Price Competition of Spreaders in Profit-maximizing Sponsored Viral Marketing

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***Abstract*—In online social networks (OSNs), celebrities are usually paid to promote products via posting or forwarding ads or related information. Imagine that one day, everyone is allowed to register as a spreader and participate in the campaign to sell influence, how much money should be claimed? Two factors play vital roles in deciding the price. One is how influence is valued by buyers (advertisers). The other is how one's price is affected by that of others. In this paper, we consider that the influence is valued as the number of final 'activations' under some existing information diffusion processes, and focus on the latter, namely, the price competition.**

We model the scenario as a pricing game where spreaders compete with each other under selection policies of the advertiser, who is trying to maximize its profit. Three cases of the advertiser are considered. i) An omniscient advertiser always selects the optimal set of spreaders. We show that the competition is so fierce that each spreader can only claim its unique influence in the Nash equilibrium (NE), and the equilibrium is also unique. ii) The greedy advertiser selects spreaders using the simple-greedy algorithm. We deduce that the unique NE exists when the number of spreaders is less than 4; however, the existence of NE cannot be guaranteed when there are at least 4 spreaders. iii) The advertiser adopts a 'double-greedy' method that greedily selects spreaders one by one in accordance with their registration order. We conclude that the unique NE exists and the utility of the platform is at least 1/2 to the optimal and also bounded by 1/2 to the influence of all spreaders.

***Index Terms*—price competition, social networks, viral marketing, game theory.**

I. Introduction

WITH the prosperity of online social networks (OSNs), like Facebook or Twitter, sponsored viral marketing has become increasingly prevalent. It sponsors influential users in OSNs, usually celebrities, as *spreaders* to publish or forward advertising information so as to promote products. New emerging advertising companies, such as TapInfluence.com, even enables anyone to register as a potential spreader with self-claimed prices ([1]). The advantage of such a marketing strategy is that it leverages the 'word-of-mouth' effect. People trust and will be influenced by the ones they follow. Once a user got influenced and forwarded the information, a.k.a. *activated*, he/she in turn influences his/her own followers and helps spread the information further. The final number of activated users in the OSN weighs the influence of those spreaders, and sponsoring them is actually buying their influence.

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Depicted in Fig. 1, the general framework of sponsored viral marketing consists of three parties: sponsor, platform, and spreader ([2]). *Sponsors* are usually companies, which initiate and pay for the viral marketing campaign of its products. Due to the lack of knowledge of viral marketing in OSNs, the sponsor delegates the task to a professional agent, possibly an advertising company or the OSN host, which we named *platform*, to carry out the campaign. The platform selects among users, who have registered for the campaign, as *spreaders* and pays them to conduct the actual promotion, posting advertising information for instance. The information will diffuse in the OSN and hopefully forwarded by a large number of users, i.e. goes 'viral'.

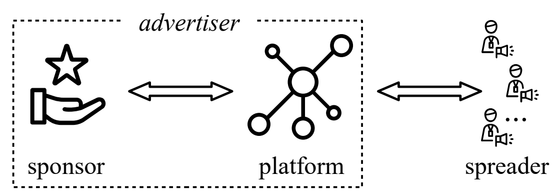

Fig. 1: The general framework of sponsored viral marketing

In real life, multiple actors exist in each party. A sponsor has multiple platforms to choose from, and conversely, a platform also has many sponsors to cooperate with. Similarly, the platform and the spreader can also choose who to work with. Moreover, these actors are autonomous and rational, meaning that they will try to maximize their own utilities facing competition and choices. However, considering the whole picture is beyond the scope of this paper. We aggregate the sponsor and the platform as a profit-maximizing *advertiser* and focus more on the pricing competition of spreaders in a single marketing campaign.

The selection of spreaders is vital in sponsored viral marketing. Actually, the problem of selecting k spreaders such that their influence is maximized, known as the *Influence Maximization* problem (IMP), has been studied for over a decade. People have proposed various information propagation models ([3]–[7]) to capture the diffusion process and proved the submodularity of the influence ([8]), i.e. the final number of activated users. IMP was thus substantially tranformed into a submodular maximzation problem subject to a cardinality constraint. It is proved to be NP-hard and the simple-greedy algorithm is the best to achieve $1-1/e$ to the optimality ([3], [9]). More efforts were focused on designing heuristics ([10]–[12]) or techniques to accelerate the computation process ([13]–[16]).

Another line of research related to the selection strategy is called Unconstrained Submodular Maximization (USM). It tries to maximize a non-negative submodular function, which is allowed to be non-monotone ([9], [17], [18]). The state-of-the-art solution is the 'double-greedy' algorithm invented by Buchbinder et al. [18], achieving $1/2$ approximation to optimality with its randomized version and $1/3$ with the deterministic version. We will show that similar to IMP and USM, the profit(utility) function of the advertiser is also submodular but not guaranteed to be non-negative, which results in no algorithms will have a guaranteed performance if no additional assumptions are made ([17]). However, both algorithms, simple-greedy algorithm for IMP and double-greedy algorithm for USM, are used in the literature on profit maximization ([19], [20]) with some minor assumptions. Therefore, we assume three cases of the selection policies: i) The ideal condition in which the advertiser always selects the optimal set of spreaders. ii) The simple-greedy algorithm is adopted ([19]). iii) The double-greedy algorithm is adopted ([20]).

With the selection policy known to the spreaders, the price competition is modeled as a game. Clearly, a higher price is more appealing to a spreader, but it reduces its chance to be selected at the same time. Spreaders will determine their own prices considering their influences, peer prices and also the selection policy. The existence and uniqueness of the Nash equilibrium (NE) is the focus of this paper.

We note that this scenario is closely related to works specifically studying pricing games ([21], [22]) and especially, case i) coincides with the model in [21]. Comparing with our previous work [23], we summarize the contributions of this work as follows:

- We explicitly address the case of an omniscient advertiser that always selects the optimal set of spreaders. Consistent with the model of [21], we show that the unique NE exists.
- In the case of simple-greedy advertiser, theorems are revised and proved with specific examples. We show that the unique NE exists when spreaders are fewer than 4, and present examples that have no NE when there are at least 4 spreaders.
- We design a selection strategy for the advertiser based on the double-greedy algorithm, and prove that the unique NE exists. We further prove that under the equilibrium, the profit for the advertiser is at least $1/2$ to the optimal and also bounded by $1/2$ to the influence of all spreaders.

The rest of this paper is organized as follows. In Section II, we formally introduce the pricing game. In Sections III, IV and V, we analyze pure NE with an omniscient platform, the platform adopting the simple-greedy algorithm, and double-greedy algorithm, respectively. Finally, we conclude this paper in Section VI.

II. Pricing Game in Sponsored Viral Marketing

In this section, we first define sponsored viral marketing and specify the utilities of each party. Then we discuss their properties and elaborate on the spreader selection algorithms. Finally, we propose the pricing game and formulate the equilibrium set of interest.

A. Sponsored Viral Marketing

We use a graph $\mathcal{G} = (V, E)$ to represent the OSN, where V and E are the set of users and directed social links, respectively. $(u, v) \in E$ means v 'follows' u and is able to see information posted by u. In other words, u has social influence on v. We use $T \subseteq V$ to represent the set of all registered potential spreaders, and their registration order is $[t_1, t_2, ..., t_{|T|}]$. Correspondingly, $V \setminus T$ represents the set of 'ordinary' users. For each potential spreader, $i \in T$, promoting information incurs a private cost c_i and its claimed price is p_i.

