the representative agent is ready to give up at the initial period to be as well off in a certain economy as he is in a stochastic one.

are the value functions under RE and RB, respectively, Δ is the compensating amount measured as a percentage of s_0 , the initial wealth,

$$\alpha_1 = \frac{r^*}{\psi}, \tilde{\alpha}_1 = \frac{\tilde{r}^*}{\psi},$$

$$\alpha_0 = \frac{\delta}{r^*} - 1 - \frac{1}{2} \frac{r^*}{\psi} \left(\gamma + \frac{\vartheta}{\psi} \right) (\sigma_s^2 - \bar{\alpha}^2 \sigma_e^2), \tilde{\alpha}_0 = \frac{\delta}{\tilde{r}^*} - 1 - \frac{1}{2} \frac{\tilde{r}^*}{\psi} \left(\gamma + \frac{\vartheta}{\psi} \right) (\tilde{\sigma}_s^2 - \bar{\alpha}^2 \sigma_e^2),$$

and r^* and \tilde{r}^* are the equilibrium interest rates in the RE and RB economies, respectively.³⁶ The following proposition summarises the result about how RB affects the welfare costs in general equilibrium.

Proposition 12. *The welfare costs due to model uncertainty are given by:*

$$\Delta = \frac{s_0 (\tilde{\alpha}_1 - \alpha_1) - \ln(\tilde{\alpha}_1/\alpha_1) + (\tilde{\alpha}_0 - \alpha_0)}{\tilde{\alpha}_1 s_0} = \left(1 - \frac{r^*}{\tilde{r}^*} \right) + \frac{\psi}{\tilde{c}_0} \ln \left(\frac{\tilde{r}^*}{r^*} \right) + \frac{\psi (\tilde{\alpha}_0 - \alpha_0)}{\tilde{c}_0}, \quad (58)$$

where $\tilde{c}_0 = \tilde{r}^* s_0$ is optimal consumption under RE.

Proof. Substituting the equilibrium condition (25) into the expressions of α_0 and $\tilde{\alpha}_0$ in the value functions under RE and RB, we obtain that:

$$\alpha_0 = \frac{1}{\psi} \bar{\alpha} \pi, \tilde{\alpha}_0 = \frac{1}{\psi} \bar{\alpha} \tilde{\pi}.$$

Combining these results with (56) yields (58). ■

Note that when $\bar{\alpha} = 0$, our multiple-asset model reduces to the benchmark model and $\Delta = \left(1 - \frac{r^*}{\tilde{r}^*} \right) + \frac{\psi}{\tilde{c}_0} \ln \left(\frac{\tilde{r}^*}{r^*} \right)$. To understand how the welfare cost varies with the degree of uncertainty

³⁶When we compare welfare in these two economies, we assume that the asset supply is the same across the two economies. See Online Appendix E for the derivation of the value functions. Note that $\Delta = 0$ if $\vartheta = 0$.

aversion, we note that:

$$\frac{\partial \Delta}{\partial \vartheta} = \frac{\partial \Delta}{\partial r^*} \frac{\partial r^*}{\partial \vartheta}.$$

The second term is negative, $\partial r^* / \partial \vartheta < 0$, for the reasons we have already discussed. The first term is

$$\frac{\partial \Delta}{\partial r^*} = -\frac{1}{\tilde{r}^*} \left(1 - \frac{\psi}{\tilde{c}_0} \right) - \frac{\bar{\alpha}}{\tilde{c}_0} \tilde{\gamma} \sigma_e (\rho_{ye} \sigma_s + \bar{\alpha} \sigma_e);$$

for reasonable values, we expect this term to be negative, so that higher model uncertainty leads to larger welfare costs.

To provide some quantitative results, we set $c_0 = \tilde{r}^* s_0 = 1$ as we did in our calibrated model. Figure 7 illustrates how the welfare cost of model uncertainty varies with ϑ for different values of $\bar{\alpha}$ and ρ_{ye} . The left panel of the figure shows that the welfare costs of model uncertainty are nontrivial. For example, when $\vartheta = 2.9$, the value we calibrate to match the data (see Table 3), the welfare cost of model uncertainty Δ is 15.1%. That is, a typical consumer is willing to sacrifice 15.1% of his initial wealth in order to get rid of such model uncertainty. Furthermore, the welfare cost rises with the degree of model uncertainty. For instance, if ϑ increases by 50% (from 2.9 to 4.35), Δ increases by about 25% (from 15.1% to 19.0%). It is worth noting that these values of welfare costs are not directly comparable to that obtained in Lucas (1987) because our welfare calculations are based on removing all model uncertainty when individual households make optimal consumption-saving-portfolio choices, while Lucas-type calculations are based on removing business cycles fluctuations in an *endowment* representative agent economy.³⁷

