Singular dividend optimization for a linear diffusion model with time-inconsistent preferences

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Abstract

With the advancement of behavioral economics, the use of exponential discounting for decision making in neoclassical economics has been questioned since it cannot provide a realistic way to explain certain decision-making behavior. The purpose of this paper is to investigate strategic decision making on dividend distribution policies of insurance companies when the management adopts a more realistic way for discounting, namely stochastic quasi-hyperbolic discounting. The use of this more realistic way for discounting is motivated by some recent developments in behavioral economics. A game theoretic approach is adopted to establish economic equilibrium results, namely subgame perfect Markov equilibrium strategies. It is shown that (1) under certain mild technical conditions, the barrier strategy with an optimal barrier, which is widely used in the traditional approach to optimal dividend problems, is a perfect Markov equilibrium strategy, (2) the optimal barrier is lower than the barrier of an optimal strategy obtained from the respective time-consistent optimal dividend problem, and (3) the solution based on the barrier strategy does not exist in some situations.

Keywords (I) Control; dividends; hyperbolic discounting; non-exponential discounting; ruin theory.

1 Introduction

Strategic decision making on dividend payments of insurance companies is one of the major research topics in actuarial science. The landmark article by (De Finetti, 1957) seems to be an early attempt to provide a scientific way to study the topic at least from the perspective of actuarial science. A key issue for the decision making is to decide the discount factor to be used in evaluating the expected present values of cash flows from future dividend payments. Discounting is used to model time preference for present over future income due to human impatience as noted in the masterpiece on the theory of interest in neoclassical economics by (Fisher, 1930), where the rate of interest was determined as the marginal rate

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of substitution for present and future goods (or the relative price between present and future goods). The scientific treatments in the traditional actuarial approach to optimal dividend payments focus on exponential discounting, which is a key tool for discounting used in utility models in neoclassical economics by (Samuelson, 1937). The rationale of adopting exponential discounting is to describe time consistency. However, with the advancement of behaviorial economics, the use of exponential discounting has been challenged. In particular, the time consistency described by exponential discounting has been questioned. An early inquiry for time inconsistency may be traced back to the monograph by (Mises, 1949). It was noted in (Thaler, 1997) that Irving Fisher in his theory of interest (Fisher, 1930) has already reckoned the significance of irrational and psychological factors in determining human's impatience and attributed the lack of impatience to that "the future is seldom considered in true proportions". There are experimental or empirical studies on human or other animals' behavior against time consistency for intertemporal choice described by exponential discounting. For example, an early experimental study conducted by (Thaler, 1981) presented empirical evidence against time consistency of discounting. Some other experimental or empirical studies supporting non-exponential discounting, such as hyperbolic discounting, are (Frederick, Loewenstein, & O'Donoghue, 2002) and the relevant literature therein.

Time-inconsistent preferences, besides playing an important role in describing behaviorial and psychological features of human's impatience, represent an intellectual challenging issue when they are applied to optimization problems in economics, finance and insurance. Specifically, certain standard approaches, such as dynamic programming, to studying optimization problems cannot be directly applied when decision makers have time-inconsistent preferences, particularly for those described by non-exponential discounting, and are choosing alternatives in the presence of uncertainty. In particular, with the time-inconsistent preferences, a strategy that is deemed to be optimal for a decision maker at the present time may no longer be optimal in the future. Contemporary approaches have been introduced to deal with optimization problems with time-inconsistent preferences. A notable example is the work by (Bjork & Murgoci, 2010), where a theoretically solid result linking decision making processes with time-inconsistent preferences and those with time-consistent preferences was established. Specifically, (Bjork & Murgoci, 2010) proved that for a general time-inconsistent problem under a general controlled Markov process, there exists "an associated time-consistent problem such that the optimal control and respective value function for the time-consistent problem coincides with the equilibrium control and respective value function for the time-inconsistent problem". Consequently, instead of attempting to seek optimal controls, one may seek equilibrium strategies. The idea is that "a decision the controller makes at every instant of time is thought of as a game problem against all the decisions the future incarnations of the controller may make".

In finance, decision making with time-inconsistent preferences has aroused interests of researchers. For example, (Ekeland & Pirvu, 2008) and (Marín-Solano & Navas, 2010) addressed an optimal consumption and portfolio management problem when preferences are time-inconsistent. (Zou, Chen, & Wedge, 2014) studied consumption and portfolio decisions with stochastic hyperbolic discounting while (Harris & Laibson, 2013) considered the optimal consumption problem under the same setting. (Zhao, Shen, & Wei, 2014) investigated a consumption-investment problem with a general discount function and a logarithmic utility function in a non-Markovian framework by treating the problem as an N-person differential game. (Dong & Sircar, 2014) studied time-inconsistent portfolio investment problems while (Grenadier & Wang, 2007) investigated optimal investment under uncertainty and stochastic hyperbolic discounting.

Decision making in insurance with time-inconsistent preferences has also attracted attention from researchers. Specifically, some recent works have adopted hyperbolic discount functions. (Zhao, Wei, & Wang, 2014) considered an optimization problem where the dividend rates are bounded, the surplus process follows a Brownian motion and the discounting function is a mixture of exponential discount functions and the pseudo-exponential discount function, respectively. (Li, Li, & Zeng, 2015) addressed the same problem but for the dual compound Poisson model. (Chen, Li, & Zeng, 2014) considered the singular dividend optimization problem for the dual risk model with exponential jumps when stochastic hyperbolic discounting was applied. (Li, Chen, & Zeng, 2015) solved a singular dividend optimization problem for a diffusion model for the surplus process with stochastic hyperbolic discounting. (Chen, Wang, Deng, & Zeng, 2016) investigated an optimal combined dividend-financing strategy in a dual risk model with exponential jumps when stochastic quasi-hyperbolic discounting was adopted.

This paper aims to investigate strategic decision making on dividend payments of insurance companies when the management adopts a more realistic way for discounting, namely stochastic quasi-hyperbolic discounting, a type of non-exponential discounting. With a view to deviating from the neoclassical approach to discounting in optimal dividend problems, this paper intends to study the impact of a behavioral approach to human's impatience described by this non-exponential discounting on optimal dividend strategies. Specifically, we discuss a theoretical treatment for the problem by investigating a singular dividend optimization problem with the time-inconsistent preferences described by stochastic quasi-hyperbolic discounting. A linear diffusion model for the surplus of an insurance company with general coefficients, where the drift and volatility coefficients are generic functions of the current surplus, is considered. Two particular cases for the diffusion model, namely the random walk process and the mean-reverting process, are considered. These two processes play an important role for modelling dynamics in economics and econometrics. To provide a theoretically solid approach to studying the optimization problem, a game theoretic approach is adopted to establish economic equilibrium results, namely subgame Markov perfect equilibrium (MPE) strategies. Our study leads to several theoretical results. Firstly, under certain mild technical conditions, the barrier strategy with an optimal barrier, which is widely used in the traditional approach to optimal dividend problems, is a MPE strategy. Secondly, the optimal barrier is lower than the barrier of an optimal strategy obtained from the respective time-consistent optimal dividend problem. Thirdly, the solution based on the barrier strategy does not exist in some situations. Our theoretical results also illustrate that the time-inconsistent preferences described by stochastic quasi-hyperbolic discounting would induce an economic incentive for the management of an insurance company, who is supposed for simplicity to have the same objective as that of the shareholders, to distribute dividends sooner than later. It is hoped that these results would shed light on understanding the impacts of non-exponential discounting on strategic decision making on dividend payments of insurance firms.

This paper also intends to contribute to the literature from the modelling and theoretical perspectives. The linear diffusion model for the surplus process considered here may include many commonly used diffusion models for surplus processes of insurance companies in the literature as particular cases. Particularly, in many of the existing works as noted above, the drift and volatility coefficients are supposed to be constant. An extension to the situation of general coefficients presented here poses a significant technical challenge from the theoretical perspective. Specifically, applying stochastic optimal control theory always involves differential equations. In the situation to be discussed here, the model coefficients are no longer constants and are general functions of the surplus process (even the forms of the functions are not specified); thus, the differential equation associated with the optimization problem has variable coefficients. Notably, the first and second-order coefficients are functions which could be of unspecified forms. This renders it difficult, if not impossible, to determine an explicit expression of the solution to the differential equation. Therefore, the standard approach used in the above references, which first attempts to determine the explicit solution of the associated differential equation, (which is often simple with an exponential structure), and then uses the explicit expression to verify that the solution is indeed the optimal solution, is no longer applicable here. We solve the problem by investigating the existence of solutions to the differential equations, and then deriving and utilizing some important properties of the solutions. Informed by one of the referees, we noticed that an extension of the impulse control problem for a linear diffusion model under the same stochastic quasi-hyperbolic discounting has been done in (Chen, Li, & Zeng, 2018), which further confined the admissible set of strategies to be of a barrier lump-sum type. In our paper, we address a singular control problem, where the admissible set of dividend strategies is any adapted càdlàg stochastic process. These two types of stochastic control problems and the regular (classical) type of control problems, are very different in terms of technical details and the properties that the optimality results have even in the same setting. There are a long and separate list of literatures for each type of control problems under exponential discounting. This paper and (Chen et al., 2018) extend the work for different types of control problems and have made their own contributions to each respective type of control problems. Neither one is an extension or a duplication of the other.

In Section 2, we formulate the optimization problem. We study a special class of dividend strategies, namely barrier strategies, and derive theoretical properties of the associated return function in Section 3. We present the optimality results in Section 4 and provide numerical illustrations in Section 5. A conclusion is provided in Section 6. All the proofs are included in Appendix.

2 Problem Formulation

Consider a complete probability space $(\Omega, \mathcal{F}, \mathrm{P})$ and let $\{W_t; t \geq 0\}$ be a standard Brownian motion on $(\Omega, \mathcal{F}, \mathrm{P})$. Let R_t represent the amount of the cash reservoir (surplus) at time t in the absence of control, which evolves according to the following diffusion process: $\mathrm{d}R_t = \mu(R_{t-})\mathrm{d}t + \sigma(R_{t-})\mathrm{d}W_t,\ t \geq 0$, where both $\mu(x)$ and $\sigma(x)$ are Lipschitz continuous on $[0,+\infty)$, $\mu(x)$ is concave on $[0,+\infty)$, and $\sigma(x)$ is positive and non-vanishing for $x \geq 0$. This dynamic is very general and it includes most of models used in the research singular control problems in the diffusion setting as special cases. Suppose R_{0-} is $\mathcal{F}$ measurable and define $\mathcal{F}_t^{0,W}$ to be the minimal P-complete σ-field generated by R_{0-} and $\{W_s; 0 \leq s \leq t\}$. Write $\mathcal{F}^{0,W} = \{\mathcal{F}_t^{0,W}; t \geq 0\}$.

The company controls dividend payments dynamically over time. Let L_t denote the cumulative amount of dividends paid out up to time t and define the dividend strategy by $L = \{L_t; t \geq 0\}$. It is natural to assume that L_t is non-decreasing and right continuous with left limits, and that L_t depends on the path of R over the time period $[0,t)$. Therefore, L is $\mathcal{F}^{0,W}$-predictable. Let R_t^L denote the surplus at time t under the dividend strategy L. It then follows the dynamics:

$$\mathrm{d}R_t^L = \mu(R_{t-}^L)\mathrm{d}t + \sigma(R_{t-}^L)\mathrm{d}W_t - \mathrm{d}L_t,\ t \geq 0. \tag{2.1}$$

Assume that for each unit of dividend paid out, the shareholders receive only β units $(0 < \beta < 1)$ due to proportional transaction costs. Further assume that decision makers have time-inconsistent preferences. More specifically, we use a time-inconsistent discounting that reflects that "people choose as if they discount future rewards at a greater rate when the delay occurs sooner in time". We use the stochastic quasi-hyperbolic model where higher discount rate (weight) is applied to dividend payments in the present period compared to the future periods. This is same as the stochastic quasi-hyperbolic discounting used in (Grenadier & Wang, 2007), (Harris & Laibson, 2013), (Zou et al., 2014) and (Chen et al., 2016). The time is split into infinite number of periods and for each period there is a decision maker who controls dividend distribution during her "present" period (this period) only. We start with the decision maker at time 0, called "self 0" here. "Self 0" is an economic self born at time $s_0 := 0$ whose life time is divided into a "present", which is from time s_0 to time $s_0 + \eta_0$, and a "future", which lasts from time $s_0 + \eta_0$ to $+\infty$. A new economic self, "self 1", is born at time $s_1 =: s_0 + \eta_0$. The present of "self 1" is the time period from s_1 to $s_1 + \eta_1$ and the future starts from $s_1 + \eta_1$ to $+\infty$. Proceeding in this manner, define $s_n = s_{n-1} + \eta_{n-1}$, for any $n = 1, \cdots$, recursively, where $\{\eta_0, \eta_1, \cdots\}$ is a sequence of $i.i.d.$ random variables that is independent of $\{W_t; t \geq 0\}$ and is exponentially distributed with mean $1/\lambda$ $(\lambda > 0)$. For any $n = 0, 1, \cdots$, "self n" is born at time s_n, whose "present" period is from s_n to $s_n + \eta_n$ and "future" period from $s_n + \eta_n$ to $+\infty$. "Self n" controls and can only control her "present", the time interval from s_n to $s_n + \eta_n$. Each "self" is present-biased. More specifically, "self n" discounts the dividends paid in her present period exponentially with discount force δ $(\delta > 0)$ and discounts the dividends in her future less than her present with the same exponential discounting force δ and an additional discount factor α $(0 < \alpha \leq 1)$. Thus, if we let $D_n(s,t)$ represent the "self n"'s present value at time s of 1 unit of dividend received at time t, then $D_n(s,t) = e^{-\delta(t-s)}$ for $s_n \leq s < t < s_n + \eta_n$ and $D_n(s,t) = \alpha e^{-\delta(t-s)}$ for $t \geq s_n + \eta_n > s \geq s_n$.