After the price set $\boldsymbol{p} := \{p_i \mid i \in T\}$ is proposed to the advertiser, a set of spreaders $X(\boldsymbol{p}) \subseteq T$ will be selected by the selection policy X to conduct the marketing campaign via spreading advertisements on the OSN site. Their influence unfolds as some of their followers sees and forwards the information so that more users can see and become a forwarder in turn. We call a user forwarding the information an *activation*, and assume it to be the standard unit for all transactions. This inherently assumes that the advertiser and the spreader agree on the valuation of a single activation and we note that it is just for ease of expression. The model can be easily extended to incorporate hierarchical valuations by multiplying different coefficients.

If we denote $\sigma(S)$ as the expected number of final activations and $p(S) = \sum_{i \in S} p_i$ as the money paid to spreaders, where S represents a set of spreaders, the expected profit of the advertiser given price set $\boldsymbol{p}$ can be expressed as:

$$u_a^X(\boldsymbol{p}) = u_a(X(\boldsymbol{p})) := \sigma(X(\boldsymbol{p})) - \sum_{i \in X(\boldsymbol{p})} p_i$$

For each spreader $i \in T$, clearly, $p_i \geq c_i$ and the utility is the profit $p_i - c_i$ if i is selected; otherwise, it is simply zero. Therefore, the utility of spreader i can be expressed as:

$$u_i^X(\boldsymbol{p}) = u_i^X(p_i | \boldsymbol{p}_{-i}) := (p_i - c_i) \cdot \mathbb{I}\{i \in X(\boldsymbol{p})\}$$

where $\boldsymbol{p}_{-i} := \boldsymbol{p} \setminus \{p_i\}$ and $\mathbb{I}\{\cdot\}$ is the indicator function. In this paper, we assume that $c_i \approx 0$, i.e. neglect the cost, and leave the case of arbitrary cost in our future work.

B. Properties of u_a^X and the Selection Policy X

The influence function σ has been proved to be non-negative, monotone and submodular within various information diffusion models ([3]–[7]). The monotone property implies that $\sigma(S)$ grows with the expansion of S, while, known as the 'diminishing return' property, submodularity implies that $\forall\ M \subseteq N \subseteq T$ and $\forall\ i \in V$,

$$\sigma(i|M) \geq \sigma(i|N)$$

where $\sigma(i|M) := \sigma(M \cup \{i\}) - \sigma(M)$ and similarly, for $\sigma(i|N)$. In the following, if no confusion will be caused, we omit the parentheses for set parameters. For instance, $\sigma(x)$ is short for $\sigma(\{x\})$.

Although computing of σ is proved to be #P-hard ([14]), it is possible to compute a metric in polynomial time ([10], [11],

[16]) to approximate it. For example in [10], a degree-based index was proposed as a substitute of Monte-Carlo evaluation with a comparable accuracy. Therefore, instead of specifying the diffusion model, we assume σ to be a *value oracle* that maps subsets of T to non-negative real numbers, and retains the properties of information diffusion models. We note that in real life, it is possible to have such an oracle under agreements of all parties.

Submodularity is the essential property that makes greedy algorithms work with guaranteed performance for IMP and USM. It can be easily proved that the utility u_a^X is still submodular. However, the non-negative and monotone properties do not hold any longer. This can be seen by a toy example with only two potential spreaders, $T = \{i,\ j\}$, where $\sigma(i) = \sigma(j) = 10$, $\sigma(T) = 15$, and $p_i = p_j = 8$. We can find that:

$$u_a(T) = \sigma(T) - p(T) = -1 < 0$$

$$u_a(i) = \sigma(i) - p_i = 2 > u_a(T)$$

The selection policy X tries to find the optimal spreader set $S^* \subseteq T$ to maximize the profit:

$$\sigma(S^*) - p(S^*)$$

As pointed out by Feige et al. [17], no efficient approximation algorithm can be found for general submodular maximization since verifying whether the maximum of the function is greater than zero is NP-hard and requires exponentially many queries.

However, it is still valuable to consider the simple-greedy algorithm as [19] have shown that, when prices of spreaders are relatively low, the performance degenerates linearly with reference to the cardinality of the selected set. The double-greedy algorithm still has its performance guarantee after a pruning process of spreaders. By ruling out those candidates with excessively high prices, the non-negativity of the utility is achieved. We refer to [20] for the detailed realization. Therefore, we address three cases of X, namely, the 'omniscient' policy X_o that always selects the optimal S^*; the policy X_s based on the simple-greedy algorithm; and the policy X_d based on the double-greedy algorithm.

C. Pricing Game of Spreaders

Given the value oracle σ and spreader selection policy $X \in \{X_o, X_s, X_d\}$, we are able to define the pricing game among spreaders. Each potential spreader i, as an autonomous and rational actor, will try to propose an optimal p_i to maximize its utility. Naturally, they will compete with each other since a higher price is more desirable while on the other hand, it also reduces its chance to be selected. We model this scenario as a game and assume that both σ and X are known to all players.

In this paper, we focus on studying the set of pure Nash equilibria of the pricing game,

$$NASH^X = \{\boldsymbol{p} \in R_+^{|T|} \mid u_i^X(p_i|\boldsymbol{p}_{-i}) \geq u_i^X(p_i'|\boldsymbol{p}_i), \\ \forall p_i' \in \mathbb{R}^+, \forall i \in T\}$$

which is the set of price vectors that no player wants to deviate unilaterally. If $NASH^X$ is not empty, we can expect spreaders' prices will converge to some price vectors other than an arbitrary one.

III. The Pricing Game with X_o

Given the price vector $\boldsymbol{p}$, the advertiser with spreader selection policy X_o always selects the set S^*:

$$S^* \in \arg\max_{S \subseteq T} u_a(S)$$

Note that such S^* may not be unique. In case of ties, we assume that X_o will choose the one with most spreaders, so that the decision is deterministic.

In [21], Babaioff et al. studied such a scenario in online combinatoral markets in which a single buyer with a combinatorial preference to purchase goods from n sellers, where each of them supplies only one product and individually decides their prices. They assumed a *demand correspondence* that selects the optimal sets of sellers to maximize the utility of the buyer and defined a *decision map* that chooses the optimal set deterministically if ties happen. We can treat spreaders in sponsored viral marketing as sellers and the advertiser as the sole buyer. Therefore, by results in [21], we can immediately draw the conclusion on $NASH^{X_o}$:

Theorem 1. *$NASH^{X_o}$ has the unique element $\boldsymbol{p}$ such that $\forall i \in T$,*

$$p_i = \sigma(i|T \setminus i)$$

and $X_o(\boldsymbol{p}) = T$.

It means that in equilibrium, each spreader i will claim the price as the influence solely brought by it, i.e. $\sigma(T) - \sigma(T \setminus i)$. It can be easily shown that anyone claiming a higher price will not be selected.

IV. The Pricing Game with X_s

Although the result with X_o is neat, it is unpractical to assume an omniscient policy. In this section, we assume the selection policy is based on the simple-greedy algorithm. Together with tie-breaking rules, we elaborate the policy in detail with pseudo-codes, following which, we analyze $NASH^{X_s}$.