In addition, the same figure shows that the welfare cost decreases with $\bar{\alpha}$, given the value of ϑ . The reason behind this result is that the risky asset provides a hedging tool for the consumer as long as $\rho_{ye} \neq 0$, and higher supply of the asset means that agents' inefficient precautionary savings

³⁷Lewis (2000) finds that the welfare gains from international risk sharing based on equity returns are around 10 to 50 percent. He argues that the large difference is due to the high volatility of equity returns and the implied intertemporal substitution in the marginal utility.

motives are weaker. The right panel of Figure 7 shows that the welfare cost of model uncertainty also decreases with the correlation between the equity return and labor income. The intuition behind this result is that the higher correlation between the two risks can make the consumers better hedge the fundamental risk, which reduces the welfare cost of uncertainty.

6.2. The Local Welfare Effects of Model Uncertainty

In this section, we examine the *local* effects of RB on welfare costs. Rather than removing all uncertainty as we did in Section 6.1, here we define a cost that measures the welfare benefits from reducing model uncertainty at the margin. This marginal analysis is useful because most economic policies would not be designed to eliminate uncertainty entirely and thus calculating the potential benefits at the marginal may be useful in and of itself.

To examine the *local* effects of RB on welfare, we follow Barro (2009) and Luo and Young (2010) to compute the marginal welfare costs due to model uncertainty at different degrees of robustness (ϑ). The basic idea of this calculation is to use the value function (57) to calculate the effects of RB on the expected lifetime utility and compare them with those from proportionate changes in the initial income level. Specifically, following Barro (2009), the marginal welfare costs (mwc) due to RB can be written as:

$$\text{mwc}(\vartheta) = -\frac{\partial f_t / \partial \vartheta}{(\partial f / \partial y_t) y_t} \Big|_{y_t=y_0} = \frac{1}{2\psi} \frac{\sigma_y^2 - \bar{\alpha}^2 \sigma_e^2 (r + \rho)^2}{(r + \rho) y_0}, \quad (59)$$

where $\partial f / \partial \vartheta$ and $\partial f / \partial y_t$ are evaluated in equilibrium for given y_0 .³⁸ The value of mwc provides the proportionate increase in initial income that compensates, at the margin, for an increase in the degree of robustness—in the sense of keeping the level of lifetime utility unchanged. From (57), it is clear that this compensating income change depends on the EIS, the properties of the income

³⁸Note that here we use the facts that $s_t = w_t + y_t / (r + \rho) + \bar{y} / (r(r + \rho))$ and $\sigma_s^2 = \sigma_y^2 / (r + \rho)^2$.

process, and the equilibrium interest rate.

To provide some quantitative results, we use the same set of parameter values in the above calibrated model: $y_0 = 1$, $\gamma = 4.5$, $\psi = 0.5$, $\rho = 0.083$, $\sigma_y = 0.182$, $\bar{x} = 10.5$, $\sigma_e = 0.16$, and $r = 1.91\%$. Based on these parameter values, we can calculate that $mwc = 0.193$, which suggests that a 10 percent increase in ϑ (from 2.9 to 3.2) requires an increase in initial income by 1.23% (i.e., $mwc \cdot 0.1 \cdot \vartheta = 0.193 \cdot 0.1 \cdot 2.9 = 1.23$) to make his lifetime utility unchanged. In other words, a typical consumer in our model economy would be willing to sacrifice 1.23% of his initial income to reduce the degree of model uncertainty by 10%. In addition, from (59) we can see that mwc is an increasing function of income volatility (σ_y), which means the marginal welfare cost will be larger if the economy is in a more volatile environment. To quantitatively see this point, let's assume σ_y increases by 20% from 0.182 to 0.22. Following the same calculation above, under this more volatile environment, the welfare costs of a 10% increase in ϑ leads to a welfare loss equivalent to a 5.61% decline in initial income, significantly larger than the 1.23% under the low volatility environment.³⁹ This highlights the potentially larger welfare losses due to model uncertainty when the economy is facing larger income volatility, such as during economic crises. It is worth noting that the higher the supply of the risky asset, the less the welfare costs of model uncertainty. The main reason behind this result is that the presence of the risky asset can hedge the income risk and help the consumers better insure the risk and uncertainty they face. In addition, the marginal welfare cost is also decreasing with the value of the EIS because the higher the value of EIS, the less the impact of RB on total welfare of uncertainty-averse households.