For any $t \geq 0$, and any $x \in \mathbb{R}$, define

$$\mathrm{P}_{t,x}(\cdot) = \mathrm{P}(\cdot | R_{t-} = x),\quad \mathrm{E}_{t,x}[\cdot] = \mathrm{E}[\cdot | R_{t-} = x],\quad \mathrm{P}_x(\cdot) = \mathrm{P}(\cdot | R_{0-} = x),\quad \mathrm{E}_x[\cdot] = \mathrm{E}[\cdot | R_{0-} = x].$$

For any strategy π, define the time to ruin of the company at time t by

$$T_t^\pi = \inf\{s \geq t : R_s^\pi \leq 0\}, \tag{2.2}$$

and write $T^\pi = T_0^\pi$. Throughout the paper, we also use L frequently to represent a control strategy, and T^L to represent the time to ruin under the strategy L.

We use $\pi^{(n,\rightarrow)(L,\tilde{L})}$ to represent the strategy when self "n" adopts L and the future selves adopt $\tilde{L}$. Let $\pi_t^{(n,\rightarrow)(L,\tilde{L})}$ represent the cumulative dividend payments from time s_n to t under $\pi^{(n,\rightarrow)(L,\tilde{L})}$. Then, $\pi_{s_n-}^{(n,\rightarrow)(L,\tilde{L})} = 0$, $\mathrm{d}\pi_t^{(n,\rightarrow)(L,\tilde{L})} = \mathrm{d}L_t$ for $t \in [s_n, s_{n+1})$ and $\mathrm{d}\pi_t^{(n,\rightarrow)(L,\tilde{L})} = \mathrm{d}\tilde{L}_t$ for $t \geq s_{n+1}$. The payoff to "self n" is the expected present value at time s_n of all the future dividends received up to the time of ruin. Given $R_{s_n-} = x$, for any $x \geq 0$, self n's payoff function of the strategy $\pi^{(n,\rightarrow)(L,\tilde{L})}$ is

$$\begin{aligned}\mathcal{P}_n(x; L, \tilde{L}) =& \mathrm{E}_{s_n,x}\Bigg[\int_{s_n}^{T_{s_n}^{\pi^{(n,\rightarrow)(L,\tilde{L})}} \wedge (s_n+\eta_n)} \beta e^{-\delta(t-s_n)}\mathrm{d}L_t \\ &+ I\{s_n + \eta_n \leq T_{s_n}^{\pi^{(n,\rightarrow)(L,\tilde{L})}}\} \int_{s_n+\eta_n}^{T_{s_n}^{\pi^{(n,\rightarrow)(L,\tilde{L})}}} \alpha\beta e^{-\delta(t-s_n)}\mathrm{d}\tilde{L}_t\Bigg].\end{aligned} \tag{2.3}$$

In the above payoff function, the first term is the present value at time s_n of all the dividends paid out from time s_n to $s_n + \eta_n$ (the "present" period of "self n") and the second term is the present value at time s_n of the dividends paid out after $s_n + \eta_n$.

"Self n" controls the dividend distribution during the present period $[s_n, s_n + \eta_n)$ and does not have control over her future. However, she does care about the dividend distribution in her future. For each self, her objective is to choose a strategy for dividend distribution during her present period so that the expectation of the total discounted dividends is maximized. Note that $\mathcal{P}_n(x; L, \tilde{L})$ involves the dividends paid during the future period of "self n", $[s_{n+1}, +\infty)$, which is controlled sequentially by the future selves $n + 1$, $n + 2$, $\cdots$. This is an intrapersonal game. We will apply the stationary Markov-perfect equilibrium concept (MPE) here. MPE is a refinement of subgame perfect equilibrium, which only considers Makov strategies, i.e., strategies with the action at any time depends on the current time and state (the current surplus) only. We assume that each "self" is sophisticated, and that each "self" selects strategies during $[s_n, s_{n+1})$ based on her anticipation that her future "selves" will take the same optimal actions. For more information on MPE, please refer to (Harris & Laibson, 2013) and the references therein.

A dividend strategy L is said to be admissible if the process L is $\{\mathcal{F}_t^{0,W}\}$-adapted, non-decreasing, right continuous with left limits, and Markovian. We use Π to denote the set of *admissible strategies.*

Notably, any admissible strategy, L, is right continuous with left limits. We can thus decompose it as follows: $L_t = L_t^c + \sum_{0\leq s\leq t}(L_s - L_{s-})$, where $\{L_t^c\}$ represents the continuous part of L.

Since $\{W_t; t \geq 0\}$ and $\{L_t; t \geq 0\}$ for any $L \in \Pi$ are $\{\mathcal{F}_t^{0,W}\}$-adapted, from (2.1) and the definition of T^L in (2.2) we can see that T^L is an $\mathcal{F}^{0,W}$-stopping time for any $L \in \Pi$.

Definition 2.1 *For any $n = 0, 1, \cdots$, the objective of "self n" is to find the optimal admissible stationary Markov-perfect equilibrium (MPE) strategy, L^*, such that $\mathcal{P}_n(x; L^*, L^*) = \sup_{L\in\Pi} \mathcal{P}_n(x; L, L^*)$.*

We can see that each strategy and its payoff function are the same for different selves. Therefore, we only consider the case for "self 0" and study the following performance/payoff function:

$$\mathcal{P}(x; L, \tilde{L}) := \mathcal{P}_0(x; L, \tilde{L})$$
$$= \mathrm{E}_x \left[\int_0^{T^{\pi^{(0,\rightarrow)(L,\tilde{L})}} \wedge \eta_0} \beta e^{-\delta t} \mathrm{d}L_t + I\{\eta_0 \le T^{\pi^{(0,\rightarrow)(L,\tilde{L})}}\} \int_{\eta_0}^{T^{\pi^{(0,\rightarrow)(L,\tilde{L})}}} \alpha\beta e^{-\delta t} \mathrm{d}\tilde{L}_t \right]. \tag{2.4}$$

If a stationary MPE (denoted by L^*) exists, we define the value function $V(x) = \mathcal{P}_0(x; L^*, L^*)$.

(Chen et al., 2018) studied a particular case of an impulse control problem with a fixed transaction cost for each dividend payment, under the same surplus model (in the absence of control) and the same formulation for discounting as those considered in this paper. However, (Chen et al., 2018) considered impulse control strategies and further confined the class of admissible strategies to be those of a barrier lump-sum type. Note that a barrier lump-sum strategy can be characterised by two parameters, say a and b, with $b > a \ge 0$ and that it prescribes to pay dividends with a view to reducing the surplus to a whenever the underlying surplus exceeds the level b. Consequently, it appears that the class of all the barrier lump-sum strategies is a small subset of the class of impulse strategies, which itself is a subset of the set of admissible strategies, Π (all adapted non-decreasing and càdlàg Markov processes), considered in this paper. In other words, the class of control strategies considered in this paper is more general than the one considered in (Chen et al., 2018).

Although the performance functional in (Chen et al., 2018), when setting $K = 0$ (i.e. assuming no fixed transaction costs), is identical to the one adopted in this paper, the problem considered there, (when $K = 0$), does not lead to a singular control problem as considered in this paper since the class of admissible strategies considered in there, as mentioned above, are limited to barrier lump-sum strategies rather than a wider class of admissible strategies Π (all adapted non-decreasing and càdlàg Markov processes) considered in this paper. When assuming $K = 0$, the optimal solution, say the MPE strategy, considered in (Chen et al., 2018) (if exists) may not be an optimal solution in this paper, say an optimal strategy in Π, and the optimization results obtained there only hold for $K > 0$.

Remark 2.1 *Because an admissible strategy is non-decreasing, for any admissible strategies,* L *and* $\tilde{L}$, $\mathcal{P}(x; L, \tilde{L}) \ge 0$ *for* $x \ge 0$.

We further assume that $\frac{\mu(y)-\mu(x)}{y-x} \le \lambda(1-\alpha) + \delta$ for $y > x \ge 0$ throughout the paper. Unlike in (Chen et al., 2018) (which requires $\mu'(x) \le \delta$), we do not require $\mu(x)$ to be differentiable here. In the special case that $\mu(x)$ is differentiable, this condition is equivalent to $\mu'(x) \le \lambda(1-\alpha) + \delta$ for $x \ge 0$, (which is weaker than the condition in (Chen et al., 2018) as $\lambda(1-\alpha)+\delta < \delta$). We can interpret $\mu'(x)$ as the growth rate for a surplus/cash reservoir of x and δ as the risk discount rate. It is indeed natural in many cases to assume that the risk discount rate equals the growth rate plus a risk margin, which implies that the growth rate is smaller than the risk discount rate. Our assumption is slightly less stringent than this as we require the growth rate to be smaller than the risk discount rate plus a positive number. Cases when such condition is violated cannot be addressed by using the same approach and will be considered in our future research.

3 The barrier strategies and their associated payoff functions

In most of the existing works concerning the singular dividend control problem for diffusion models with consistent time preferences (i.e., exponential discounting), the optimal strategy either exists and is a barrier strategy or does not exist. For the special cases with hyperbolic discounting that have been addressed, the optimal strategies are of a similar type to the consistent counterparts (see for instance, (Li et al., 2015) and (Chen et al., 2014)). This motivates us to study the barrier strategies first. We start by defining the barrier strategies and then move on to investigate the associated payoff functions.

Definition 3.1 *For any $b \geq 0$, (i) let L^b denote the barrier strategy that prescribes to pays no dividends when the surplus is below b and to pay out all of the excess surplus over b as dividends if the surplus is above b, to keep the controlled surplus, X^{L^b}, reflected at b after the initial time; and (ii) define*

$$V_b(x) = \mathcal{P}(x; L^b, L^b), \ \ x \geq 0. \tag{3.1}$$

For $b > 0$, the strategy L^b prescribes to pay a dividend $(R_{0-} - b)^+$ at time 0 and then to keep the controlled surplus R^{L^b} a diffusion reflected at b. If $b = 0$, the strategy L^b prescribes to pay all surplus out as dividends at time 0, and ruin occurs immediately. Therefore, the payoff is βx if the initial surplus is x. If the initial surplus is less than or equal to 0, ruin occurs immediately at time 0 and there are no dividend payments at all, which results in a payoff of 0. Therefore,

$$V_0(x) = \beta x \ \text{ for } x \geq 0, \ \ \text{and } V_b(x) = 0, \ \text{for } b \geq 0 \text{ and } x \leq 0. \tag{3.2}$$

Furthermore, we can show that $V_b(x)$ is non-negative and non-decreasing.

Lemma 3.1 *For any $b \geq 0$, $V_b(x) \geq 0$ for $x \geq 0$ and $V_b(x)$ is non-decreasing on $[0, +\infty)$.*

To assist with the investigation of the stationary MPE strategy, we will study the following functions.

Definition 3.2 *For any fixed $b \geq 0$ and $L \in \Pi$, define the functions $\mathcal{P}^E(\cdot; L)$ and $V_b^E(\cdot)$ by*

$$\mathcal{P}^E(x; L) = \mathrm{E}_x\left[\int_0^{T^L} \beta e^{-\delta t} \mathrm{d}L_t\right], \quad V_b^E(x) = \mathcal{P}^E(x; L^b), \ \ x \geq 0. \tag{3.3}$$

The functions colorblue as defined above are the return functions (the expected present value of the total dividends) when the exponential discounting $e^{-\delta t}$ is used. These functions have the following useful properties.

Throughout the paper, we define $f'(0)$ and $f''(0)$ to be, respectively, the first and second-order right derivatives of $f(\cdot)$ at 0.

Lemma 3.2 *(i) For any $b > 0$, the function $V_b^E(x)$ is continuously differentiable and twice continuously differentiable for $x \geq 0$, except for the point $x = b$, and it satisfies*

$$\frac{\sigma^2(x)}{2} V_b^{E''}(x) + \mu(x) V_b^{E'}(x) - \delta V_b^E(x) = 0, \ \ 0 \leq x < b, \tag{3.4}$$

$$V_b^E(0) = 0, \quad V_b^{E'}(b) = \beta, \quad V_b^{E'}(x) = \beta, \ \ x > b, \quad \lim_{b \downarrow 0} V_b^E(b) = 0, \quad \lim_{b \downarrow 0} V_b^E(x) = V_0^E(x). \tag{3.5}$$

For any $b \geq 0$, the function $V_b^E(x)$ is non-negative and strictly increasing on $[0, +\infty)$.
(ii) For any $b > 0$, if $V_b^{E''}(b-) \leq 0$, then $V_b^{E''}(x) \leq 0$ for $x \in [0, b)$.

We will now present some theoretical properties of the payoff functions associated with barrier strategies. Those properties will be useful in the analysis of some later results. We will first show that the payoff function of a barrier strategy is the solution to a boundary-value problem.