A. The Simple-Greedy Policy X_s

The detailed process is described in Algorithm 1. Similar to the widely used simple-greedy algorithm in IMP ([3]), X_s iteratively selects the candidate that is currently 'optimal'. We address the differences with the original algorithm as follows:

- Trying to maximize u_a, in each iteration, X_s should compare marginal utility of adding each candidate spreader into S: $u_a(i|S)$, which is the marginal influence $\sigma(i|S)$ deduced by p_i (line 3).
- X_s shall not terminate as long as there is a candidate spreader to provide positive marginal gain. Moreover, a spreader providing zero marginal gain should be selected as well if others provide negative marginal gains. Therefore, the algorithm terminates when all available potential spreaders generate negative marginal gains (lines 5-7).
- Since we are considering pure NE, there should be a clear tie-breaking rule for the case that multiple spreaders bring the same maximal marginal gain in a single iteration. We adopt the rule that the bigger σ the better. If ties still happen, the lexicographically first will be selected (lines 8-14).

Algorithm 1 The simple-greedy policy X_s

Input: $\mathcal{G} = (V,\ E)$, T, $\sigma(\cdot)$, $\boldsymbol{p}$
Output: $S \subseteq T$
Initialisation: $S \leftarrow \emptyset$
while $T \setminus S \neq \emptyset$ **do**
  for all $i \in T \setminus S$ **do**
    calculate $u_a(i|S) = \sigma(i|S) - p_i$
  end for
  if $\max_{i \in T\setminus S}\ u_a(i|S) < 0$ **then**
    break
  end if
  $\text{max\_set} = \{\arg\max_{i \in T\setminus S}\ u_a(i|S)\}$
  if $|\text{max\_set}| > 1$ **then**
    $\text{max\_set} \leftarrow \{\arg\max_{j \in \text{max\_set}}\ \sigma(j)\}$
    if $|\text{max\_set}| > 1$ **then**
      $\text{max\_set} \leftarrow \{lexicographically \text{ first in max\_set}\}$
    end if
  end if
  $S \leftarrow S \cup \text{max\_set}$
end while
return S

B. $NASH^{X_s}$ when $|T| \leq 3$

In our previous work ([23]), we analyzed $NASH^{X_s}$ with the size of T. In the following, we revise the theorems while giving some better proofs. Before proceeding to the main conclusions, we specify several notations for ease of representation: $\forall i, j \in T$,

- $\delta_i := \sigma(i \mid T \setminus i)$. It represents the influence (number of activations) provided by i given all of its competitors are selected. In other words, it represents the *unique* influence of i. It should be noted that δ_i is strictly greater than 0 since even if i has no influence on any others, it can still bring one single activation, namely, itself. Similar as the price set, we use bold symbol $\boldsymbol{\delta}$ to denote the set $\{\delta_i \mid i \in T\}$, and use $\delta(i)$ interchangeably with δ_i.
- $\mu_{ij} := \sigma(i) + \sigma(j) - \sigma(\{i, j\})$. Its physical meaning is the *mutual* influence between i and j. As a natural extension, $\mu_{ij|S}$ is calculated with σ conditioned on S, and we use $\mu(i, j)$ and μ_{ij} interchangeably.
- If $\exists B_i(\boldsymbol{p}_{-i}) > 0$ such that:
$$u_i^X(p_i|\boldsymbol{p}_{-i}) = \begin{cases} p_i, & 0 < p_i < B_i(\boldsymbol{p}_{-i}) \\ 0, & p_i \geq B_i(\boldsymbol{p}_{-i}) \end{cases} \tag{1}$$
We call $B_i(\boldsymbol{p}_{-i})$ the *edge-price* of i and use $B_i(\boldsymbol{p}_{-i}) - \epsilon$ to denote the highest price that i will claim, where $\epsilon \to 0_+$ has its physical meaning as the smallest change in price.

We start with the following observations:

Lemma 1. *$\forall\ i \in T$, claiming price of δ_i implies $i \in X_s(\boldsymbol{p})$. Moreover, if $\exists\ \boldsymbol{p} \in NASH^{X_s}$, then $X_s(\boldsymbol{p}) = T$ and $p_i \geq \delta_i$.*

Proof: In any iteration t that the spreader i has not been chosen, if S_t is the set of selected spreaders, naturally $S_t \subseteq T \setminus i$. By the submodularity of u_a,
$$u_a(i \mid S_t) \geq u_a(i \mid T \setminus i)$$
Claiming δ_i implies $u_a(i \mid S_t) \geq 0$. Note that it satisfies any iteration, therefore, i must be selected before X_s terminates.

No one will claim a price lower than its δ in any NE since increasing to δ obviously is a better choice. Also, everyone will be selected in an NE because for any $i \notin X_s(\boldsymbol{p})$, choosing price δ_i increases its utility. ■

We can define the *social welfare* $W(\boldsymbol{p}, X)$ as the sum utility of all parties, i.e.
$$W(\boldsymbol{p}, X) := u_a(X(\boldsymbol{p})) + \sum_{i \in T} u_i^X(p_i \mid \boldsymbol{p}_{-i}) = \sigma(X(\boldsymbol{p}))$$
then if the Nash equilibria $NASH^{X_s}$ is not empty, Lemma 1 implies that
$$W(\boldsymbol{p}, X_s) = \sigma(T)$$
for any $\boldsymbol{p}$ being an NE price vector. That is to say, the maximal welfare is achieved by any NE, and thus both the *Price of Anarchy* and the *Price of Stability* equal one. In fact, such pure NE exists and is unique when $|T| \leq 3$.

Theorem 2. *A unique pure Nash equilibrium price set $\boldsymbol{p}$ exists when $|T| \leq 3$. In particular, $\boldsymbol{p} = \boldsymbol{\delta}$ when $|T| = 2$; while $|T| = 3$, if $T = \{x, y, z\}$,*
$$\begin{cases} p_x = \delta_x + [\min\{\mu_{xy}, \mu_{xz}\} - \mu_{yz}]^+ \\ p_y = \delta_y + [\min\{\mu_{yz}, \mu_{yx}\} - \mu_{zx}]^+ \\ p_z = \delta_z + [\min\{\mu_{zx}, \mu_{zy}\} - \mu_{xy}]^+ \end{cases} \tag{2}$$
and minus ϵ if any price is an edge-price.

The superscript $[\ \cdot\ ]^+$ represents taking non-negative values only. It is worthy noting that when $|T| = 2$, the conclusion coincides with Theorem 1 that each player is forced to claim the lowest price δ, i.e., its unique influence, in the NE. However, when one more spreader exists, one player may get more utilities than δ in NE. We adapt our proofs in [23] for the consistency.

Proof: First, consider the case of $T = \{x, y\}$. Clearly, $\boldsymbol{\delta}$ is an NE because
$$u_a(x) = u_a(y) = \mu_{xy}$$
then either one in T claiming a higher price alone will not get selected in the second iteration (it fails in the first iteration).

Now suppose $\exists \boldsymbol{p'} \neq \boldsymbol{\delta}$ is also an NE, then both of p'_x and p'_y must be higher than their δ prices. The one left in the second iteration will not be selected by X_s since it provides a negative marginal profit. This contradicts Lemma 1 that all players will be selected in an NE. Therefore, $\boldsymbol{\delta}$ is the only NE.

Then for the case of $T = \{x, y, z\}$, we can immediately state that an NE price vector must contain at least two δ prices. Assuming the NE prices, after the first iteration, say x was selected, the remaining two boil down to the case of solely two spreaders, except that all values of σ are conditioned on x. Therefore, p_y will be $\sigma(\{y, z\}|x) - \sigma(z|x) = \sigma(y|\{x, z\}) = \delta_y$. Similarly, we know $p_z = \delta_z$.