In addition, we can also examine the *local* effects of income uncertainty on the welfare cost of model uncertainty for different degrees of RB. Specifically, the marginal welfare costs (mwc) due

³⁹Reducing the value of ρ can generate a similar result because either an increase in σ_y or a reduction in ρ lead to greater income uncertainty facing the consumers.

to income uncertainty can be written as:

$$\text{mwc} \left(\sigma_y^2 \right) = - \frac{\partial f_t / \partial \sigma_y^2}{(\partial f / \partial y_t) y_t} \Big|_{y_t = y_0} = \frac{1}{2} \frac{\tilde{\gamma}}{(r + \rho) y_0}, \quad (60)$$

where $\partial f / \partial \sigma_y^2$ and $\partial f / \partial y_t$ are evaluated in equilibrium for given y_0 . The value of mwc gives us the proportionate increase in y_0 to compensate for a small increase in σ_y^2 in the sense of keeping the level of lifetime utility unchanged. This formula can help us evaluate the importance of economic policies that aim to reduce income uncertainty of households. It is also clear from (60) that the higher the effective coefficient of risk aversion ($\tilde{\gamma}$), the greater the welfare costs of income uncertainty. The intuition behind this result is that more risk- and uncertainty-averse consumers suffer more from income uncertainty.

Let's consider the following simple policy experiment. The government is implementing a macro policy to reduce the variance of household income by 10 percent, from σ_y^2 to $0.9 \cdot \sigma_y^2$. Holding all the other parameters fixed, we can calculate that the 10% reduction in income variance leads to a welfare improvement equivalent to a 16.9% increase in the initial income. In other words, a typical household is willing to reduce his initial income by 16.9% to reduce the variance of his income process by 10%. As a comparison, this welfare gain is only 7.4% of the initial income if there is no model uncertainty. One policy implication stemming from this finding is that macro policies aiming to reduce income volatility and inequality are more beneficial in an economy in which consumers have a greater aversion to model uncertainty, both because they reduce risk and because they mitigate costly precautionary saving.⁴⁰ Finally, although our benchmark model has no business cycle dynamics, the above welfare calculations can still help us infer some insight about the welfare costs of business cycles under RB. Note that one key fact about the US business

⁴⁰Ellison and Sargent (2015) find that idiosyncratic consumption risk has a greater effect on the cost of business cycles when agents fear model misspecification. In addition, they showed that endowing agents with fears about misspecification leads to greater welfare costs caused by the idiosyncratic consumption risk. The underlying reasons are the same: the enhanced risk aversion created by uncertainty aversion.

cycles is that income volatility is countercyclical.⁴¹ Specifically, when the economy moves from an expansion into a recession, σ_y will increase. At the same time, consumers may become more concerned about model misspecification and will thus suffer more from model uncertainty. Consequently, a macro policy aiming to reduce the aggregate fluctuations would be more beneficial to this economy.

7. Conclusions and Future Research

We develop a tractable continuous-time recursive utility version of the Huggett (1993) model to explore how the preference for robustness affects the equilibrium interest rate and the dynamics of consumption and income. We find that for moderate risk aversion and plausibly calibrated fear of model misspecification, our benchmark model can generate the observed low risk-free rate in the US economy. Unfortunately, the model cannot generate the observed high relative dispersion of consumption to income. But if we allow for a positive net supply of a risky asset, our model is able to reconcile low interest rates, moderate risk aversion, and relatively high dispersion of consumption to income. The welfare costs of model uncertainty are sizable: a 10% increase in the degree of model uncertainty leads to a welfare cost equivalent to a 1.23% reduction in initial income.

To better illustrate the key effects of robustness on the equilibrium interest rate and relative consumption dispersion, we choose a framework that can analytically show the key mechanisms and mathematically is as simple as possible. However, our key insights can also be carried to more complicated cases. For example, our framework can be extended to study implications of robustness in a hidden-state model in which the consumers cannot perfectly observe the growth of their stochastic labor income. As discussed in Hansen and Sargent (2007, ch. 17 and ch. 18), in

⁴¹As discussed in Bloom (2014), unemployment rises during a recession, so the volatility of income at the household level will increase as well.

this case, agents' preference for robustness not only affects their optimal control problem but also affects their optimal filtering problem. The effect of robustness on optimal filtering also provides additional information that could be used to further distinguish ambiguity aversion (a preference for robustness) from risk aversion. We leave these extensions for future research.