Lemma 3.3 *(i) For any $b > 0$, there is a unique solution that is twice continuously differentiable on $(0, +\infty)$ to the following boundary value problem,*

$$\frac{\sigma^2(x)}{2} f''(x) + \mu(x) f'(x) - (\lambda + \delta) f(x) + \lambda \alpha V_b^E(x) = 0 \text{ for } 0 \leq x < b, \ \ f(0) = 0, \quad f'(b) = \beta. \tag{3.6}$$

(ii) Let $S_b(x)$ denote the above solution. Then, for any $b > 0$,

$$V_b(x) = \mathcal{P}(x; L^b, L^b) = \begin{cases} S_b(x) & 0 \leq x \leq b \\ S_b(b) + \beta(x - b) & x > b, \end{cases} \quad \text{and} \quad \lim_{b \downarrow 0} V_b(b) = 0. \tag{3.7}$$

Remark 3.1 *From the last lemma it follows immediately that $V_b(x)$ is continuously differentiable on $[0, +\infty)$ and twice continuously differentiable on $[0, b) \cup (b, +\infty)$; additionally,*

$$\frac{\sigma^2(x)}{2} V_b''(x) + \mu(x) V_b'(x) - (\lambda + \delta) V_b(x) + \lambda \alpha V_b^E(x) = 0 \text{ for } 0 \leq x < b, \tag{3.8}$$

$$V_b(0) = 0, \quad V_b'(b) = \beta, \quad V_b'(x) = \beta, \quad V_b''(x) = 0 \text{ for } x > b. \tag{3.9}$$

We now define two quantities that will play key roles in the characterization of MPE strategies.

Definition 3.3 *(i) Define $b^E = \inf\{b > 0 : V_b^{E''}(b-) \geq 0\}$, and $b^E = +\infty$ if $V_b^{E''}(b-) < 0$ for all $b > 0$.*
(ii) Define $b^* = \inf\{b > 0 : V_b''(b-) \geq 0\}$, and $b^* = +\infty$ if $V_b''(b-) < 0$ for all $b > 0$.

From the last lemma and the last remark, we observe that $V_b(x)$ may not be twice differentiable at $x = b$. The quantity b^* is the barrier of a barrier strategy under which the payoff function with stochastic quasi-hyperbolic discounting is also twice differentiable at b and thus V_b (see (3.7)) is a smooth pasting of the solution of an ODE and the linear function, and it is said to satisfy the "heuristic principle of smooth fit". Similarly, the quantity, b^E , is the barrier of a barrier strategy under which the payoff function in the exponential discounting case satisfies the "heuristic principle of smooth fit".

Using the definitions for b^E and b^* and the ordinary differential equations that they satisfy, we can derive the following relationship between b^E and b^*.

Lemma 3.4 *The following holds: $0 \leq b^* \leq b^E$.*

We derive the following properties of b^* and the payoff function associated with L^{b^*}, when $b^* < +\infty$.

Theorem 3.5 *If $\mu(0) \leq 0$, then $b^* = 0$. If $\mu(0) > 0$, then $b^* > 0$.*

Lemma 3.6 *If $b^* < +\infty$, the function $V_{b^*}^E$ is concave on $[0, b^*)$.*

The above results will be used to derive the optimality results in the next section.

4 Optimality results

In this section we will derive the optimality results. We will first study the properties of the payoff function associated with barrier strategies for the cases when b^* is finite and when b^* is infinite separately. We will show that in the first case the barrier strategy, L^{b*}, is a MPE strategy and no barrier strategies are MPE strategies in the second case. We will also provide two examples which consider two widely used surplus models. For the two examples, the optimal strategies will be obtained and an analysis for the degree of the impact of time-inconsistency on the optimal strategies will be provided.

Theorem 4.1 *Suppose* $\mu(0) > 0$. *(i) For any finite* $b \in [0, b^*]$, $V_b(x)$ *is concave on* $(0, b]$. *(ii) If* $b^* < +\infty$, *the function* $V_{b^*}(x)$ *is twice continuously differentiable and concave on* $(0, +\infty)$. *(iii) If* $0 < b^* < +\infty$, *the following hold:*

$$\frac{\sigma^2(x)}{2} V''_{b^*}(x) + \mu(x) V'_{b^*}(x) - (\lambda + \delta) V_{b^*}(x) + \lambda\alpha V^E_{b^*}(x) = 0 \text{ for } 0 < x \leq b^* \tag{4.1}$$

$$V_{b^*}(0) = 0, \quad V'_{b^*}(b^*) = \beta, \quad V_{b^*}(x) = V_{b^*}(b^*) + \beta(b^* - x) \text{ for } x \geq b^*, \tag{4.2}$$

$$V'_{b^*}(x) = \beta, \quad V''_{b^*}(x) = 0 \text{ for } x \geq b^*, \quad V'_{b^*}(x) \geq \beta \ \text{ for } x \geq 0. \tag{4.3}$$

We show below that if there is an optimal barrier strategy, then the strategy is also a MPE stratey.

Theorem 4.2 *If* $b^* < +\infty$, *then the strategy* L^{b^*} *is a MPE strategy, that is,* $V_{b^*}(x) = \mathcal{P}(x; L^{b^*}, L^{b^*}) = \sup_{L \in \Pi} \mathcal{P}(x; L, L^{b^*})$, $x \geq 0$.

We will show below that if the optimal barrier does not exist (e.g., $b^* = +\infty$), then the payoff function for the barrier strategy increases as the barrier increases and no barrier strategies are MPE strategies.

Lemma 4.3 *If* $b^* = +\infty$, *then (i)* $V^E_{b_1}(x) \leq V^E_{b_2}(x)$ *for* $x \geq 0$ *and any* $0 \leq b_1 < b_2$; *and (ii)* $V_{b_1}(x) \leq V_{b_2}(x)$ *for* $x \geq 0$ *and any* $0 \leq b_1 < b_2$.

Theorem 4.4 *If* $b^* = +\infty$, *then no barrier strategy is a MPE strategy.*

As an immediate result of the above lemmas and theorem, we have the following corollary.

Corollary 4.5 *(i) If* $\mu(0) \leq 0$, *then* $b^* = 0$, L^0 *is a MPE strategy and* $V(x) = V_0(x) = \beta x$ *for* $x \geq 0$. *(ii) If* $\mu(0) > 0$ *and* $b^* < +\infty$, *then* L^{b^*} *is a MPE strategy and* $V(x) = V_{b^*}(x)$. *(iii) If* $\mu(0) > 0$ *and* $b^* = +\infty$, *then no barrier strategy is a MPE strategy.*

We can see that when $\mu(0) \leq 0$, the MPE strategy is to distribute all surplus as dividends immediately, which results in immediate ruin. This is the so-called take-the-money-and-run strategy. For cases with $\mu(0) > 0$, if b^* is finite, the barrier strategy with barrier b^* is a MPE strategy. In such case, the optimal barrier b^* and the value function $V_{b^*}(x)$ can be obtained by solving $(b, f(\cdot))$ to

$$\frac{\sigma^2(x)}{2} f''(x) + \mu(x) f'(x) - (\lambda + \delta) f(x) + \lambda\alpha V^E_b(x) = 0, \ 0 \leq x < b, \tag{4.4}$$

$$f(0) = 0, \quad f'(b) = \beta, \quad f''(b) = 0, \quad f'(x) = \beta, \ x \geq b, \tag{4.5}$$

where $V_b^E(x)$ is the solution to $\frac{\sigma^2(x)}{2}f''(x)+\mu(x)f'(x)-\delta f(x)=0$ for $0\leq x<b$, with $f(0)=0$, $f'(b)=\beta$, $f''(b)=0$, and $f'(x)=\beta$ for $x\geq b$.

By Lemma 3.4, we have $0\leq b^*\leq b^E$. Thus, if the optimal strategy in the corresponding optimization problem with exponential discounting is a barrier strategy (which means $b^E<+\infty$), then $b^*<+\infty$ and the optimal solution under stochastic quasi-hyperbolic discounting is also the barrier strategy with barrier b^*. Note that the barrier strategy, L^{b^E}, is the optimal strategy in the exponential discounting case, and L^{b^*} is the MPE strategy in the time-inconsistent case. This may convey some economic insights into understanding the impact of behaviorial impatience described by non-exponential discounting, particularly stochastic quasi-hyperbolic discounting, on strategic dividend payments policies. Specifically, under the premise of stochastic quasi-hyperbolic discounting in the time-inconsistent situation, the decision maker, say the management of an insurance company here, would value not too distant future income higher than distant future income. This would induce an economic incentive to distribute dividends sooner than later with a view to maximizing the profit arising from the expected discounted aggregate dividend payments. Of course, it goes without much saying that we do not take into account an agency problem here, so that the management and the shareholders of the insurance company are deemed to have the same objective of maximizing the total expected discounted dividends. In practice, the agency problem may exist.

From the above discussion we can see that b^* may be infinite only if $\mu(0)>0$ and $b^E=+\infty$. However, this is a sufficient condition only since $b^*\leq b^E$. According to the definition of b^*, a necessary condition for b^* to be infinitely large is $\sup_{b\geq 0}V_b''(b)<0$, which is equivalent to

$$\sup_{b>0}\left\{(\lambda+\delta)V_b(b)-\mu(b)\beta-\lambda\alpha V_b^E(b)\right\}<0 \text{ by (4.1).}$$

In the following two examples we will show that the barrier strategy is indeed a MPE strategy for the most widely studied model in the dividend optimization literature (the Brownian motion model) and for a more general model, in which the drift is a linear function of the state of the model. From the perspectives of economics and econometrics, the Brownian motion model (Example 1 below) describes the situation where the surplus process follows a random walk process while the Ornstein-Uhlenbeck model (Example 2 below) describes the situation where the surplus process follows a mean-reverting process. In a discrete-time situation, the former may be described by a white noise process while the latter may be described by an autoregressive process.

Example 1: The Brownian Motion Model. Assume $\mu(x)\equiv\mu(\geq 0)$ and $\sigma(x)\equiv\sigma(>0)$. The controlled surplus process is a controlled Brownian motion: $\mathrm{d}X_t^L=\mu\mathrm{d}t+\sigma\mathrm{d}W_t-\mathrm{d}L_t,\ t\geq 0$. It follows by Lemma 3.2 that $V_b^E(\cdot)$ is continuously differentiable and satisfies $\frac{\sigma^2}{2}V_b^{E''}(x)+\mu V_b^{E'}(x)-\delta V_b(x)=0$ for $0<x<b$, $V_b^E(0)=0$, $V_b^{E'}(b)=\beta$, and $V_b^{E'}(x)=\beta$ for $x>b$. Let r_1 and $-r_2$ represent the positive and negative roots, respectively, of the equation $\frac{1}{2}\sigma^2x^2+\mu x-\delta x=0$. Solving the above equations, we have

$$V_b^E(x)=\begin{cases}\frac{\beta(e^{r_1x}-e^{-r_2x})}{r_1e^{r_1b}+r_2e^{-r_2b}}, & 0\leq x\leq b,\\ \frac{\beta(e^{r_1b}-e^{-r_2b})}{r_1e^{r_1b}+r_2e^{-r_2b}}+\beta(x-b), & x>b.\end{cases}$$

By Definition 3.3, $b^E=\inf\{b>0:V_b^{E''}(b)\geq 0\}$. Then, by using the expression for V_b^E as above we can obtain $b^E=\frac{2\ln(\frac{r_2}{r_1})}{r_1+r_2}$. Because $b^*\leq b^E$, we can conclude that b^* is finite and that the barrier strategy L^{b^*} is a MPE strategy.

Example 2: The Ornstein-Uhlenbeck Model. Consider the Ornstein-Uhlenbeck type model where $\mu(x) = p + rx$ and $\sigma(x) = \sigma$ with $p \geq 0$, $\sigma > 0$ and $0 < r < \delta$. The controlled process is $\mathrm{d}X_t^L = (p + rX_t^L)\mathrm{d}t + \sigma \mathrm{d}W_t - \mathrm{d}L_t$, $t \geq 0$. Here, we can interpret r as the force of interest that the surplus is earning. It is natural to assume that r is lower than the risk discount rate, δ, that shareholders use to discount dividend payment cash flows, as δ is generally r plus a risk margin. It follows by Lemma 3.2 that $V_b^E(\cdot)$ is continuously differentiable and satisfies the following equations

$$\frac{\sigma^2}{2} V_b^{E''}(x) + (p + rx) V_b^{E'}(x) - \delta V_b(x) = 0 \text{ for } 0 < x < b, \quad V_b^{E'}(x) = \beta \text{ for } x > b. \tag{4.6}$$

and the conditions: $V_b^E(0) = 0$ and $V_b^{E'}(b) = \beta$. Consider the homogeneous equation

$$\frac{\sigma^2}{2} g''(x) + (p + rx) g'(x) - \delta g(x) = 0, \ x > 0. \tag{4.7}$$

Following the same procedure used in (Cai, Gerber, & Yang, 2006), we can convert the above equation into a Kummer's confluent hypergeometric equation. Let $t = -\frac{(p+rx)^2}{r\sigma^2}$ and $h(t) := g(x)$. We can see that the above equation is equivalent to the following Kummer's confluent hypergeometric equation: $th''(t) + (c - t)h'(t) - a_1 h(t) = 0$, for $t < -\frac{p^2}{r\sigma^2}$ with $a_1 = -\frac{\delta}{2r}$ and $c = \frac{1}{2}$. Two independent solutions to the above equation are (see page 505 of (Abramowitz & Stegun, 1968)) $h_1(t) = U(a_1, c, t)$ and $h_2(t) = e^t U(c - a_1, c, -t)$, where the function $U(\cdot, \cdot, \cdot)$ is the confluent hypergeometric function of the second kind. Hence, if we define, $g_1(x) = U(a_1, \frac{1}{2}, -\frac{(p+rx)^2}{r\sigma^2})$ and $g_2(x) = e^{-\frac{(p+rx)^2}{r\sigma^2}} U(\frac{1}{2} - a_1, \frac{1}{2}, \frac{(p+rx)^2}{r\sigma^2})$, then g_1 and g_2 are two independent solutions to (4.7). Therefore, for some constants, K_1 and K_2, $V_b^E(x) = K_1 g_1(x) + K_2 g_2(x)$. By noting $V_b^E(0) = 0$ and $V_b^{E'}(b) = \beta$, we obtain $K_1 g_1(0) + K_2 g_2(0) = 0$ and $K_1 g_1'(b) + K_2 g_2'(b) = \beta$. Let $K_1(b)$ and $K_2(b)$ represent K_1 and K_2 that solve the above equations. Then, $V_b^E(x) = K_1(b) g_1(b) + K_2(b) g_2(b)$, where $K_1(b) = \frac{g_2(0)\beta}{g_2(0)g_1'(b) - g_1(0)g_2'(b)}$ and $K_2(b) = -\frac{g_1(0)\beta}{g_2(0)g_1'(b) - g_1(0)g_2'(b)}$.