By symmetry, we can assume $\mu_{xy} \geq \mu_{yz} \geq \mu_{zx}$ and further separate it into two situations:

1) $\mu_{xy} \geq \mu_{yz} = \mu_{zx}$. In other words, equations in (2) reduce to $\boldsymbol{p} = \boldsymbol{\delta}$. We prove that $\boldsymbol{\delta}$ is the unique NE.

By Lemma 1, we know all of them will be selected. Particularly in the first iteration of X_s, we have

$$u_a(x) = \sigma(x) - \sigma(x|yz) = \gamma - \mu_{yz}$$

where $\gamma := \sum_{i \in T} \sigma(i) - \sigma(T)$. By symmetry and the conditions of μ, we have

$$u_a(x) = u_a(y) \geq u_a(z)$$

Therefore, x or y will be selected according to the tie-breaking rule. If either one, say x, is selected, then neither of the remaining two, namely, y or z, will change its price. We can easily see x will not change its price either; otherwise, y and z will be selected one by one and x will not be selected eventually.

For the uniqueness, if we assume $\exists \boldsymbol{p'} \neq \boldsymbol{\delta}$ is also an NE, $\boldsymbol{p'}$ must have at least two prices greater than the corresponding δ. By the statement above, we know $\boldsymbol{p'}$ cannot be an NE.

2) $\mu_{xy} \geq \mu_{yz} > \mu_{zx}$. Equations in (2) can be restated as:

$$\begin{cases} p_x = \delta_x \\ p_y = \delta_y + \mu_{yz} - \mu_{zx} \\ p_z = \delta_z \end{cases}$$

Then we have

$$u_a(y) = u_a(x) = \gamma - \mu_{yz} \geq u_d(z) = \gamma - \mu_{xy}$$

We note that $\delta_y + \mu_{yz} - \mu_{zx}$ becomes an edge-price B_y if y fails the first iteration by tie-breaking rules. Because if so, $p_y \geq \delta_y + \mu_{yz} - \mu_{zx}$ implies y will fail the first iteration, and the subsequent two iterations as well, which can be seen by the arguments of $|T| = 2$. Therefore, y is selected firstly and cannot unilaterally increase its price to get a higher utility. By the same arguments in situation 1), x and z will not increase their prices unilaterally. Therefore, the price set is an NE.

For the uniqueness, we argue that if another NE $\boldsymbol{p'}$ exists, it must be that x or z has a price greater than δ, say $p_x > \delta_x$. Note that in this case, y and z will be selected in the first two iterations, and x will not be selected in the third iteration, which violates Lemma 1.

By the arguments above, we can see the unique NE exists when $|T| \leq 3$ and the detailed prices are stated by equations (2). ■

To better understand Theorem 2, we can illustrate the case of $T = \{x, y, z\}$ as shown in Fig. 2. Circles with different colors represent the influences of different spreaders, and the intersection of two circles means their mutual influence μ. The one opposite the smallest μ can take an extra utility in the NE. In the situation 2) of the proof, i.e. $\mu_{xy} \geq \mu_{yz} > \mu_{zx}$, y can claim a price $\mu_{yz} - \mu_{zx}$ higher than δ_y in the NE. Intuitively, it is because y has the largest 'substituting power': $\mu_{xy} + \mu_{yz}$.

However, in the case of $|T| \geq 4$, which is much more common in real life, the existence of NE cannot be guaranteed.

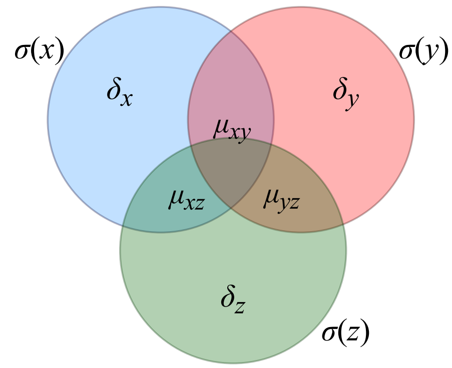

Fig. 2: Illustration of the case of $T = \{x, y, z\}$

C. $NASH^{X_s}$ when $|T| \geq 4$

Theorem 3. *The existence of NE cannot be guaranteed when $|T| \geq 4$.*

We prove the theorem by constructing cases in which NE does not exits. Especially, we consider the value oracle that

$$\sigma(S) = |neighbor(S) \cup S|$$

i.e. the node coverage function. It is non-negative, monotone, submodular and can be treated as a naive representation of influence. Under such settings, we can state that:

Proposition 1. *If $\mathbf{p}$ is an NE price set with graph $\mathcal{G}$, such that spreaders in T are selected in order $(s_1, s_2, s_3...)$, then $\mathbf{p}_{-s_1}$ is an NE in graph $\mathcal{G}_{-s_1}$, which is a subgraph of $\mathcal{G}$ with s_1 and its covered nodes excluded, and the selection order still holds.*

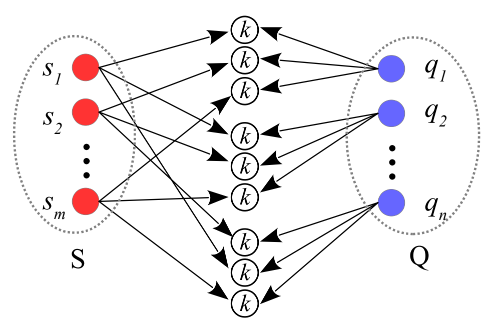

Fig. 3: Topology for Lemma 2

We first observe the following example shown in Fig. 3. The spreaders are separated into two categories: S and Q, which are represented by red and blue circles, respectively. For each pair of (s, q), $\forall s \in S$ and $\forall q \in Q$, $\mu(s, q) = k$, i.e. they mutually cover $k \in \mathbb{Z}^+$ ordinary users, who are illustrated by a hollowed circle with letter k. We further assume $\sigma(s) > \sigma(q)$, meaning s always wins when they are in a tie. Each q has no more mutual connections with any other spreader, while spreaders in S can have some mutual coverages, which are not shown in the figure. Moreover, each spreader may have its own distinct coverage.

Lemma 2. *In the graph shown in Fig. 3. If for any $m \leq m_0$ and $n = n_0$, with $m_0, n_0 \in \mathbb{Z}^+$, NE exists and for each NE,* *spreaders in Q are selected after S, then for $n = n_0 + 1$, NE still exists, and the selection order holds as well.*

Proof: We use $\boldsymbol{p}^{m,n}$ to represent the price set of spreaders when $|S| = m, |Q| = n$. First, we can see if $\boldsymbol{p}^{m_0,n_0}$ is an NE and Q is selected after S, then $\boldsymbol{p}^{m_0,n_0}(q_i) = \delta(q_i)$.

Then when q_{n_0+1} is added, if $\exists\ \boldsymbol{p}^{m_0,n_0+1}$ which is an NE, we know $\forall q \in Q$ will not be selected before any $s \in S$. Otherwise, supposing q_s is the first such spreader in Q, and $S' \subset S$ is selected before q_s, we can extract a subgraph with S' and its coverage excluded, in which $\boldsymbol{p}_{-S'}^{m_0-|S'|,n_0+1}$ is an NE (by Proposition 1) and q_s is selected firstly. It yields a contradiction. Therefore, Q is still selected after S, and we can also have $\boldsymbol{p}^{m_0,n_0+1}(q_i) = \delta(q_i)$.