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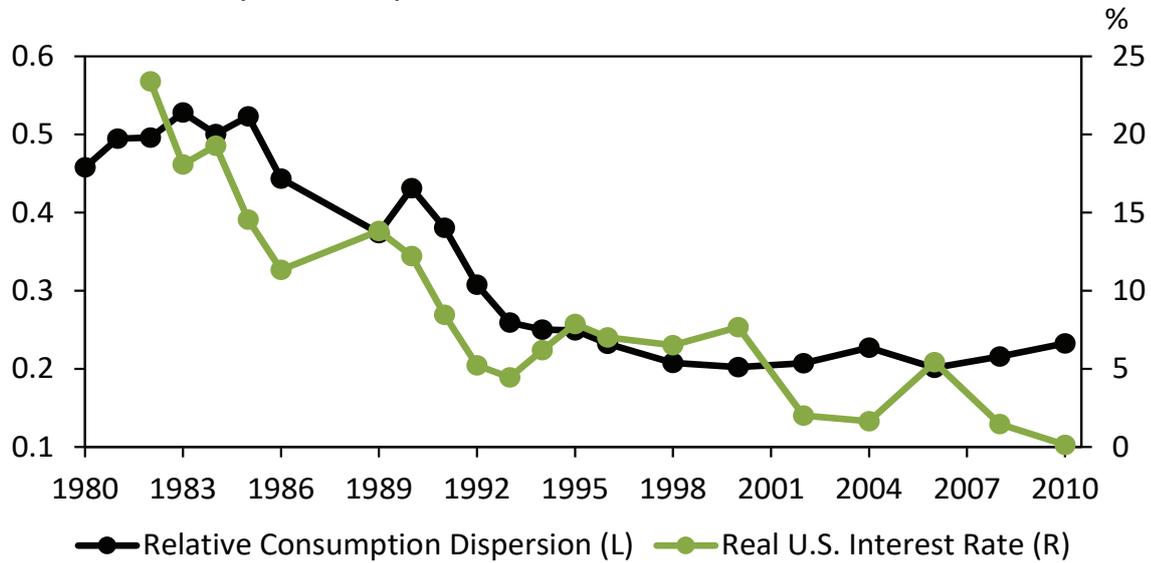
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Note: "Relative consumption dispersion" is defined as the standard deviation of consumption changes to the standard deviation of income changes; "Real U.S. interest rate" is defined as the 3-month Treasury yield deflated by the PCE index

Figure 1. Real Risk-free Rates and Relative Consumption Dispersion.