For any f and g, we say $f(x) \sim g(x)$ iff $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1$. It follows from Eq. (13.5.2) on page 508 of (Abramowitz & Stegun, 1968) that $g_1(x) = U(a_1, \frac{1}{2}, -\frac{(p+rx)^2}{r\sigma^2}) \sim (-\frac{(p+rx)^2}{r\sigma^2})^{-a_1}$ and $g_2(x) = e^{-\frac{(p+rx)^2}{r\sigma^2}} U(\frac{1}{2} - a_1, \frac{1}{2}, \frac{(p+rx)^2}{r\sigma^2}) \sim e^{-\frac{(p+rx)^2}{r\sigma^2}} (\frac{(p+rx)^2}{r\sigma^2})^{a_1 - 0.5}$. Note $\frac{\partial}{\partial z} U(a, b, z) = -aU(a+1, b+1, z)$. Thus,

$$\begin{aligned} g_1'(x) &= a_1 \frac{2(p+rx)}{\sigma^2} U(a_1 + 1, 1.5, -\frac{(p+rx)^2}{r\sigma^2}) \sim a_1 \frac{2}{\sigma^2} (-\frac{1}{r\sigma^2})^{-a_1 - 1} (p+rx)^{-2a_1 - 1} \\ g_2'(x) &= -\frac{(p+rx)}{\sigma^2} e^{-\frac{(p+rx)^2}{r\sigma^2}} \left(2U(\frac{1}{2} - a_1, \frac{1}{2}, \frac{(p+rx)^2}{r\sigma^2}) + (1 - 2a_1) U(1.5 - a_1, 1.5, \frac{(p+rx)^2}{r\sigma^2}) \right) \\ &\sim -\frac{2}{\sigma^2} e^{-\frac{(p+rx)^2}{r\sigma^2}} (\frac{1}{r\sigma^2})^{-\frac{1}{2} + a_1} (p+rx)^{2a_1}. \end{aligned}$$

Furthermore, $g_1''(x) \sim \left(a_1 \frac{2r}{\sigma^2} (-\frac{1}{r\sigma^2})^{-a_1 - 1} + (\frac{2}{\sigma^2})^2 a_1 (a_1 + 1)(-\frac{1}{r\sigma^2})^{-a_1 - 2} \right) (p+rx)^{-2a_1 - 2}$, and $g_2''(x) \sim (1.5 - a_1)(\frac{2}{\sigma^2})^2 e^{-(\frac{(p+rx)^2}{r\sigma^2})} (\frac{1}{r\sigma^2})^{a_1 - \frac{1}{2}} (p+rx)^{2a_1 + 1}$. Then, $K_1(b) g_1''(b) = \frac{\beta g_2(0) g_1''(b)}{g_2(0)g_1'(b) - g_1(0)g_2'(b)} \sim \frac{\beta(\delta - r)}{p + rb}$ and

$$K_2(b) g_2''(b) = \frac{-g_1(0)\beta g_2''(b)}{g_2(0)g_1'(b) - g_1(0)g_2'(b)} \sim -(-1)^{a_1 + 1} \frac{g_1(0)\beta(1.5 - a_1)\frac{2}{\sigma^2} e^{-(\frac{(p+rb)^2}{r\sigma^2})} (\frac{1}{r\sigma^2})^{2a_1 + 1} (p+rb)^{4a_1 + 2}}{g_2(0) a_1}.$$

As a result, $V_b^E(b) = K_1(b) g_1''(b) + K_2(b) g_2''(b) \sim \frac{\beta(\delta - r)}{p + rb}$. Noticing that $\delta - r > 0$, we can therefore conclude that there exists $b_1 > 0$ so that $V_{b_1}^{E''}(b_1) > 0$. It follows from (4.6) and $V_b^E(0) = 0$ that $V_b^{E''}(0) = \frac{2}{\sigma^2}(-pV_b^{E'}(0+) + \delta V_b^E(0+)) = -\frac{2p}{\sigma^2} pg'(0+) < 0$. Then, $\lim_{b\downarrow 0} V_b^{E''}(b) = K_1(0) g_1''(0) + K_2(0) g_2''(0) =$

$\lim_{b\downarrow 0}(K_1(b)g_1''(0)+K_2(b)g_2''(0)) = \lim_{b\downarrow 0} V_b^{E''}(0) < 0$. Define $h(b) = V_b^{E''}(b) = K_1(b)g_1''(b)+K_2(b)g_2''(b)$ for $b>0$. We can see that $h(b)$ is continuous on $(0,+\infty)$. By noticing $\lim_{b\downarrow 0} h(b) = \lim_{b\downarrow 0} V_b^{E''}(b) < 0$, $h_{b_1}(b_1) > 0$ and the continuity of $h(\cdot)$, we can conclude that there exists a positive b such that $V_b^{E''}(b) = 0$. Hence, from Definition 3.3 (i), we can see that $b^E < +\infty$ and is the smallest positive solution to $V_b^{E''}(b) = 0$. As $b^* \le b^E$ (see (3.4)), we conclude that $b^* < \infty$. Thus, L^{b^*} is a MPE strategy.

5 Numerical illustration

We will provide numerical illustrations by considering the two examples studied in the last section. It is intended that the numerical examples would illustrate how the optimal strategies may be implemented and that they would provide concrete illustrations for the economic insights that may be conveyed by the theoretical results on optimality established in the last section.

Example 1 (continued): The Brownian Motion Model. Now assume $\mu = 1$, $\sigma = 2$, $\delta = 0.1$, $\beta = 0.95$, $\lambda = 1$ and $\alpha = 0.95$. It follows by the formula for b^E in Example 1 in the last section that $b^E = 5.738786$. It follows by Remark 3.1 that

$$\frac{\sigma^2}{2}V_b''(x)+\mu V_b'(x)-(\delta+\lambda)V_b(x)+\lambda\alpha V_b^E(x)=0 \text{ for } 0<x<b, \tag{5.1}$$
$$V_b(0)=0, \quad V_b'(b)=\beta, \quad V_b'(x)=\beta \text{ for } x>b.$$

Let θ_1 and $-\theta_2$ represent the positive and negative roots, respectively, of the equation $\frac{1}{2}\sigma^2x^2+\mu x-(\lambda+\delta) = 0$. We can see that $e^{\theta_1 x}$ and $e^{-\theta_2 x}$ are two linearly independent solutions to $\frac{\sigma^2}{2}f''(x)+\mu f'(x)-(\lambda+\delta)f(x) = 0$. Define $f_1(x) = e^{\theta_1 x}$ and $f_2(x) = e^{-\theta_2 x}$. The Wronskian of f_1 and f_2 is $W(x) = f_1(x)f_2'(x)-f_2(x)f_1'(x) = -(\theta_1+\theta_2)e^{(\theta_1-\theta_2)x}$. By applying the variation of coefficients method, we know that any solution to (5.1) has the following general form: $C_1f_1(x)+C_2f_2(x)-f_1(x)\int_0^x \frac{-2\lambda\alpha f_2(y)V_b^E(y)}{W(y)\sigma^2}dy + f_2(x)\int_0^x \frac{-2\lambda\alpha f_1(y)V_b^E(y)}{W(y)\sigma^2}dy$. Hence, for some constants C_1 and C_2,

$$V_b(x) = C_1e^{\theta_1 x}+C_2e^{-\theta_2 x}-\frac{2\lambda\alpha}{\sigma^2(\theta_1+\theta_2)}e^{\theta_1 x}\int_0^x \frac{e^{-\theta_2 y}V_b^E(y)}{e^{(\theta_1-\theta_2)y}}dy+\frac{2\lambda\alpha}{\sigma^2(\theta_1+\theta_2)}e^{-\theta_2 x}\int_0^x \frac{e^{\theta_1 y}V_b^E(y)}{e^{(\theta_1-\theta_2)y}}dy.$$

Since $V_b(0) = 0$, we have $C_2 = -C_1$. By plugging in $V_b^E(y) = \frac{\beta(e^{r_1y}-e^{-r_2y})}{r_1e^{r_1b}+r_2e^{-r_2b}}$ (see Example 1 in the last section) and $C_2 = -C_1$, we obtain

$$V_b(x) = C_1(e^{\theta_1 x}-e^{-\theta_2 x})+M(b)\left(-\frac{e^{r_1x}-e^{\theta_1 x}}{r_1-\theta_1}-\frac{e^{-r_2x}-e^{\theta_1 x}}{r_2+\theta_1}++\frac{e^{r_1x}-e^{-\theta_2 x}}{r_1+\theta_2}+\frac{e^{-r_2x}-e^{-\theta_2 x}}{r_2-\theta_2}\right),$$

where $M(b) = \frac{2\lambda\alpha\beta}{(\theta_1+\theta_2)\sigma^2(r_1e^{r_1b}+r_2e^{-r_2b})}$. Hence, $V_b'(x) = C_1(\theta_1e^{\theta_1 x}+\theta_2e^{-\theta_2 x})+M(b)N(x)$ with $N(x) = \left(-\frac{r_1e^{r_1x}-\theta_1e^{\theta_1 x}}{r_1-\theta_1}+\frac{r_2e^{-r_2x}+\theta_1e^{\theta_1 x}}{r_2+\theta_1}+\frac{r_1e^{r_1x}+\theta_2e^{-\theta_2 x}}{r_1+\theta_2}-\frac{r_2e^{-r_2x}-\theta_2e^{-\theta_2 x}}{r_2-\theta_2}\right)$. By letting $V_b'(b) = \beta$, we have $C_1 = \frac{\beta-M(b)N'(b)}{T(b)}$ with $T(b) = (\theta_1e^{\theta_1 b}+\theta_2e^{-\theta_2 b})$. Furthermore, $V_b''(x) = (\theta_1^2e^{\theta_1 x}-\theta_2^2e^{-\theta_2 x})C_1(b)+M(b)N''(x)$. Thus, b^* is the positive solution to the equation:, $(\theta_1^2e^{\theta_1 b}-\theta_2^2e^{-\theta_2 b})C_1(b)+M(b)N''(b) = 0$. By plugging in the numerical value of the model parameters provided above and solving the above equation we obtain $b^* = 5.243$. Thus, the barrier strategy, $L^{5.243}$, is a MPE strategy.

Example 2 (continued): The Ornstein-Uhlenbeck Model. Suppose $\sigma = 1$, $p = 1$, $r = 0.05$, $\delta = 0.1$ and $\beta = 0.95$. It has been shown in the last section that b^E is finite and is the smallest positive solution of

b to $V_b^{E\prime\prime}(b) = 0$. Furthermore, V_b^E is the solution to (4.6). Hence, b^E and $V_{b^E}^E(\cdot)$ when restricted to $(0, b^E)$ is the solution of $(b, f(\cdot))$ to $\frac{\sigma^2}{2} f''(x) + (p + rx) f'(x) - \delta f(x) = 0$ for $0 < x < b$ with $f(0) = 0$, $f'(b) = \beta$ and $f''(b) = 0$. By plugging in the numerical values of the parameters and then solving these equations numerically, we obtain $b^E = 3.101$.

It was shown in the last section that L^{b^*} is a MPE strategy. From the paragraph following Corollary 4.5, we know that b^* and V_{b^*} when restricted to $(0, b^*)$ are the solutions of $(b, f(\cdot))$ to the equation $\frac{\sigma^2}{2} f''(x) + (p + rx) f'(x) - (\lambda + \delta) f(x) + \lambda \alpha V_b^E(x) = 0$ for $0 < x < b$ with boundary conditions $f(0) = 0$, $f'(b) = \beta$ and $f''(b) = 0$, where V_b^E is the solution to the equation $\frac{\sigma^2}{2} V_b^{E\prime\prime}(x) + (p+rx) V_b^{E\prime} - \delta V_b^E(x) = 0$ for $0 < x < b$ with $V_b^E(0) = 0$ and $V_b^{E\prime}(b) = \beta$. By plugging the numerical values of the parameters provided as well as using the numerical values of α and λ provided in Tables 1 and 2 to solve the boundary-value problem, we obtain b^*, which is presented in Tables 1 and 2.

Table 1: The values of the optimal barrier with $\lambda = 1$ and varying values of α

α	0.50	0.55	0.60	0.65	0.70	0.75	0.80	0.85	0.90	0.95	1.00
b^*	1.101	1.158	1.222	1.294	1.375	1.469	1.581	1.719	1.905	2.197	3.101

In Table 1, we can see that a higher α corresponds to a higher optimal barrier b^*. This may make intuitive sense. Say a higher α implies a lower discount placed on dividends receivable in the future periods and hence higher present values for future dividend payments. Therefore, it is optimal to raise the dividend barrier so that ruin will occur later; this produces dividends over a longer period.

Notice that when $\alpha = 1$, $b^* = 3.101 = b^E$. This can be explained in that when $\alpha = 1$, the decision maker is applying exponential discounting; the optimization problem then reduces to the time-consistent problem, and therefore b^* is the same as b^E. These observations are in line with the economic intuition that is conveyed by the theoretical results established in this paper.

Table 2: The values of the optimal barrier with $\alpha = 0.9$ and varying values of λ

λ	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
b^*	3.101	2.925	2.739	2.574	2.433	2.314	2.210	2.120	2.040	1.969	1.905

In Table 2, we can see that when λ increases, b^* decreases. This is because, with a higher λ, the future periods will arrive earlier. There is a higher discount for dividends in the future periods than the present period. Thus, the decision maker needs to pay dividends earlier and reduce the dividend barrier to achieve a higher expected present value of dividends.