Furthermore, we can conclude that $\boldsymbol{p}^{m_0,n_0+1}$ with

$$\begin{cases} \boldsymbol{p}^{m_0,n_0+1}(s) = \boldsymbol{p}^{m_0,n_0}(s) + k \\ \boldsymbol{p}^{m_0,n_0+1}(q) = \delta(q) \end{cases} ,\forall s \in S, q \in Q$$

is an NE where $\boldsymbol{p}^{m_0,n_0}$ is an NE for $|Q| = n_0$. The existence of q_{n+1} adds k coverages to each $\sigma(s)$, therefore, s can increase its price by k and still stay in equilibrium. ■

With Lemma 2, we derive the NE in several special cases for ease of calculations afterwards.

Case 1: $m = 1$. When $n = 1$, by Theorem 2, we know the unique NE is $\boldsymbol{\delta}$, and the selection order is (s_1, q_1). When $n \geq 1$, Lemma 2 implies that the price set with

$$\begin{cases} \boldsymbol{p}^{1,n}(s_1) = \delta(s_1) + (n-1) * k \\ \boldsymbol{p}^{1,n}(q_i) = \delta(q_i),\ i = 1, 2, ..., n \end{cases}$$

is an NE with selection order (s_1, Q). It can be easily seen that the NE is also unique.

Case 2: $m = 2$, $\mu(s_1, s_2) \geq k$. If $n = 1$, it is the situation 1) in the proof of Theorem 2. Therefore, the unique NE is also $\boldsymbol{\delta}$, and the selection order is (S, q_1). When $n \geq 1$, Lemma 2 together with Case 1 imply that the price set with

$$\begin{cases} \boldsymbol{p}^{2,n}(s_j) = \delta(s_j) + (n-1) * k,\ j = 1, 2 \\ \boldsymbol{p}^{2,n}(q_i) = \delta(q_i),\ i = 1, 2, ..., n \end{cases}$$

is an NE with selection order (S, Q), where the order of s_1 and s_2 is decided by the tie-breaking rule. With a proof by contradiction, we can see the NE is unique as well.

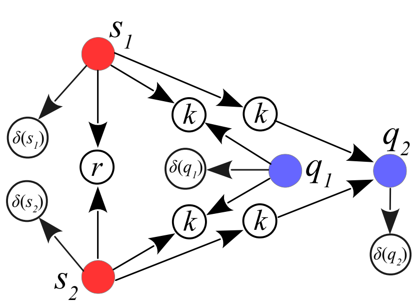

Fig. 4: Cases 2 & 3 with $m = 2, n = 2$

Case 3: $m = 2$, $\mu(s_1, s_2) = r < k$. It is a little different from Case 2. When $n = 1$, it is the situation 2) in the proof of Theorem 2. We can see the unique NE is with

$$\begin{cases} \boldsymbol{p}^{2,1}(s_i) = \delta(s_i), i = 1, 2 \\ \boldsymbol{p}^{2,1}(q_1) = \delta(q_1) + k - r - \epsilon \end{cases} \tag{3}$$

The selection order is (q_1, S), and thus Lemma 2 is not directly applicable.

Consider the case of $n = 2$, which is illustrated in Fig. 4. Suppose q_1 is selected firstly in an NE $\boldsymbol{p}^{2,2}$, by Proposition 1, we know $\boldsymbol{p}^{2,2}(q_2) = \delta(q_2) + k - r - \epsilon$ and $\boldsymbol{p}^{2,2}(s_i) = \delta(s_i)$, which implies $\boldsymbol{p}^{2,2}(q_1) < \sigma(q_1) - u_d(s_1) = \delta(q_1) - r < \delta(q_1)$. It contradicts Lemma 1. Therefore, q_1 will not be selected firstly in any NE, and neither will q_2 due to the symmetry.

Now suppose s_1 is selected firstly in an NE $\boldsymbol{p}^{2,2}$, by Proposition 1 and Case 1, we know

$$\begin{cases} \boldsymbol{p}^{2,2}(s_2) = \delta(s_2) + k \\ \boldsymbol{p}^{2,2}(q_i) = \delta(q_i),\ i = 1, 2 \end{cases} \tag{4}$$

Therefore, the highest price that s_1 could claim is

$$\sigma(s_1) - \max[u_d(s_2), u_d(q_1), u_d(q_2)] = \delta(s_1) + r$$

In fact, such price set is indeed an NE with $\boldsymbol{p}^{2,2}(s_1) = \delta(s_1) + r$: fixing prices in (4), if $p'_{s_1} > \delta(s_1) + r$, q_1 and q_2 will be selected in the first two iterations, and neither of s_1 or s_2 will be selected afterwards. In addition, the selection order in $\boldsymbol{p}^{2,2}$ is (s_1, s_2, Q).

By symmetry, we know another NE is with $p_{s_2} = \delta(s_2) + r$ and $p_{s_1} = \delta(s_1) + k$, and the selection order is (s_2, s_1, Q). Therefore, utilizing Lemma 2, we know when $n \geq 2$, there are two equilibria of Case 3 that

$$\begin{cases} \boldsymbol{p}^{2,n}(s_1) = \delta(s_1) + r + (n-2) * k \\ \boldsymbol{p}^{2,n}(s_2) = \delta(s_2) + (n-1) * k \\ \boldsymbol{p}^{2,n}(q_i) = \delta(q_i),\ i = 1, 2, ..., n \end{cases}$$

with selection order (s_1, s_2, Q), and

$$\begin{cases} \boldsymbol{p}^{2,n}(s_1) = \delta(s_1) + (n-1) * k \\ \boldsymbol{p}^{2,n}(s_2) = \delta(s_2) + r + (n-2) * k \\ \boldsymbol{p}^{2,n}(q_i) = \delta(q_i),\ i = 1, 2, ..., n \end{cases}$$

with selection order (s_2, s_1, Q). Moreover, we can see these are the only two NE. It should be noted as well that r is allowed to be 0.

Case 4: $m = 3, \mu(s_1, s_2) \geq \mu(s_1, s_3) > \mu(s_2, s_3) > k$.

If $n = 1$, the topology is shown in Fig. 5, and $r_1 \geq r_3 > r_2 > k$. Suppose q_1 is selected firstly in some NE, by Proposition 1 and Theorem 2, the NE must have $p_{s_2} = \delta(s_2)$, which entails that $p_{q_1} < \delta(q_1)$. Then suppose s_2 is selected firstly in some NE $\boldsymbol{p}^{3,1}$, we can infer that $\boldsymbol{p}^{3,1}_{-s_2} = \boldsymbol{\delta}_{-s_2}$, which yields that

$$p_{s_2} \leq \sigma(s_2) - u_d(s_1) = \delta(s_2) + r_2 - r_3 < \delta(s_2)$$

and similarly, assuming s_3 being firstly selected in some NE also yields $p_{s_3} < \delta(s_3)$. Finally, if s_1 is selected firstly in some NE $\boldsymbol{p}^{3,1}$, we have $\boldsymbol{p}^{3,1}(s_1) = \delta(s_1) + r_3 - r_2$ and $\boldsymbol{p}^{3,1}_{-s_1} = \boldsymbol{\delta}_{-s_1}$. We can examine that such a $\boldsymbol{p}^{3,1}$ is an NE and also the unique one, and the selection order is $(s_1, \{s_2, s_3\}, q_1)$. The order of s_2 and s_3 is decided by the tie-breaking rule and ϵ should be subtracted from p_{s_1} in case of edge-price.