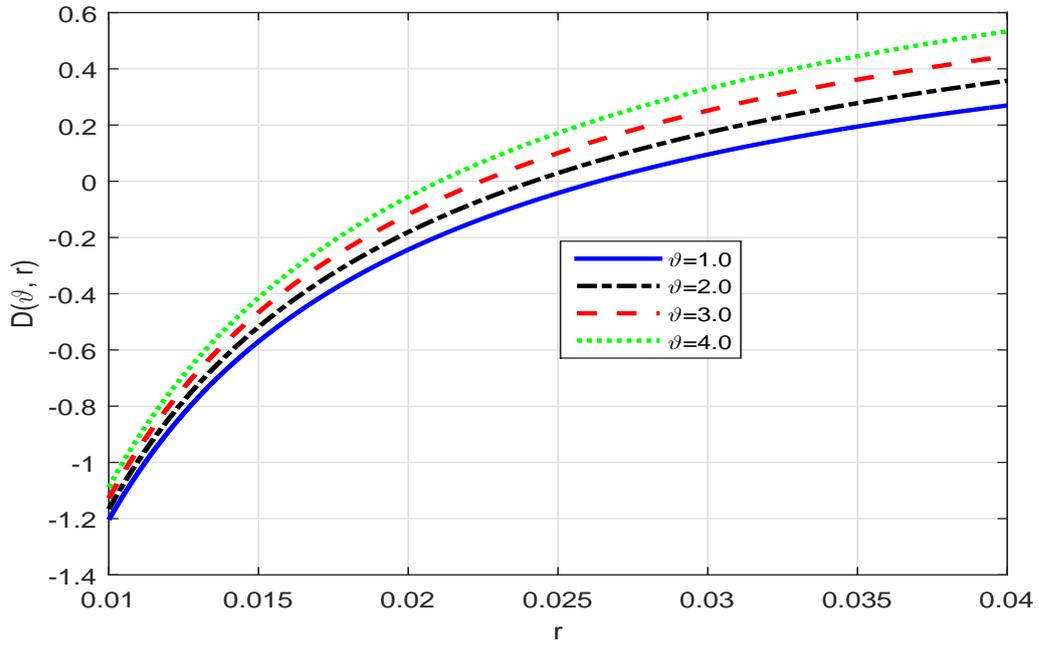


Figure 2. Effects of RB on Aggregate Savings.

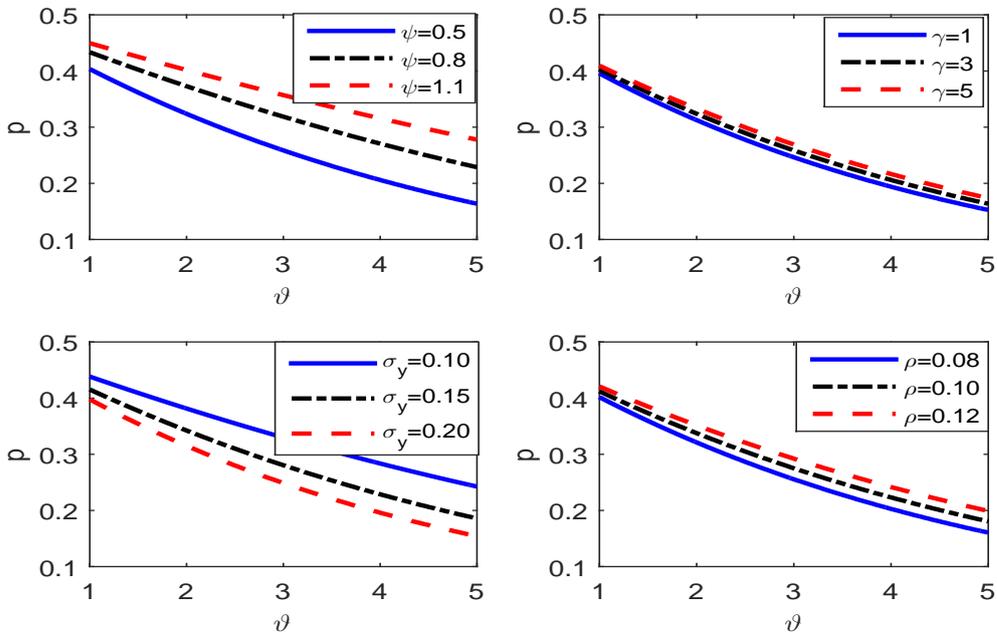


Figure 3. Relationship between ϑ and p .