When $\lambda = 0$, $b^* = 3.101 = b^E$. This is because λ represents the arrival rate of the future period and $\lambda = 0$ means that the present period lasts forever. In this example, the decision maker is using the exponential discounting at the same discount rate δ for all future dividends, and hence the optimization problem reduces to the time-consistent case. As a result, $b^* = b^E$.

6 Conclusion

It is the intention of this paper to provide a theoretical study on the impact of behaviorial and psychological features of human's impatience on strategic decision making of dividend payments of insurance companies. Specifically, these features of human's impatience manifest themselves as time-inconsistent preferences, which are technically described by stochastic quasi-hyperbolic discounting. A linear diffusion model was considered for the surplus process of an insurance company. The model generalizes those in the relevant literature. We studied the optimality of a class of widely used strategies, say barrier strategies, which were shown to be the optimal strategy in most cases with similar setting when decision makers are time-consistent. We showed that a barrier strategy with an optimal barrier is a perfect Markov equilibrium (PME) strategy. We also provided a condition under which the barrier strategy is no longer a PME strategy.

Appendix

Proof of Lemma 3.1. The non-negativity of $V_b(x)$ on $[0,\infty)$ follows immediately from its definition in (3.1) and Remark 2.1. We now proceed to show that $V_b(x)$ is non-decreasing on $[0,b]$. Let L^b represent the barrier strategy with the barrier, b. For any $x>0$, let $R_t^{x,b}$ represent the controlled stochastic process $\mathrm{d}R_t^{x,b}=\mu(R_{t-}^{x,b})\mathrm{d}t+\sigma(R_{t-}^{x,b})\mathrm{d}W_t-\mathrm{d}L_t^b$ with $R_{0-}^{x,b}=x$. Now consider $R_t^{x_1,b}$ and $R_t^{x_2,b}$ with $0\le x_1<x_2\le b$. By extending the comparison theorem (Theorem 1.1 in (Ikeda & Watanabe, 1977)) slightly we can show that with probability 1, $R_t^{x_2,b}\ge R_t^{x_1,b}$ for all $t\ge 0$, and thus, when $R_t^{x_2,b}$ is distributing dividends, $R_t^{x_1,b}$ may or may not distribute dividends, and when $R_t^{x_1,b}$ is distributing dividends, $R_t^{x_2,b}$ also distributes dividends at the same rate with probability 1. As a result, $V_b(x_1)\le V_b(x_2)$ for $0\le x_1\le x_2\le b$. For any x_1 and x_2 with $x_1\le b<x_2$, we obtain $V_b(x_1)\le V_b(b)\le V_b(b)+x_2-b=V_b(x_2)$. For any x_1 and x_2 with $b\le x_1<x_2$, $V_b(x_1)\le V_b(b)+x_1-b\le V_b(b)+x_2-b=V_b(x_2)$. Thus, $V_b(\cdot)$ is non-decreasing. □

Proof of Lemma 3.2 (i) Throughout this proof, we use $V_U(\cdot)$ to represent the same function defined in Equation (4.3) of (Shreve, Lehoczky, & Gaver, 1984). Now let us set some of the quantities in that reference as follows: $U=b$, $P=0$, $\beta=\delta$ and $a(\cdot)\equiv\mu(\cdot)$. We can see that the process ξ_U involved in the definition for V_U in (Shreve et al., 1984) coincides with the barrier strategy, L^U, in this paper, and therefore, from the definitions for V_b^E in this paper we observe $V_b^E(\cdot)\equiv\beta V_b(\cdot)$. From the paragraph following Equation (4.3) in (Shreve et al., 1984), we know that for any $b\ge 0$, $V_b(\cdot)$ satisfies $\frac{\sigma^2(x)}{2}V_b{''}(x)+a(x)V_b{'}(x)-\beta V_b(x)=0$ for $x\in(0,b)$, $V_b(0)=P=0$ and $V_b'(b)=1$. Further note that $V_b^{E\prime}(x)=\beta$ for $x>b$ (see (3.5)). The above implies that (3.4) and (3.5) hold. We conclude that $\lim_{b\downarrow 0}V_b^E(b)=0$ and $\lim_{b\downarrow 0}V_b^E(x)=V_0^E(x)$ by taking limits on the expression for $V_U(x)$ in Equation (4.8) of (Shreve et al., 1984) and noting $V_0^E(x)=\beta x$. Since, $V_b^E(x)=\beta V_b(x)$ and $V_0^E(x)=\beta x$, the last two equations in (3.5) follow immediately. Noting that $V_b^E(0)=0$ and $V_b^{E\prime}(b)=1$, it follows by Lemma 4.1 in (Shreve et al., 1984) that $V_E'(x)=\beta V_b'(x)>0$ for $x\in[0,b]$. Hence, by noting $V_b^{E\prime}(x)=\beta$ for $x>b$ (see (3.5)), we conclude that V_b^E is strictly increasing on $[0,+\infty)$. (ii) Alternatively, all of the statements in (i) follow immediately by noticing that $V_b^E(x)/\beta$ is identical to the function $\mathcal{T}_b(f,1)$ (defined in (Zhu & Chen, 2013)) with $f\equiv 0$ and using the property of $\mathcal{T}_b(f,1)$ in (Zhu & Chen, 2013). The statement in (ii) follows by Lemma 3.6 in (Zhu & Chen, 2013). □

Following are some important results that will be used to prove some later lemmas and theorems.

Lemma A.1 *For any $v \geq 0$, any $L \in \Pi$ and any $b \geq 0$, let $\bar{\pi}^{v,L,L^b}$ represent the strategy such that $\mathrm{d}\bar{\pi}_t^{v,L,L^b} = \mathrm{d}L_t$ for $0 \leq t < v$ and $\mathrm{d}\bar{\pi}_t^{v,L,L^b} = \mathrm{d}L_t^b$ for $t \geq v$. The following hold:*

$$\mathrm{E}_x\left[\int_0^{T^L\wedge\eta_0} \beta e^{-\delta t}\mathrm{d}L_t\right] = \mathrm{E}_x\left[\beta\int_0^{T^L} e^{-(\lambda+\delta)t}\mathrm{d}L_t\right],\ x \geq 0, \tag{A.1}$$

$$\mathrm{E}_x\left[I\{\eta_0 \leq T^L\}\alpha e^{-\delta\eta_0}V_b^E(R_{\eta_0-}^L)\right] = E_x\left[\int_0^{T^L} e^{-(\lambda+\delta)s}\lambda\alpha V_b^E(R_{s-}^L)\mathrm{d}s\right],\ x \geq 0, \tag{A.2}$$

$$\mathrm{E}_x\left[I\{\eta_0 \leq T^{\bar{\pi}^{\eta_0,L,L^b}}\}\int_{\eta_0}^{T^{\bar{\pi}^{\eta_0,L,L^b}}} \beta e^{-\delta t}\mathrm{d}\bar{\pi}_t^{\eta_0,L,L^b}\right] = \mathrm{E}_x\left[\int_0^{T^L} e^{-(\lambda+\delta)s}\lambda V_b^E(R_{s-}^L)\mathrm{d}s\right],\ x \geq 0. \tag{A.3}$$

Proof. Note that η_0 is an exponential random variable with mean $1/\lambda$. By conditioning on η_0 and using the independence between η_0 and (L, T^L) (due to the independence between η_0 and R^L), we obtain that for $x \geq 0$,

$$\begin{aligned}\mathrm{E}_x\left[\int_0^{T^L\wedge\eta_0} \beta e^{-\delta t}\mathrm{d}L_t\right] &= \int_0^\infty \mathrm{E}_x\left[\beta\int_0^{T^L\wedge s} e^{-\delta t}\mathrm{d}L_t\right]\lambda e^{-\lambda s}\mathrm{d}s = \mathrm{E}_x\left[\int_0^\infty \beta\lambda e^{-\lambda s}\left(\int_0^{T^L\wedge s} e^{-\delta t}\mathrm{d}L_t\right)\mathrm{d}s\right] \\ &= \mathrm{E}_x\left[\beta\int_0^{T^L} e^{-\delta t}\left(\int_t^\infty \lambda e^{-\lambda s}\mathrm{d}s\right)\mathrm{d}L_t\right] = \mathrm{E}_x\left[\beta\int_0^{T^L} e^{-(\lambda+\delta)t}\mathrm{d}L_t\right],\end{aligned} \tag{A.4}$$

$$\begin{aligned}\mathrm{E}_x\left[I\{\eta_0 \leq T^L\}e^{-\delta\eta_0}V_b^E(R_{\eta_0-}^L)\right] &= \int_0^\infty \mathrm{E}_x\left[I\{s \leq T^L\}e^{-\delta s}V_b^E(R_{s-}^L)\right]\lambda e^{-\lambda s}\mathrm{d}s \\ &= \mathrm{E}_x\left[\int_0^{T^L} e^{-(\lambda+\delta)s}\lambda V_b^E(R_{s-}^L)\mathrm{d}s\right],\ x \geq 0.\end{aligned} \tag{A.5}$$

Now we proceed to prove (A.3). We can also see that η_0 is independent of $(\bar{\pi}^{s,L,L^b}, R^{\bar{\pi}^{s,L,L^b}})$ and thus, it is also independent of $T^{\bar{\pi}^{s,L,L^b}}$. Hence, by conditioning on η_0 and then taking expectation we have

$$\begin{aligned}&\mathrm{E}_x\left[I\{\eta_0 \leq T^{\bar{\pi}^{\eta_0,L,L^b}}\}\int_{\eta_0}^{T^{\bar{\pi}^{\eta_0,L,L^b}}} e^{-\delta t}\mathrm{d}\bar{\pi}_t^{\eta_0,L,L^b}\right] = \int_0^\infty \mathrm{E}_x\left[I\{s \leq T^{\bar{\pi}^{s,L,L^b}}\}\int_s^{T^{\bar{\pi}^{s,L,L^b}}} e^{-\delta t}\mathrm{d}\bar{\pi}_t^{s,L,L^b}\right]\lambda e^{-\lambda s}\mathrm{d}s \\ &= \int_0^\infty \mathrm{E}_x\left[I\{s \leq T^{\bar{\pi}^{s,L,L^b}}\}\mathrm{E}_x\left[\int_s^{T^{\bar{\pi}^{s,L,L^b}}} e^{-\delta t}\mathrm{d}\bar{\pi}_t^{s,L,L^b}\middle|\mathcal{F}_{s-}^{0,W}\right]\right]\lambda e^{-\lambda s}\mathrm{d}s,\ x \geq 0,\end{aligned} \tag{A.6}$$

where the last equality follows by noticing that $\{s \leq T^{\bar{L}^{s,L,L^b}}\} = \{R_v^{T^{\bar{L}^{s,L,L^b}}} > 0 \text{ for all } v \in [0,s)\}$ given $R_0 > 0$ and therefore $I\{s \leq T^{\bar{\pi}^{s,L,L^b}}\}$ is $\mathcal{F}_{s-}^{0,W}$ measurable. It follows by the strong Markov property of $R^{\bar{\pi}^{s,L,L^b}}$ and $\bar{\pi}^{s,L,L^b}$ that for $x \geq 0$,

$$\begin{aligned}&\mathrm{E}_x\left[\int_s^{T^{\bar{\pi}^{s,L,L^b}}} \beta e^{-\delta t}\mathrm{d}\bar{\pi}_t^{s,L,L^b}\middle|\mathcal{F}_{s-}^{0,W}\right] = e^{-\delta s}\mathrm{E}_x\left[\int_s^{T^{\bar{\pi}^{s,L,L^b}}} \beta e^{-\delta(t-s)}\mathrm{d}\bar{\pi}_t^{s,L,L^b}\middle|\mathcal{F}_{s-}^{0,W}\right] \\ &= e^{-\delta s}\mathrm{E}_{R_{s-}^{\bar{\pi}^{s,L,L^b}}}\left[\int_0^{T^{\bar{\pi}^{0,L,L^b}}} \beta e^{-\delta t}\mathrm{d}\bar{\pi}_t^{0,L,L^b}\right] = e^{-\delta s}\mathrm{E}_{R_{s-}^{\bar{\pi}^{s,L,L^b}}}\left[\int_0^{T^{L^b}} \beta e^{-\delta t}\mathrm{d}L_t^b\right] = e^{-\delta s}V_b^E(R_{s-}^{\bar{\pi}^{s,L,L^b}}),\end{aligned} \tag{A.7}$$

where the second to the last equality follows by noting $\bar{\pi}^{0,L,L^b} = \bar{L}^b$ and the last equality follows by Definition 3.2 for $V_b^E(\cdot)$. Notice that $\bar{\pi}^{s,L,L^b}$ coincides with L for the time period $[0,s)$. Therefore,

$$R_v^{\bar{\pi}^{s,L,L^b}} = R_v^L, 0 \leq v < s, \tag{A.8}$$

$$I\{s \leq T^{\bar{\pi}^{s,L,L^b}}\} = I\{R_v^{\bar{\pi}^{s,L,L^b}} > 0 \text{ for all } v \in [0,s)\} = I\{R_v^L > 0 \text{ for all } v \in [0,s)\} = I\{s \leq T^L\}. \tag{A.9}$$