If $n = 2$, similar as arguments in Case 3, we can see that q_i cannot be selected firstly in NE. On the other hand, assuming $\forall s \in S$ being selected firstly will reduce the calculation of NE to Case 2. Following a similar process, we can deduce that the unique NE is with

$$\begin{cases} \boldsymbol{p}^{3,2}(s_1) = \delta(s_1) + r_3 - r_2 + k \\ \boldsymbol{p}^{3,2}(s_j) = \delta(s_j) + k, \ \ j = 2, 3 \\ \boldsymbol{p}^{3,2}(q_i) = \delta(q_i), \ \ i = 1, 2 \end{cases}$$

Therefore, we can utilize Lemma 2 to conclude that when $n \geq 1$, the price vector with

$$\begin{cases} \boldsymbol{p}^{3,n}(s_1) = \delta(s_1) + r_3 - r_2 + (n-1) * k \\ \boldsymbol{p}^{3,n}(s_j) = \delta(s_j) + (n-1) * k, \ \ j = 2, 3 \\ \boldsymbol{p}^{3,n}(q_i) = \delta(q_i), \ \ i = 1, 2, ..., n \end{cases} \quad (5)$$

is NE and the selection order is $(s_1, \{s_2, s_3\}, Q)$. This is also the unique NE.

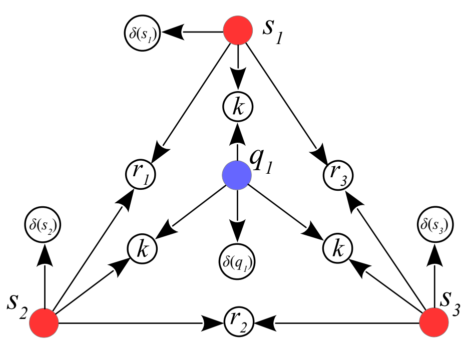

Fig. 5: Cases 4 & 5 with $m = 3, n = 1$

Case 5: $m = 3, \mu(s_1, s_2) \geq \mu(s_1, s_3) > k, \mu(s_2, s_3) = 0$.

If $n = 1$, the topology is shown in Fig. 5, and $r_1 \geq r_3 > k, r_2 = 0$. Following the same arguments in Case 4, we know it must be s_1 to be selected firstly in NE. Then we can further calculate the NE with equations of (3):

$$\begin{cases} \boldsymbol{p}^{3,1}(s_1) = \delta(s_1) + r_3 \\ \boldsymbol{p}^{3,1}(s_j) = \delta(s_j), \ \ j = 2, 3 \\ \boldsymbol{p}^{3,1}(q_i) = \delta(q_i) + k - \epsilon \end{cases}$$

and the selection order is $(s_1, q_1, \{s_2, s_3\})$.

If $n = 2$, same as Case 4, we can still argue that only $s \in S$ can be selected firstly in NE. If s_2 or s_3 is selected firstly in NE, we can finally conclude its prices must be lower than δ utilizing Proposition 1 and Case 2. Therefore, it must be s_1 to be selected firstly in NE as well. Then we can utilize Case 3 to have two price sets, and prove them to be the only two equilibria with selection order S, Q. Leveraging Lemma 2, we can conclude that when $n \geq 2$, there exists two NE:

$$\begin{cases} \boldsymbol{p}^{3,n}(s_1) = \delta(s_1) + r_3 + (n-2) * k \\ \boldsymbol{p}^{3,n}(s_2) = \delta(s_2) + (n-2) * k \\ \boldsymbol{p}^{3,n}(s_3) = \delta(s_3) + (n-1) * k \\ \boldsymbol{p}^{3,n}(q_i) = \delta(q_i), \ \ i = 1, 2, ..., n \end{cases} \quad (6)$$

with selection order (s_1, s_2, s_3, Q) and

$$\begin{cases} \boldsymbol{p}^{3,n}(s_1) = \delta(s_1) + \min[r_1, r_3 + k] + (n-2) * k \\ \boldsymbol{p}^{3,n}(s_2) = \delta(s_2) + (n-1) * k \\ \boldsymbol{p}^{3,n}(s_3) = \delta(s_3) + (n-2) * k \\ \boldsymbol{p}^{3,n}(q_i) = \delta(q_i), \ \ i = 1, 2, ..., n \end{cases} \quad (7)$$

with selection order (s_1, s_3, s_2, Q). Moreover, we can see these are the only two NE.

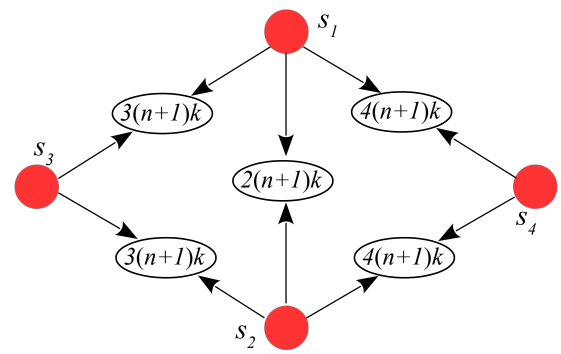

Fig. 6: Topology of S for Theorem 3

Now, we can prove Theorem 3 by presenting an example with no NE.

Proof: Firstly, Case 2 and 3 show that when $|T| \geq 4$, it is possible to have multiple Nash equilibria, i.e. the uniqueness is not guaranteed. Then by showing examples with no NE, we prove the existence of NE is not guaranteed either.

Utilizing the same topology in Fig. 3, we construct the example with $m = 4$, and some specific $\mu(s_i, s_j)$, whose topology is shown in Fig. 6. We neglect $\delta(s_i)$ and Q in the figure to clearly show the topology of S. In addition, we set $\sigma(s_1) > \sigma(s_2) > \sigma(s_4) > \sigma(s_3) > \sigma(q_i)$. This can be satisfied by adjusting δ properly.

The way to prove the non-existence is like this: If an NE exists, then with such NE, a spreader must be deterministically selected in the first iteration. By assuming a spreader as the firstly chosen one, we reversely calculate the NE price set (utilizing Proposition 1). Then we show such a price set cannot be NE. After inspecting all spreaders and finding no one will be selected firstly, we conclude that no NE exists.

First, consider $n = 0$, i.e. $|T| = 4$. Assuming that with an NE $\boldsymbol{p}$, in the first iteration,

- s_4 is selected, by Proposition 1 and Theorem 2, we can infer that

$$\boldsymbol{p}(s_4) < \sigma(s_4) - u_d(s_1) = \delta(s_4) - k < \delta(s_4)$$

- s_3 is selected, similarly, we have

$$\boldsymbol{p}(s_3) < \sigma(s_3) - u_d(s_1) = \delta(s_3) - 3k < \delta(s_3)$$

- s_1 is selected, the only possible $\boldsymbol{p}$ is

$$\begin{cases} \boldsymbol{p}(s_1) = \delta(s_1) + k \\ \boldsymbol{p}(s_2) = \delta(s_2) + 3k \\ \boldsymbol{p}(s_i) = \delta(s_i), i = 3, 4 \end{cases} \quad (8)$$

with selection order (s_1, s_2, s_4, s_3)

Clearly, neither s_3 or s_4 will be selected firstly in any NE. For the price vector set by (8), consider $\boldsymbol{p'}$ with $\boldsymbol{p'}(s_1) = \delta(s_1) + 2k$ and others the same. In the first iteration, s_4 is selected since it provides the largest marginal profit of $8k$. Subsequently, s_3 is selected for providing largest $u_a(\cdot|s_4)$ of $6k$. Thereafter, s_1 will be selected providing zero profit while that of s_2 is $-k$. Finally, s_2 is not selected. In other words, s_1 individually increases its price and gains a higher utility. Therefore, $\boldsymbol{p}$, as specified by (8) is not an NE. The same happens if we assume s_2 is being selected firstly. That is to say, no NE exists.