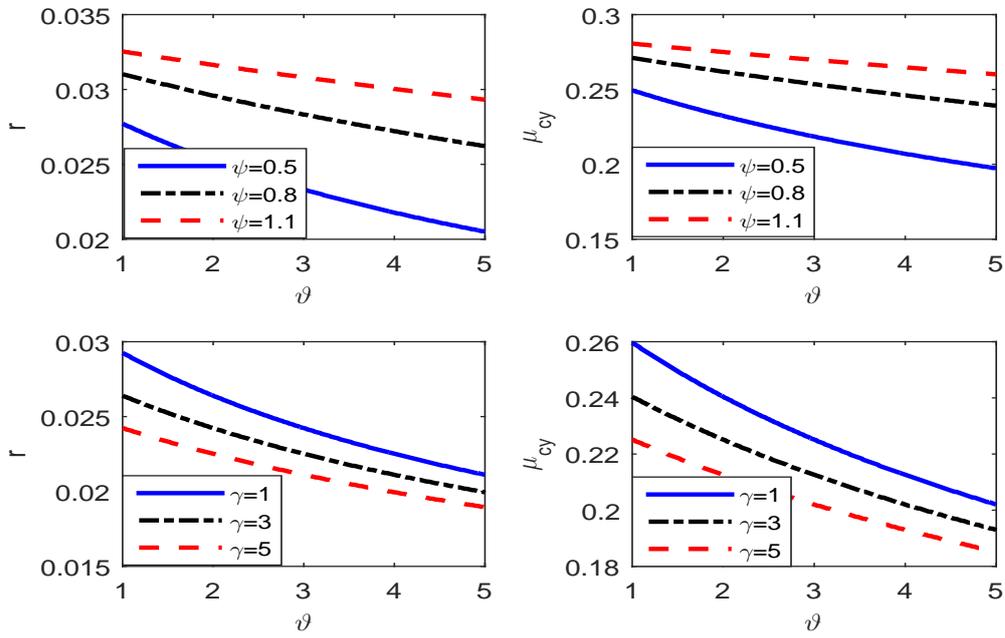


Figure 4. Effects of RB on the Interest Rate and Consumption Volatility.

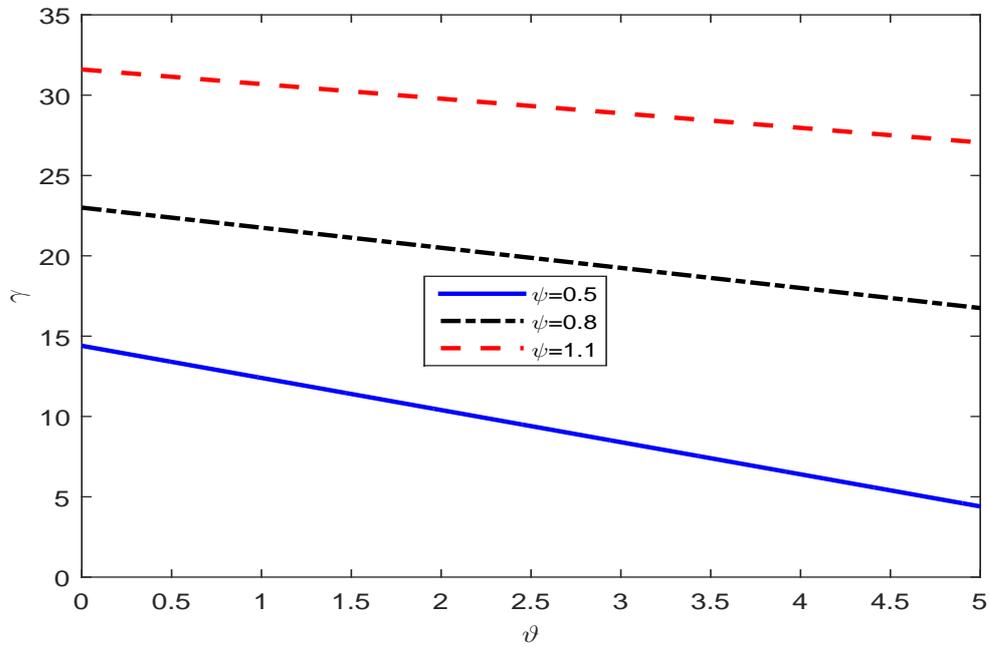


Figure 5. Relationship between γ and ϑ .

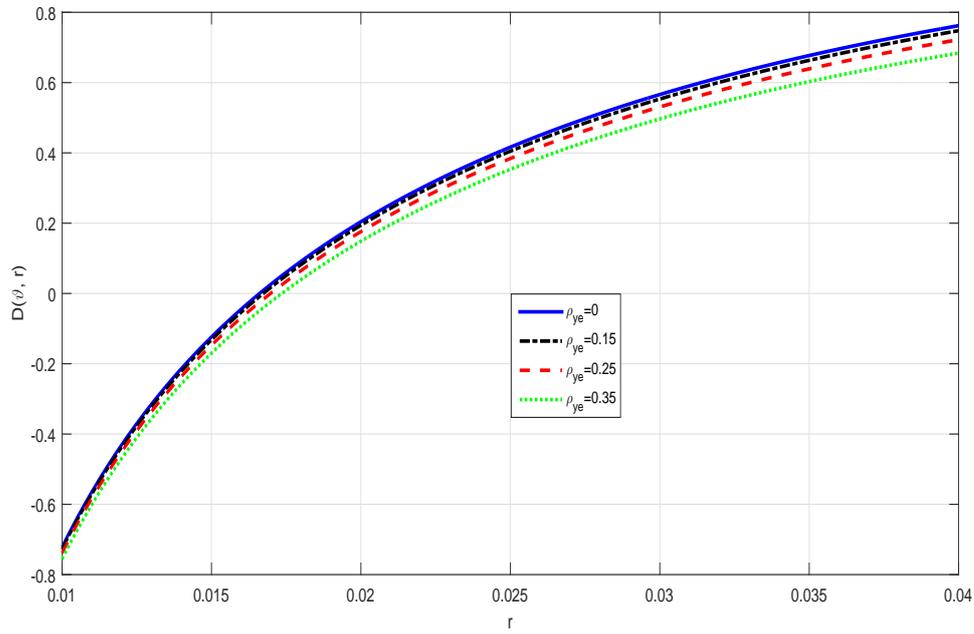


Figure 6. Effects of RB on Aggregate Savings.