Multiplying (A.6) by β and the combining it with (A.7) yields
$\mathrm{E}_x\left[I\{\eta_0 \le T^{\bar{\pi}^{\eta_0,L,L^b}}\}\int_{\eta_0}^{T^{\bar{\pi}^{\eta_0,L,L^b}}}\beta e^{-\delta t}\mathrm{d}\bar{\pi}_t^{\eta_0,L,L^b}\right] = \int_0^{+\infty}\lambda e^{-(\lambda+\delta)s}\times \mathrm{E}_x\left[I\{s\le T^{\bar{\pi}^{\eta_0,L,L^b}}\}V_b^E(R_{s-}^{\bar{\pi}^{\eta_0,L,L^b}})\right]\mathrm{d}s$
$= \mathrm{E}_x\left[\int_0^{T^L} e^{-(\lambda+\delta)s}\lambda V_b^E(R_{s-}^L)\mathrm{d}s\right]$ for $x\ge 0$, where the last equality follows by (A.8) and (A.9). □

Define the operator $\mathcal{G}\in\mathcal{C}^2(\mathbb{R}^+)$ by,

$$\mathcal{G}_f(x) = \frac{\sigma^2(x)}{2}f''(x) + \mu(x)f'(x) - (\delta+\lambda)f(x). \tag{A.10}$$

Proof of Lemma 3.3 (i) As in (Zhu & Yang, 2016), we use the Variation of Parameters method to construct solutions to the non-constant coefficient differential equation, (3.4). Let $v_1(\cdot)$ represent the unique classical solution to $\frac{\sigma^2(x)}{2}f''(x)+\mu(x)f'(x)-(\lambda+\delta)f(x)=0$ with initial value conditions $f(0)=1$ and $f'(0)=1$, and $v_2(\cdot)$ represent the unique classical solution to the same equation with initial value conditions $f(0)=1$ and $f'(0)=0$. Then, the pair $v_1(x)$ and $v_2(x)$ form a set of linearly independent solutions to the homogeneous equation corresponding to (3.6). Define $W(x)=v_1(x)v_2'(x)-v_2(x)v_1'(x)$. The function, $W(x)$, is the Wronskian of the fundamental set of solutions v_1 and v_2 and, is always non-zero. Further define $B(x)=v_1(x)\int_0^x\frac{v_2(y)}{W(y)}\frac{2\lambda\alpha V_b^E(y)}{\sigma^2(y)}\mathrm{d}y - v_2(x)\int_0^x\frac{v_1(y)}{W(y)}\frac{2\lambda\alpha V_b^E(y)}{\sigma^2(y)}\mathrm{d}y$. Then, any solution to the equation (3.6) can be expressed in the form $K_1v_1(\cdot)+K_2v_2(\cdot)+B(x)$, where K_1 and K_2 are constants. Define $g_b(x)=K_1(b)v_1(x)+K_2(b)v_2(x)+B(x)$, where $K_1(b)$ and $K_2(b)$ are the solutions, say K_1 and K_2, of the following equations, $K_1v_1(0)+K_2v_2(0)+B(0)=0$ and $K_1v_1'(b)+K_2v_2'(b)+B'(b)=\beta$. Noting that $v_1(0)=v_2(0)=1$ and $B(0)=0$, we can solve

$$K_1(b)=\frac{\beta-B'(b)}{v_1'(b)-v_2'(b)},\quad K_2(b)=-K_1(b)=\frac{B'(b)-\beta}{v_1'(b)-v_2'(b)}. \tag{A.11}$$

We can verify that $g_b(0)=0$, that $g_b(\cdot)$ is twice continuously differentiable, and that $g_b'(b)=\beta$. Hence, the existence of a twice continuously differentiable solution to (3.6) is proven.

(ii) From (i) we note $S_b(x)\equiv g_b(x)$, and therefore $S_b(0)=0$. Define

$$w_b(x)=S_b(x)\text{ for }0\le x\le b\text{ and }w_b(x)=\beta(x-b)+S_b(b)\text{ for }x>b. \tag{A.12}$$

We can observe that $w_b(x)$ is continuously differentiable on $[0,+\infty)$ and twice continuously differentiable on the same interval except for the point b. Using Itô's formula and (2.1), we obtain that for any $t>0$,

$$\begin{aligned}
&\mathrm{E}_x\left[e^{-(\lambda+\delta)(T^{L^b}\wedge t)}w_b(R^{L^b}_{T^{L^b}\wedge t})-w_b(R^{L^b}_{0-})\right]\\
=\;&\mathrm{E}_x\Bigg[\int_0^{T^{L^b}\wedge t}e^{-(\lambda+\delta)s}\mathcal{G}_{w_b}(R^{L^b}_{s-})\mathrm{d}s+\int_0^{T^{L^b}\wedge t}e^{-(\lambda+\delta)s}\sigma(R^{L^b}_{s-})w_b'(R^{L^b}_{s-})\mathrm{d}W_s\\
&-\int_0^{T^{L^b}\wedge t}e^{-(\lambda+\delta)s}w_b'(R^{L^b}_{s-})\mathrm{d}L_s^{b,c}+\sum_{0\le s\le T^{L^b}\wedge t}e^{-(\lambda+\delta)s}\left(w_b(R_s^{L^b})-w_b(R^{L^b}_{s-})\right)\Bigg],
\end{aligned}\tag{A.13}$$

where $\{L_s^{b,c}\}$ is the continuous part of $\{L_s^b\}$. Notice by the definition of L^b, $R^{L^b}_{s-}\in[0,b]$ for $0<s\le T^{L^b}$. Further note that $w_b(y)=S_b(y)$ for $y\in[0,b]$ and $S_b(y)$ satisfies (3.6). Hence,

$$\mathcal{G}_{w_b}(R^{L^b}_{s-})=-\lambda\alpha V_b^E(R^{L^b}_{s-})\quad\text{for }0<s\le T^{L^b}. \tag{A.14}$$

Because $R^{L^b}_{s-} \in [0,b]$ for $0 < s \le T^{L^b}$, the process, $\left\{\int_0^{T^{L^b}\wedge t} e^{-(\lambda+\delta)s}\sigma(R^{L^b}_{s-})w_b'(R^{L^b}_{s-})\mathrm{d}W_s; t\ge 0\right\}$, is a P_x-martingale, which implies

$$\mathrm{E}_x\left[\int_0^{T^{L^b}\wedge t} e^{-(\lambda+\delta)s}\sigma(R^{L^b}_{s-})w_b'(R^{L^b}_{s-})\mathrm{d}W_s\right] = 0. \tag{A.15}$$

It follows by the definition of L^b in Definition 3.1(i) that

$$L^b_s I\{R^{L^b}_{s-} \le b\} = L^b_{s-} I\{R^{L^b}_{s-} \le b\}, \quad R^{L^b}_s I\{R^{L^b}_{s-} \le b\} = R^{L^b}_{s-} I\{R^{L^b}_{s-} \le b\}, \tag{A.16}$$

$$L^b_s - L^b_{s-} = (R^{L^b}_{s-} - b)I\{R^{L^b}_{s-} > b\}, \quad \mathrm{d}L^{b,c}_s = I\{R^{L^b}_{s-} = b\}\mathrm{d}L^{b,c}_s. \tag{A.17}$$

Note that by (A.12) we have $w_b(y) - w_b(b) = \beta(y-b)$ for $y > b$. As a result of this and (A.16),

$$\begin{aligned}\sum_{0\le s\le T^{L^b}\wedge t} e^{-(\lambda+\delta)s}(w_b(R^{L^b}_s) - w_b(R^{L^b}_{s-})) = \sum_{0\le s\le T^{L^b}\wedge t} e^{-(\lambda+\delta)s}(w_b(R^{L^b}_s) - w_b(R^{L^b}_{s-}))I\{R^{L^b}_{s-} > b\} \\ = -\sum_{0\le s\le T^{L^b}\wedge t} e^{-(\lambda+\delta)s}\beta(R^{L^b}_{s-} - b)I\{R^{L^b}_{s-} > b\} = -\sum_{0\le s\le T^{L^b}\wedge t} e^{-(\lambda+\delta)s}\beta(L^b_s - L^b_{s-}),\end{aligned} \tag{A.18}$$

where the last equality follows by (A.17). By (A.17), we can also obtain

$$\int_0^{T^{L^b}\wedge t} e^{-(\lambda+\delta)s}w_b'(R^{L^b}_{s-})\mathrm{d}L^{b,c}_s = \int_0^{T^{L^b}\wedge t} e^{-(\lambda+\delta)s}w_b'(b)I\{R^{L^b}_{s-} = b\}\mathrm{d}L^{b,c}_s = \int_0^{T^{L^b}\wedge t} e^{-(\lambda+\delta)s}\beta\mathrm{d}L^{b,c}_s, \tag{A.19}$$

where the last equality follows by $w_b'(b) = \beta$ (by (A.12) and $S_b'(b) = g_b'(b) = \beta$) and (A.17) again. It follows from (A.13), (A.14), (A.15), (A.18) and (A.19) that $\mathrm{E}_x\left[e^{-(\lambda+\delta)(T^{L^b}\wedge t)}w_b(R^{L^b}_{T^{L^b}\wedge t}) - w_b(R^{L^b}_{0-})\right] = -\mathrm{E}_x\left[\int_0^{T^{L^b}\wedge t} e^{-(\lambda+\delta)s}\lambda\alpha V^E_b(R^{L^b}_{s-})\mathrm{d}s + \beta\int_0^{T^{L^b}\wedge t} e^{-(\lambda+\delta)s}\mathrm{d}L^b_s\right]$, which implies

$$w_b(x) = \mathrm{E}_x\left[\int_0^{T^{L^b}\wedge t} e^{-(\lambda+\delta)s}\lambda\alpha V^E_b(R^{L^b}_{s-})\mathrm{d}s + \beta\int_0^{T^{L^b}\wedge t} e^{-(\lambda+\delta)s}\mathrm{d}L^b_s + e^{-(\lambda+\delta)(T^{L^b}\wedge t)}w_b(R^{L^b}_{T^{L^b}\wedge t})\right].$$

Recall that $R^{L^b}_{s-} \in [0,b]$ for $s \in (0, T^{L^b}]$ and that L^b_s is non-decreasing in s. By taking $t \to +\infty$ on both sides of the above equation and then using the dominated convergence for the first and third terms in the right-hand side of the equation as well as the monotone convergence for the second term, we obtain

$$\begin{aligned} w_b(x) =& \mathrm{E}_x\left[\int_0^{T^{L^b}} e^{-(\lambda+\delta)s}\lambda\alpha V^E_b(R^{L^b}_{s-})\mathrm{d}s\right] + \mathrm{E}_x\left[\beta\int_0^{T^{L^b}} e^{-(\lambda+\delta)s}\mathrm{d}L^b_s\right] + \mathrm{E}_x\left[e^{-(\lambda+\delta)T^{L^b}}w_b(R^{L^b}_{T^{L^b}})\right], \\ =& \mathrm{E}_x\left[\int_0^{T^{L^b}} e^{-(\lambda+\delta)s}\lambda\alpha V^E_b(R^{L^b}_{s-})\mathrm{d}s\right] + \mathrm{E}_x\left[\beta\int_0^{T^{L^b}} e^{-(\lambda+\delta)s}\mathrm{d}L^b_s\right], \end{aligned} \tag{A.20}$$

where the last equality follows by noting that $w_b(R^{L^b}_{T^{L^b}}) = w_b(0) = S_b(0) = 0$.

Noting that the strategy $\pi^{(0,\to)(L^b,L^b)}$ coincides with $\bar{\pi}^{\eta_0,L^b,L^b}$, we have

$$T^{\pi^{(0,\to)(L^b,L^b)}} = T^{\bar{\pi}^{\eta_0,L^b,L^b}}. \tag{A.21}$$

Further note that $\pi^{(0,\to)(L^b,L^b)}$ coincides with L^b during the time period $[0,\eta_0)$. Hence, $R^{\pi^{(0,\to)(L^b,L^b)}}_t = R^{L^b}_t$ for $0 \le t < \eta_0$. Therefore,

$$\begin{aligned} &I\{T^{\pi^{(0,\to)(L^b,L^b)}} \ge \eta_0\} = I\{R^{\pi^{(0,\to)(L^b,L^b)}}_t > 0 \text{ for all } t\in[0,\eta_0)\} \\ =& I\{R^{L^b}_t > 0 \text{ for all } t\in[0,\eta_0)\} = I\{T^{L^b} \ge \eta_0\}, \end{aligned} \tag{A.22}$$

$$T^{\pi^{(0,\to)(L^b,L^b)}} I\{T^{\pi^{(0,\to)(L^b,L^b)}} < \eta_0\} = \inf_{t\in[0,\eta_0)} \{R_t^{\pi^{(0,\to)(L^b,L^b)}} \le 0\} I\{T^{\pi^{(0,\to)(L^b,L^b)}} < \eta_0\}$$
$$= \inf_{t\in[0,\eta_0)} \{R_t^{L^b} \le 0\} I\{T^{\pi^{(0,\to)(L^b,L^b)}} < \eta_0\} = \inf_{t\in[0,\eta_0)} \{R_t^{L^b} \le 0\}(1 - I\{T^{\pi^{(0,\to)(L^b,L^b)}} \ge \eta_0\})$$
$$= \inf_{t\in[0,\eta_0)} \{R_t^{L^b} \le 0\}(1 - I\{T^{L^b} \ge \eta_0\}) = \inf_{t\in[0,\eta_0)} \{R_t^{L^b} \le 0\} I\{T^{L^b} \le \eta_0\} = T^{L^b} I\{T^{L^b} \le \eta_0\}, \qquad (A.23)$$

where the second last equality follows by (A.22). As a result of (A.22) and (A.23),

$$T^{\pi^{(0,\to)(L^b,L^b)}} \wedge \eta_0 = T^{\pi^{(0,\to)(L^b,L^b)}} I\{T^{\pi^{(0,\to)(L^b,L^b)}} < \eta_0\} + \eta_0 I\{T^{\pi^{(0,\to)(L^b,L^b)}} \ge \eta_0\}$$
$$= T^{L^b} I\{T^{L^b} < \eta_0\} + \eta_0 I\{T^{L^b} \ge \eta_0\} = T^{L^b} \wedge \eta_0. \qquad (A.24)$$

It follows by (2.4) that

$$\mathcal{P}(x; L^b, L^b) = \mathrm{E}_x\Big[\int_0^{T^{\pi^{(0,\to)(L^b,L^b)}} \wedge\eta_0} \beta e^{-\delta t} \mathrm{d}L_t^b + I\{\eta_0 \le T^{\pi^{(0,\to)(L^b,L^b)}}\} \int_{\eta_0}^{T^{\pi^{(0,\to)(L^b,L^b)}}} \alpha\beta e^{-\delta t} \mathrm{d}L_t^b\Big]$$
$$= \mathrm{E}_x\Big[\int_0^{T^{L^b}\wedge\eta_0} \beta e^{-\delta t} \mathrm{d}L_t^b + I\{\eta_0 \le T^{\bar{\pi}^{\eta_0,L^b,L^b}}\} \int_{\eta_0}^{T^{\pi^{(0,\to)(L^b,L^b)}}} \alpha\beta e^{-\delta t} \mathrm{d}L_t^b\Big], \ x \ge 0, \qquad (A.25)$$

where the last equality follows by (A.24) and (A.21). Combining (A.25), (A.1) and (A.3) yields

$$\mathcal{P}(x; L^b, L^b) = \mathrm{E}_x\left[\beta \int_0^{T^{L^b}} e^{-(\lambda+\delta)s} \mathrm{d}L_s^b\right] + \mathrm{E}_x\left[\int_0^{T^{L^b}} e^{-(\lambda+\delta)s} \lambda\alpha V_b^E(R_{s-}^{L^b}) \mathrm{d}s\right] = w_b(x), \ x \ge 0,$$

where the last equality is due to (A.20). By further noting (A.12), we complete the proof for (3.7).