Then consider $n = 1$. In this case, q_1 will not be selected firstly in any NE, because no NE exists in the case of $n = 0$. Assuming that with an NE $\boldsymbol{p}^{4,1}$, in the first iteration,

- s_4 is selected, utilizing Case 4, we know the remaining four spreaders have prices specified by (5). In particular, $\boldsymbol{p}^{4,1}(s_1) = \delta(s_1)$. Therefore,
$$\boldsymbol{p}^{4,1}(s_4) < \sigma(s_4) - u_d(s_1) = \delta(s_4) - 2k < \delta(s_4)$$
- s_3 is selected, similarly, we have
$$\boldsymbol{p}^{4,1}(s_3) < \delta(s_3) - u_d(s_1) = \delta(s_3) - 6k < \delta(s_3)$$
- s_1 is selected, the remaining 4 spreaders are identical to Case 5. Therefore, the only possible $\boldsymbol{p}^{4,1}$ is
$$\begin{cases} \boldsymbol{p}^{4,1}(s_1) = \delta(s_1) + 2k \\ \boldsymbol{p}^{4,1}(s_2) = \delta(s_2) + 6k \\ \boldsymbol{p}^{4,1}(s_i) = \delta(s_i), i = 3, 4 \\ \boldsymbol{p}^{4,1}(q_1) = \delta(s_1) + k - \epsilon \end{cases} \tag{9}$$
with selection order $(s_1, s_2, q_1, s_4, s_3)$.

In fact, for any $n \geq 1$, s_4 being selected firstly will result in a price less than $\delta(s_4)$. By equation (5) in Case 4, we know $p_{s_1} = \delta(s_1) + (n-1)k$, which implies that

$$p_{s_4} < \sigma(s_4) - u_d(s_1) = \delta(s_4) - 2k < \delta(s_4)$$

Similarly, for s_3, we have

$$p_{s_3} < \sigma(s_3) - u_d(s_1) = \delta(s_3) - (2n+4)k < \delta(s_3)$$

Therefore, neither s_3 or s_4 will be selected firstly in any NE.

We note that $\boldsymbol{p}^{4,1}$ specified in (9) is still not an NE. Suppose that s_1 unilaterally increases its price to $\delta(s_1) + 4k$, then s_4, s_3, s_1, q_1 will be selected one by one in the first four iterations, while s_2 will not be selected eventually. Therefore, $\boldsymbol{p}^{4,1}$ in (9) is not an NE. With the same arguments of s_2, we can conclude that no NE exists when $n = 1$.

Now, suppose we know that when $n = n_0 \geq 1$, there is no NE. Then when $n = n_0 + 1$, we immediately get that for any NE, $q_i, i = 1, 2, ..., n_0 + 1$ will not be firstly selected. Therefore, we just need to assume that with an NE $\boldsymbol{p}^{4,n_0+1}$, s_1 is selected in the first iteration. Utilizing equations (6) and (7) in Case 5, we get two possible NE:

$$\begin{cases} \boldsymbol{p}^{4,n_0+1}(s_1) = \delta(s_1) + (2n_0+1)k \\ \boldsymbol{p}^{4,n_0+1}(s_2) = \delta(s_2) + (4n_0+5)k \\ \boldsymbol{p}^{4,n_0+1}(s_3) = \delta(s_4) + n_0 k \\ \boldsymbol{p}^{4,n_0+1}(s_4) = \delta(s_4) + (n_0-1)k \\ \boldsymbol{p}^{4,n_0+1}(q_i) = \delta(q_i), \ i = 1, 2, ..., n_0+1 \end{cases}$$

with selection order (s_1, s_2, s_4, s_3, Q) and

$$\begin{cases} \boldsymbol{p}^{4,n_0+1}(s_1) = \delta(s_1) + (2n_0+2)k \\ \boldsymbol{p}^{4,n_0+1}(s_2) = \delta(s_2) + (4n_0+6)k \\ \boldsymbol{p}^{4,n_0+1}(s_3) = \delta(s_3) + (n_0-1)k \\ \boldsymbol{p}^{4,n_0+1}(s_4) = \delta(s_4) + n_0 k \\ \boldsymbol{p}^{4,n_0+1}(q_i) = \delta(q_i), \ i = 1, 2, ..., n_0+1 \end{cases}$$

with selection order (s_1, s_2, s_3, s_4, Q).

However, in both cases, s_1 can unilaterally increase its price to $\delta(s_1) + (3n_0+3)k$, such that s_4, s_3, s_1, Q will be selected one by one, and eventually, s_2 fails to be selected. This can been seen by going through X_s by iterations. Therefore, neither of the two is NE and no NE exists for $n = n_0 + 1$.

Therefore, we have constructed examples for $|T| \geq 4$ that have no NE. ■

V. The Pricing Game with X_d

In this section, we consider the scenario that the advertiser adopts spreader selection policy based on the double-greedy algorithm. Different from the case of simple-greedy algorithm, in which NE is not guaranteed to exist when $|T| \geq 4$, there exists an unique NE with an arbitrary number of spreaders. We will first prove the existence and characterise the NE, and then discuss the profit bounds for the advertiser under the equilibrium.

A. The Double-Greedy Policy X_d

We adopt the deterministic version of the algorithm invented by Buchbinder et al. [18]. Spreaders are processed one by one, which is assumed to be in the registration order $[t_1, t_2, ..., t_{|T|}]$ for the advertising campaign. Actually, whatever order is used, we just need to assume that each celebrity is aware of its own number in the sequence, as well as that of its competitors.

The algorithm unfolds iteratively as well but maintains two sets of candidates, A and B. A starts from $A_0 = \emptyset$, while B starts from the full set $B_0 = T$. In each iteration, two marginal gains will be calculated, one of which is obtained by adding the focus spreader into A and the other is by removing it from B. The option providing the larger marginal gain will be chosen and the algorithm terminates after processing all celebrities in T. The policy X_d favors including more celebrities, i.e. the player will be added if its two indices are equal. Obviously, A and B will coincide after the last iteration and there is no need to specify additional tie-breaking rules. X_d is formally stated by pseudo codes in Algorithm 2.

B. $NASH^{X_d}$

Assume the spreaders are in the order $[t_1, t_2, ..., t_{|T|}]$. For ease of notations, we set

$$\varphi_i := \frac{\delta_i + \sigma(t_i|S_{i-1})}{2}$$

where $S_i = \{t_1, \ t_2, \ ..., \ t_i\}$ and $S_0 = \emptyset$. We use bold symbol $\boldsymbol{\varphi}$ to denote the set $\{\varphi_i | i \in T\}$ and use $\varphi(i)$ interchangeably with φ_i. We directly present the main conclusion here.