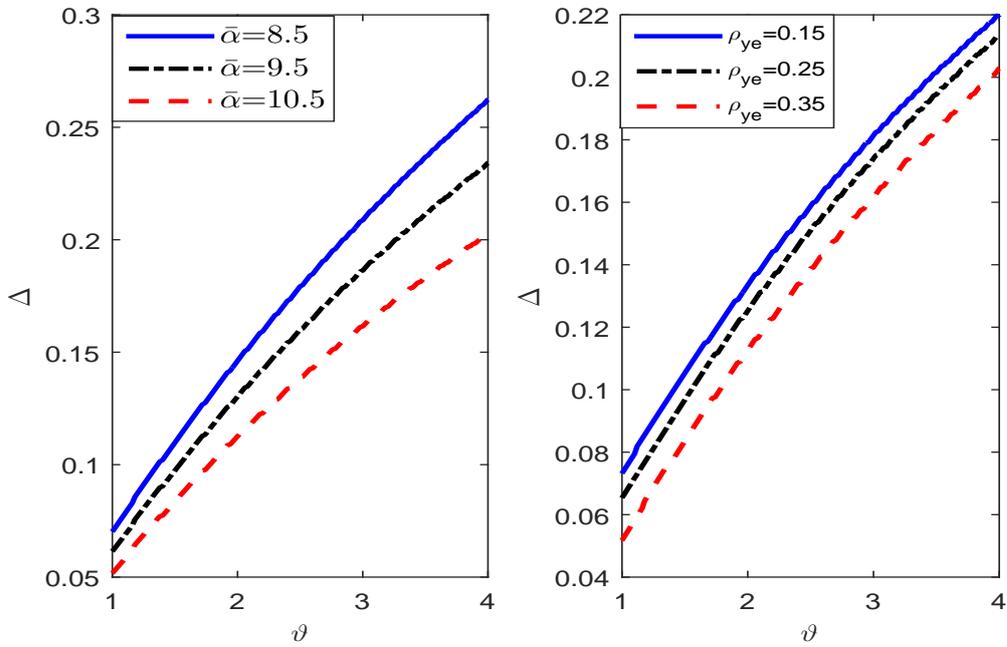


Figure 7. Effects of RB on the Welfare Cost.

Table 1. Measures of the Risk Free Rate

	Three-month Nominal T-Bond	One-year Nominal T-Bond
CPI Inflation (1981 – 2010)	1.87%	2.33%
PCE Inflation (1981 – 2010)	1.96%	2.42%
CPI Inflation (1981 – 2015)	1.37%	1.78%
PCE Inflation (1981 – 2015)	1.75%	2.16%

Table 2. Estimation and Calibration Results

Labor Income	Parameter	Values
Discrete time specification		
Constant	ϕ_0	0.0005
Persistence	ϕ_1	0.919
Std. of shock	σ	0.175
Continuous-time specification		
Persistence	ρ	0.083
Std. of income changes	σ_y	0.182

Table 3. Model Comparison with Key Parameter Values

Parameters	Data	RE	RB (Benchmark)	RB ($\bar{\alpha} = 10.5$)
Moments				
γ		14.5	4.5	4.5
ϑ		0	5	2.9
p		0.5	0.17	0.19
r^*	1.91%	1.91	1.91	1.91
π^*	7.2%	n.a.	n.a.	7.2%
μ	0.31	0.19	0.19	0.31

Table 4. Explaining the Decline in Relative Consumption Dispersion

Period	Key Parameter		DEP	Data		Model	
	$\bar{\alpha}$	ϑ	p	r	μ_{cy}	r^*	μ_{cy}^*
1980 – 1995	7.3	1	0.41	3.1%	0.39	3.1%	0.39
1996 – 2010	7.3	13.5	0.02	1.5%	0.22	1.5%	0.21