Notice from the proof for (i) that $V_b(b) = S_b(b) = K_1(b)v_1(b) + K_2(b)v_2(b) + B(b)$ for $b > 0$. It follows by (A.11) that $K_1(b) = \frac{\beta - B'(b)}{v_1'(b) - v_2'(b)}$ and $K_2(b) = -K_1(b) = \frac{B'(b)-\beta}{v_1'(b)-v_2'(b)}$. Note that $v_1(x)$, $v_2(x)$ and $B(x)$ are all continuous differentiable on $[0, +\infty)$. Hence, $\lim_{b\downarrow 0} K_1(b)$ and $\lim_{b\downarrow 0} K_2(b)$ exist. Therefore, $\lim_{b\downarrow 0} V_b(b) = \lim_{b\downarrow 0} K_1(b) \lim_{b\downarrow 0} v_1(b) + \lim_{b\downarrow 0} K_2(b) \lim_{b\downarrow 0} v_2(b) + \lim_{b\downarrow 0} B(b) = \lim_{b\downarrow 0}(K_1(b)v_1(0) + K_2(b)v_2(0) + B(0)) = \lim_{b\downarrow 0}(K_1(b) + K_2(b)) = 0$. □

Proof of Lemma 3.4 The inequality $b^* \ge 0$ is obvious according to Definition 3.3(ii). Consider any $b > 0$. It follows by Lemma 3.2 that V_b^E satisfies the equation (3.4) and that $V_b^{E\prime}(b) = \beta$. As a result,

$$\frac{\sigma^2(b)}{2} V_b^{E\prime\prime}(b-) = -\mu(b)\beta + \delta V_b^E(b) = -\mu(b)\beta + (\lambda\alpha + \delta)V_b^E(b) - \lambda\alpha V_b^E(b). \qquad (A.26)$$

It follows by Lemma 3.3 that V_b satisfies the equation (3.6), and $V_b'(b) = \beta$. Hence, $\frac{\sigma^2(b)}{2} V_b''(b-) = -\mu(b)\beta + (\lambda + \delta)V_b(b) - \lambda\alpha V_b^E(b)$, which along with (A.26) implies

$$\frac{\sigma^2(b)}{2} V_b''(b-) - \frac{\sigma^2(b)}{2} V_b^{E\prime\prime}(b-) = (\lambda + \delta)V_b(b) - (\lambda\alpha + \delta)V_b^E(b). \qquad (A.27)$$

Note by (2.4), (3.1) and $\bar{\pi}^{\eta_0,L^b,L^b} = L^b$ that $V_b(b) = \mathrm{E}_b\Big[\int_0^{\eta_0\wedge T^{L^b}} \beta e^{-\delta t} \mathrm{d}L_t^b + I\{\eta_0 \le T^{L^b}\} \int_{\eta_0}^{T^{L^b}} \alpha\beta e^{-\delta t} \mathrm{d}L_t^b\Big]$, and by (3.3) that $V_b^E(b) = \mathrm{E}_b\Big[\int_0^{\eta_0\wedge T^{L^b}} \beta e^{-\delta t} \mathrm{d}L_t^b + I\{\eta_0 \le T^{L^b}\} \int_{\eta_0}^{T^{L^b}} \beta e^{-\delta t} \mathrm{d}L_t^b\Big]$. Therefore,

$$V_b(b) - V_b^E(b) = -(1 - \alpha)\beta \mathrm{E}_b\left[I\{\eta_0 \le T^{L^b}\} \int_{\eta_0}^{T^{L^b}} e^{-\delta t} \mathrm{d}L_t^b\right]. \qquad (A.28)$$

It follows by noticing $\bar{\pi}^{\eta_0,L^b,L^b} = L^b$ and using (A.3) that

$$\beta \mathrm{E}_b\left[I\{\eta_0 \le T^{L^b}\}\int_{\eta_0}^{T^{L^b}} e^{-\delta t}\mathrm{d}L_t^b\right] = \mathrm{E}_b\left[I\{\eta_0 \le T^{\bar{\pi}^{\eta_0,L^b,L^b}}\}\int_{\eta_0}^{T^{\bar{\pi}^{\eta_0,L^b,L^b}}} \beta e^{-\delta t}\mathrm{d}L_t^b\right]$$

$$= \mathrm{E}_b\left[\int_0^{T^{L^b}} e^{-(\lambda+\delta)t}\lambda V_b^E(R_{t-}^{L^b})\mathrm{d}t\right] \le V_b^E(b)\int_0^{+\infty}\lambda e^{-(\lambda+\delta)s}\mathrm{d}t = \frac{\lambda}{\lambda+\delta}V_b^E(b), \tag{A.29}$$

where the inequality in the second to the last step follows by noting that the controlled surplus will never exceed b under the barrier strategy L^b, conditional on $R_{0-} = b$, and noting that the function V_b^E is increasing for $b > 0$ by Lemma 3.2. By combining (A.28) and the equation (A.29), we obtain $V_b(b) - V_b^E(b) \ge -(1-\alpha)\frac{\lambda}{\lambda+\delta}V_b^E(b)$, which implies $V_b(b) \ge \frac{\lambda\alpha+\delta}{\lambda+\delta}V_b^E(b)$. This, along with (A.27), implies

$$\frac{\sigma^2(b)}{2}\left(V_b{}''(b-) - V_b^{E}{}''(b-)\right) \ge 0,\ b > 0. \tag{A.30}$$

(a) Suppose $0 < b^E < +\infty$. It follows by the definition of b^E in Definition 3.3(i) that $V_{b^E}^{E}{}''(b^E-) = 0$. Therefore, from (A.30) we conclude $\frac{\sigma^2(b^E)}{2}V_{b^E}{}''(b^E-) \ge 0$. It then follows by the definition for b^* in Definition (3.3)(ii) that $0 \le b^* \le b^E$. (b) Suppose $b^E = 0$. It follows by the definition for b^E in Definition 3.3(i) that $\limsup_{b\downarrow 0}\frac{\sigma^2(b)}{2}V_b^{E}{}''(b-) \ge 0$. Then by taking $\liminf_{b\downarrow 0}$ on both sides of (A.30) and using the last inequality, we arrive at $\liminf_{b\downarrow 0}\frac{\sigma^2(b)}{2}V_b^{E}{}''(b-) \ge 0$, which together with the definition for b^* in Definition 3.3(ii), implies that $b^* = 0 = b^E$. (c) If $b^E = +\infty$, the inequality $b^* \le b^E$ is trivial. □

Proof of Theorem 3.5 Note that the function V_b satisfies (3.6) on $[0, b)$ (by Lemma 3.3). Therefore,

$$\lim_{x\downarrow 0} V_b''(x) = \frac{-\mu(0)\lim_{x\downarrow 0}V_b'(x) + (\lambda+\delta)V_b(0) - \lambda\alpha V_b^E(0)}{\frac{\sigma^2(0)}{2}} = -\frac{2\mu(0)\lim_{x\downarrow 0}V_b'(x)}{\sigma^2(0)}, \tag{A.31}$$

where the last equality follows by $V_b(0) = 0$ in (3.9) and $V_b^E(0) = 0$ in (3.5). Note that $\lim_{x\downarrow 0}V_b'(x) \ge 0$ (due to the non-decreasing property of V_b). It then follow from (A.31) that $\lim_{x\downarrow 0}V_b''(x) \ge 0$ if and only if $\mu(0) \le 0$. As a result, by combining the definition of b^* in Definition 3.3(ii) and the inequality $\lim_{x\downarrow 0}V_b''(x) \ge 0$ in (A.31), we can conclude that $b^* = 0$ if $\mu(0) \le 0$ and $b^* > 0$ if $\mu(0) > 0$. □

Proof of Lemma 3.6 Recall by Lemma 3.4 that $0 \le b^* \le b^E$. Then by Definition 3.3(i), $V_{b^*}^{E}{}''(b^*-) \le 0$. Hence, by Lemma 3.2(ii), $V_{b^*}^{E}{}''(x) \le 0$ for any $x \in [0, b^*]$. Thus, $V_{b^*}^E$ is concave on $(0, b^*)$. □

Proof of Theorem 4.1 It follows by Theorem 3.5 that $b^* > 0$. (i) Note from Lemma 3.3 that $V_b(x)$ is continuously differentiable on $(0, +\infty)$ and twice continuously differentiable on $(0, +\infty)\backslash\{b\}$. For $b = 0$, $V_b(x) = V_0(x) = \beta x$ for $x \ge 0$ (see (3.7)). The concavity of V_b is obvious. We now consider any finite $b \in (0, b^*]$. It follows by Definition 3.3(ii) that

$$V_b''(b-) = \lim_{x\uparrow b}\frac{\mathrm{d}^2}{\mathrm{d}x^2}V_b''(x) \le 0,\ b \in (0, b^*). \tag{A.32}$$

Because the function V_b satisfies (3.6) on $[0, b)$ (by Lemma 3.3), we have

$$\lim_{x\downarrow 0} V_b''(x) = \frac{-\mu(0)\lim_{x\downarrow 0}V_b'(x) + (\lambda+\delta)V_b(0) - \lambda\alpha V_b^E(0)}{\frac{\sigma^2(0)}{2}} = -\frac{2\mu(0)\lim_{x\downarrow 0}V_b'(x)}{\sigma^2(0)} \le 0, \tag{A.33}$$

where the last equality follows by $V_b(0) = 0$ in (3.9) and $V_b^E(0) = 0$ in (3.5) and the last inequality by $\mu(0) \le 0$ and $\lim_{x\downarrow 0}V_b'(x) \ge 0$ (due to the non-deceasing property of V_b). We now use proof by contradiction to show that the function V_b is always concave on $(0, b]$. Suppose this is not true. Then there exists some point in the interval $(0, b]$ such that the second order derivative of V_b at this point is strictly greater than 0. Recall that $V_b''(0+) \le 0$ (see (A.33)) and $V_b''(b-) \le 0$ (see (A.32)). Hence, if we use $V_b''(0)$ and $V_b''(b)$ to represent $V_b''(0+)$ and $V_b''(b-)$, respectively, it follows by the continuity of $V_b''(x)$ on $(0, b)$ that there exist x_1 and x_2 with $0 \le x_1 < x_2 \le b$ such that $V_b''(x_1) = 0 = V_b''(x_2)$ and $V_b''(x) > 0$ for $x \in (x_1, x_2)$. Recall that the function V_b satisfies (3.6) on $[0, b)$ (see Lemma 3.3). Thus, we have $\frac{\sigma^2(x)}{2} V_b''(x) = -\mu(x)V_b'(x) + (\lambda+\delta)V_b(x) - \lambda\alpha V_b^E(x)$. It follows by the above equations that for $x \in (x_1, x_2)$, $-\mu(x)V_b'(x) + (\lambda+\delta)V_b(x) - \lambda\alpha V_b^E(x) > 0 = -\mu(x_l)V_b'(x_l) + (\lambda+\delta)V_b(x_l) - \lambda\alpha V_b^E(x_l)$, $l = 1, 2$. As a result, $(\mu(x_l)V_b'(x_l) - \mu(x)V_b'(x)) + \lambda\alpha(V_b^E(x_l) - V_b^E(x)) > (\lambda+\delta)(V_b(x_l) - V_b(x))$, $x \in (x_1, x_2)$, $l = 1, 2$. Now, by dividing both sides of the above equation by $x_l - x$ and then taking $\limsup_{x\downarrow x_1}$ if $l = 1$ and $\limsup_{x\uparrow x_2}$if $l = 2$, we arrive at

$$\mu(x_1)V_b''(x_1) + \lambda\alpha V_b^{E'}(x_1) + (\limsup_{x\downarrow x_1} \frac{\mu(x) - \mu(x_1)}{x - x_1} - \lambda - \delta)V_b'(x_1) \le 0, \tag{A.34}$$

$$\mu(x_2)V_b''(x_2) + \lambda\alpha V_b^{E'}(x_2) + (\limsup_{x\uparrow x_2} \frac{\mu(x) - \mu(x_2)}{x - x_2} - \lambda - \delta)V_b'(x_2) \ge 0. \tag{A.35}$$