Algorithm 2 The double-greedy policy X_d

Input: $\mathcal{G} = (V, E)$, T, $\sigma(\cdot)$, $\boldsymbol{p}$
Output: $A_{|T|} \subseteq T$
Initialisation: $A_0 \leftarrow \emptyset$, $B_0 \leftarrow T$
for $i = 1$ to $|T|$ **do**
  $a_i \leftarrow u_a(t_i|A_{i-1}) = \sigma(t_i|A_{i-1}) - p(t_i)$
  $b_i \leftarrow -u_a(t_i|B_{i-1} \setminus t_i) = p(c_i) - \sigma(t_i|B_{i-1} \setminus t_i)$
  if $a_i \geq b_i$ **then**
    $A_i \leftarrow A_{i-1} \cup \{t_i\}$, $B_i \leftarrow B_{i-1}$
  else
    $A_i \leftarrow A_{i-1}$, $B_i \leftarrow B_{i-1} \setminus \{t_i\}$
  end if
end for
return $A_{|T|}$

Theorem 4. $NASH^{X_d} = \{\boldsymbol{\varphi}\}$

Before giving the detailed proof of Theorem 5, we introduce a lemma which is similar to Lemma 1.

Lemma 3. *$\forall\ i \in T$, claiming price of φ_i implies $i \in X_d(\boldsymbol{p})$. Moreover, if $\exists\ \boldsymbol{p} \in NASH^{X_d}$, then $X_d(\boldsymbol{p}) = T$ and $p_i \geq \delta_i$.*

Proof: $\forall\ i \in T$, we have $A_{i-1} \subseteq \{t_1,\ t_2,\ ...,\ t_{i-1}\} = S_{i-1}$, $B_{i-1} \subseteq T$. Therefore, by submodularity,

$$a_i = u_a(t_i|A_{i-1}) \geq u_a(t_i|S_{i-1}) = \frac{\sigma(t_i|S_{i-1}) - \delta_i}{2}$$

$$b_i = -u_a(t_i|B_{i-1}\backslash\{t_i\}) \leq -u_a(t_i|C\backslash\{t_i\}) = \frac{\sigma(t_i|S_{i-1}) - \delta_i}{2}$$

i.e. $a_i \geq b_i \Rightarrow i \in X_d(\boldsymbol{p})$.

With NE price vector $\boldsymbol{p}$, if spreader $i \notin X_d(\boldsymbol{p})$, then $u_i^{X_d}(p_i|\boldsymbol{p} \setminus i) = 0$. Choosing $p_i' = \varphi_i$ will result in a higher utility, which yields a contradiction. Therefore, $X_d(\boldsymbol{p}) = T$, and obviously, no one will claim a price lower than φ. ■

Lemma 3 also implies that the maximal welfare is achieved:

$$W(\boldsymbol{p}, X_d) = \sigma(X_d(\boldsymbol{p})) = \sigma(T)$$

and both the *Price of Anarchy* and the *Price of Stability* equal one. Recall that in the case of the simple-greedy advertiser, the existence of NE cannot be guaranteed in most cases ($|T| \geq 4$), which renders that claim on welfare meaningless. Now we turn to the detailed proof of Theorem 4.

Proof: By Lemma 3, we know that $\forall\ i \in T$, $u_i^{X_d}(\varphi_i|\boldsymbol{\varphi} \setminus i) = \varphi_i$. We first prove that no spreader will unilaterally increase its price.

Assume $p_j' > \varphi_j$ and others remain the same. By Lemma 3, $S_{j-1} \subseteq X_d(\boldsymbol{p}')$. Therefore, $A_{j-1} = S_{j-1}$ and $B_{j-1} = T$,

$$a_j = u_a(t_j|A_{j-1}) = \sigma(t_j|S_{j-1}) - p_j' < \frac{\sigma(t_i|S_{i-1}) - \delta_i}{2}$$

$$b_j = -u_a(t_j|B_{j-1} \setminus t_i) > \varphi_j - \delta_j = \frac{\sigma(t_i|S_{i-1}) - \delta_i}{2}$$

which yields that $j \notin X_2(\boldsymbol{p}')$. Note that it is also true for $j = 1$. Therefore, $\boldsymbol{\varphi} \in NASH^{X_d}$.

Assume $\exists\ \boldsymbol{p}' \in NASH^{X_d}$ that $\boldsymbol{p}' \neq \boldsymbol{\varphi}$. There must exist $j \in T$ such that $p_j' > \varphi_j$. Assume l is the first one. By Lemma 3, $p_i = \varphi_i$ for $i < l$. Following the same arguments above, we have $l \notin X_d(\boldsymbol{p}')$ which contradicts Lemma 3. Therefore, $NASH^{X_d} = \{\boldsymbol{\varphi}\}$. ■

C. The Profit of the Advertiser with Price Set $\boldsymbol{\varphi}$

By Theorem 4, when in the equilibrium $\boldsymbol{p} = \boldsymbol{\varphi}$, all spreaders participate in the advertisement campaign, bringing the maximal revenue, $\sigma(T)$, for the advertiser. But how about the profit? Although it has been prevously stated that the double-greedy algorithm does not have a guaranteed performance for arbitrary price vectors, we can find both the upper and the lower bounds for this specific scenario.

Theorem 5. *Under the NE, i.e. $\boldsymbol{p} = \boldsymbol{\varphi}$, if S^* is the optimal set of spreaders that achieves the maximum profits,*

$$\frac{1}{2}u(S^*) \leq u_a(X_d(\boldsymbol{p})) \leq \frac{1}{2}\sigma(T)$$

Proof: The upper bound can be shown by the lower bound of the total payments to spreaders.

$$\begin{aligned} p(X_d(\boldsymbol{p})) = p(T) = &\ \textstyle\sum_{i \in T} \varphi_i \\ = &\ \textstyle\sum_{i \in T} \delta_i/2 + \sum_{i \in T} \sigma(i|S_{i-1})/2 \\ \geq &\ \sigma(T)/2 \end{aligned}$$

Therefore,

$$u_a(X_2(\boldsymbol{p})) = \sigma(T) - p(T) \leq \frac{1}{2}\sigma(T)$$

Equality holds when $\boldsymbol{\delta} = \boldsymbol{0}$, i.e., no spreader has unique influences. The upper bound is thus tight.

If we define

$$S_i^* := S^* \cup S_i,$$

then we have

$$\begin{aligned} u_a(S_{i-1}^*) - u_a(S_i^*) = &\ u_a(t_i|S^* \cup S_{i-1}) \\ \leq &\ u_a(t_i|S_{i-1}) \\ = &\ u_a(S_i) - u(S_{i-1}) \end{aligned}$$

Therefore, if we take series sum for both sides,

$$u(S_0^*) - u(S_{|t|}^*) \leq u(S_{|T|}) - u(S_0)$$

Note that $S_0 = \emptyset$, $S_0^* = S^*$, and $S_{|T|}^* = S_{|T|} = T$, then we have

$$u_a(X_d(\boldsymbol{p})) = u_a(T) \geq \frac{1}{2}u_a(S^*)$$

■

VI. Conclusion

In this paper, we analyzed the Nash equilibrium of the pricing game among spreaders in the framework of sponsored viral marketing. We conclude that if the advertiser is ideal, i.e., alsways selects the optimal set of spreaders, or adopts the double-greedy based spreader selection policy, the unique NE exists. In the equilibrium, spreaders in the former case will claim their prices as their unique influences, while in the latter, their prices are higher and related with previously registered competitors, and moreover, we can find the bounds for the profits of the advertiser. If the advertiser adopts a simple-greedy algorithm, unique NE exists when there are fewer than four spreaders; otherwise, the existence of NE cannot be guaranteed.

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