It follows by (A.34) and $V_{b^*}''(x_1) = 0$ that

$$(\lambda + \delta - \limsup_{x\downarrow x_1} \frac{\mu(x) - \mu(x_1)}{x - x_1})V_b'(x_1) \ge \lambda\alpha V_b^{E'}(x_1) \ge \lambda\alpha V_b^{E'}(x_2) \ge (\lambda + \delta - \limsup_{x\uparrow x_2} \frac{\mu(x) - \mu(x_2)}{x - x_2})V_b'(x_2), \tag{A.36}$$

where the second to the last inequality follows by the concavity of V_b^E on $(0, b)$ (Lemma 3.6), and the last inequality by (A.35) and $V_b''(x_2) = 0$ It follows by $V_b''(x) > 0$ for $x \in (x_1, x_2)$ and the non-decreasing property of V_b that

$$0 \le V_b'(x_1) < V_b'(x_2). \tag{A.37}$$

Note that we are considering b that is finite and smaller than b^*. It follows by Lemma 3.4 that $b \le b^* \le b^E$, which together with Definition 3.3(i) implies $V_b^{E''}(b-) \le 0$. This along with Lemma 3.2(ii) leads to $V_b^{E''}(x) \le 0$ for $x \in [0, b)$. Then by $V_b^{E'}(b) = \beta$ (see (3.7)) and $0 \le x_1 < b$, we obtain

$$V_b^{E'}(x_1) \ge V_b^{E'}(b) = \beta. \tag{A.38}$$

Recall that $\frac{\mu(y)-\mu(x)}{y-x} \le \lambda(1-\alpha) + \delta \le \lambda + \delta$ for $y > x \ge 0$ and that $\mu(x)$ is concave. The concavity of μ implies

$$0 \le \lambda + \delta - \limsup_{x\downarrow x_1} \frac{\mu(x) - \mu(x_1)}{x - x_1} \le \lambda + \delta - \limsup_{x\uparrow x_2} \frac{\mu(x) - \mu(x_2)}{x - x_2}, \tag{A.39}$$

which, along with (A.37), implies $(\lambda+\delta-\limsup_{x\downarrow x_1} \frac{\mu(x)-\mu(x_1)}{x-x_1})V_b'(x_1) < (\lambda+\delta-\limsup_{x\uparrow x_2} \frac{\mu(x)-\mu(x_2)}{x-x_2})V_b'(x_2)$ if $\lambda + \delta - \limsup_{x\downarrow x_1} \frac{\mu(x)-\mu(x_1)}{x-x_1} \ne 0$. If $\lambda + \delta - \limsup_{x\downarrow x_1} \frac{\mu(x)-\mu(x_1)}{x-x_1} = 0$, then by (A.39) we have $(\lambda + \delta - \limsup_{x\downarrow x_1} \frac{\mu(x)-\mu(x_1)}{x-x_1})V_b'(x_1) = 0 < \lambda\beta \le \lambda\alpha V_b^{E'}(x_1)$, where the last inequality is due to (A.38). We have now obtained a contradiction to (A.36) in each case.

(ii) From Remark 3.1 (i), we already know that $V_{b^*}(x)$ is continuously differentiable on $(0,+\infty)$ and twice continuously differentiable on $(0,+\infty)\backslash\{b^*\}$. To prove the twice continuous differentiability at b^*, it suffices to show that the left second-order derivative of $V^S_{b^*}(x)$ at $x=b^*$ equals the right second-order derivative, which is true by noticing $V''_{b^*}(b^*-)=0$ (by the definition for b^* in Definition 3.3(ii) and $b^*>0$) and $V''_{b^*}(b^*+)=0$ (by (3.7)). It further follows that $V''_{b^*}(b^*)=0$. We now proceed to prove the concavity. From (i) we know that V_{b^*} is concave on $(0,b^*]$ and $V''_{b^*}(b^*)=0$. Hence by noting that $V_{b^*}(x)\equiv 0$ for $x\geq b^*$ (by Lemma 3.3), we conclude V_{b^*} is also concave on $[b^*,+\infty)$ and $V''_{b^*}(x)\equiv 0$ for $x\geq b^*$. The equation (4.1) follows from (3.8), and all the equations in (4.2) follow from (3.9) and (3.9). The equations in (4.3) are immediate results of the twice continuous differentiability of V_{b^*} on $[0,+\infty)$ and (4.2). Since V_{b^*} is concave on $[0,+\infty)$ and $V'_{b^*}(x)=\beta$ for $x\geq b^*$, we conclude $V'_{b^*}(x)\geq\beta$ for $x\geq 0$. □

Remark A.1 *Suppose that L is an admissible strategy so that $V_b'(R^L_{s-})\geq\beta$ for $0\leq s\leq T^L$. Recall that $\{L^c_s\}$ is the continuous part of $\{L_s\}$. Notice that for any $L\in\Pi$, if $R^L_s\leq R^L_{s-}$ and $R^L_{s-}-R^L_s=L_s-L_{s-}$, and that $V_b(\cdot)$ is non-decreasing. Thus, for any stopping time τ, $\sum_{0\leq s\leq T^L\wedge\tau}e^{-(\lambda+\delta)s}(V_b(R^L_s)-V_b(R^L_{s-}))\leq \sum_{0\leq s\leq T^L\wedge\tau}e^{-(\lambda+\delta)s}\beta(R^L_s-R^L_{s-})=-\sum_{0\leq s\leq T^L\wedge\tau}\beta e^{-(\lambda+\delta)s}(L_s-L_{s-})$ and $\int_0^{T^L\wedge\tau}e^{-(\lambda+\delta)s}V_b'(R^L_{s-})\mathrm{d}L^c_s\geq\int_0^{T^L\wedge t_n}e^{-(\lambda+\delta)s}\beta\mathrm{d}L^c_s$. As a result,*

$$\begin{aligned}&\sum_{0\leq s\leq T^L\wedge\tau}e^{-(\lambda+\delta)s}(V_b(R^L_s)-V_b(R^L_{s-}))-\int_0^{T^L\wedge\tau}e^{-(\lambda+\delta)s}V_b'(R^L_{s-})\mathrm{d}L^c_s\\&\leq-\beta\sum_{0\leq s\leq T^L\wedge\tau}e^{-(\lambda+\delta)s}(L_s-L_{s-})-\int_0^{T^L\wedge t_n}e^{-(\lambda+\delta)s}\beta\mathrm{d}L^c_s=-\beta\int_0^{T^L\wedge t_n}e^{-(\lambda+\delta)s}\mathrm{d}L_s.\end{aligned}$$

Proof of Theorem 4.2 Note by Definition 3.1 (ii) that $V_{b^*}(x)=\mathcal{P}(x;L^{b^*},L^{b^*})(x)\leq\sup_{L\in\Pi}\mathcal{P}(x;L,L^{b^*})$. According to the definition for a MPE strategy, we can see that it is sufficient to show that $V_{b^*}(x)\geq\sup_{L\in\Pi}\mathcal{P}(x;L,L^{b^*})$, $x\geq 0$. Theorem 4.1 shows that $V_{b^*}(\cdot)$ is twice continuously differentiable on $(0,+\infty)$. Note that for any $b\geq 0$, $V_b(x)$ is continuously differentiable on $[0,+\infty)$ and twice continuously differentiable on $[0,b)\cup(b,+\infty)$. Define for any $b\geq 0$,

$$h_b(x)=\frac{\sigma^2(x)}{2}V_b''(x)+\mu(x)V_b'(x)-(\lambda+\delta)V_b(x)+\lambda\alpha V^E_b(x),\ x\in[0,b)\cup(b,+\infty).\tag{A.40}$$

It follows immediately by (3.8) that

$$h_b(x)=0,\ x\in[0,b).\tag{A.41}$$

Suppose $b^*<+\infty$. We now show that

$$h_{b^*}(x)\leq 0,\ x>0.\tag{A.42}$$

We distinguish two cases: (a) $0<b^*<+\infty$ and (b) $b^*=0$. (a) Suppose $0<b^*<+\infty$. By (A.41) and the continuous differentiability of $V''_{b^*}(x)$ we have $h_{b^*}(x)=0$ for $x\in[0,b^*]$. For any $x>0$, let $y^-_{x,n}$ and $y^+_{x,n}$ represent the sequences such that $y^-_{x,n}\uparrow x$ and $y^+_{x,n}\downarrow x$ as $n\to+\infty$, $\limsup_{y\uparrow x}\frac{h_{b^*}(y)-h_{b^*}(x)}{y-x}=\limsup_{n\to+\infty}\frac{h_{b^*}(y^-_{x,n})-h_{b^*}(x)}{y^-_{x,n}-x}$ and $\limsup_{y\downarrow x}\frac{h_{b^*}(y)-h_{b^*}(x)}{y-x}=\limsup_{n\to+\infty}\frac{h_{b^*}(y^+_{x,n})-h_{b^*}(x)}{y^+_{x,n}-x}$. Since $\mu(\cdot)$ and $\sigma(\cdot)$

are Lipschitz continuous, for any $x\geq 0$, we can find sub-sequences of $y^-_{x,n}$ and $y^+_{x,n}$, say y^-_{x,n_k} and y^+_{x,n_k}, such that $\lim_{k\to+\infty}\frac{\mu(y^+_{x,n_k})-\mu(x)}{y^+_{x,n_k}-x}$, $\lim_{k\to+\infty}\frac{\sigma(y^+_{x,n_k})-\sigma(x)}{y^+_{x,n_k}-x}$, $\lim_{k\to+\infty}\frac{\mu(y^-_{x,n_k})-\mu(x)}{y^-_{x,n_k}-x}$, and $\lim_{k\to+\infty}\frac{\sigma(y^-_{x,n_k})-\sigma(x)}{y^-_{x,n_k}-x}$ exist. This, along with $0=\lim_{n\to+\infty}\frac{h_{b^*}(y^-_{x,n_k})-h_{b^*}(x)}{y^-_{x,n_k}-x}$ for $x\in(0,b^*]$ (by (A.41)), (A.41) and the differentiability of V_{b^*} and $V^E_{b^*}$, implies the existence of $\lim_{k\to+\infty}\frac{V''_{b^*}(y^-_{x,n_k})-V''_{b^*}(x)}{y^-_{x,n_k}-x}$ and

$$0=\lim_{k\to+\infty}\frac{h_{b^*}(y^-_{x,n_k})-h_{b^*}(x)}{y^-_{x,n_k}-x}=\frac{\sigma^2(x)}{2}\lim_{k\to+\infty}\frac{V''_{b^*}(y^-_{x,n_k})-V''_{b^*}(x)}{y^-_{x,n_k}-x}+\left(\mu(x)+\sigma(x)\lim_{k\to+\infty}\frac{\sigma(y^-_{x,n_k})-\sigma(x)}{y^-_{x,n_k}-x}\right)$$
$$\times V''_{b^*}(x)+\left(\lim_{k\to+\infty}\frac{\mu(y^-_{x,n_k})-\mu(x)}{y^-_{x,n_k}-x}-\lambda-\delta\right)V'_{b^*}(x)+\lambda\alpha V^{E\prime}_{b^*}(x),\ x\in(0,b^*]. \tag{A.43}$$

Note that V_{b^*} is concave (Theorem 4.1(ii)). Thus, $V''_{b^*}(x)\leq 0$ for $0<x<b^*$. From (4.3) further notice

$$V''_{b^*}(x)=0 \text{ for } x\geq b^*, \text{ and thus, } V^{(3)}_{b^*}(x)=0\leq\lim_{k\to+\infty}\frac{V''_{b^*}(y_{b^*,n_k})-V''_{b^*}(b^*)}{y_{b^*,n_k}-b^*} \text{ for } x>b^*. \tag{A.44}$$

By using (A.40) and (A.44), and noting $V'_{b^*}(x)=\beta$ for $x\geq b^*$ (see (4.3)), we have $h_{b^*}(x)=\mu(x)\beta-(\lambda+\delta)V_{b^*}(x)+\lambda\alpha V^E_{b^*}(x)$ for $x\geq b^*$. Hence, by using $V'_{b^*}(x)=\beta$ for $x\geq b^*$ (see (4.3)), and $V^{E\prime}_{b^*}(x)=\beta$ for $x\geq b^*$ (see (3.5)) we get

$$\begin{aligned}\lim_{k\to+\infty}\frac{h_{b^*}(y^+_{x,n_k})-h_{b^*}(x)}{y^+_{x,n_k}-x}&=\left(\lim_{k\to+\infty}\frac{\mu(y^+_{x,n_k})-\mu(x)}{y^+_{x,n_k}-x}-\lambda-\delta\right)\beta+\lambda\alpha\beta,\ x\geq b^*\\&\leq\left(\lim_{n_k\to+\infty}\frac{\mu(y^-_{b^*,n})-\mu(b^*)}{y^-_{b^*,n_k}-b^*}-\lambda-\delta\right)\beta+\lambda\alpha\beta,\end{aligned} \tag{A.45}$$

where the last inequality follows by the concavity of $\mu(x)$. Furthermore, by setting $x=b^*$ in (A.43) and then using $V'_{b^*}(x)=\beta$ for $x\geq b^*$, $V^{E\prime}_{b^*}(x)=\beta$ for $x\geq b^*$, and $V''_{b^*}(x)=0$ for $x\geq b^*$ again we obtain

$$\begin{aligned}0&=\frac{\sigma^2(b^*)}{2}\lim_{k\to+\infty}\frac{V''_{b^*}(y^-_{b^*,n_k})-V''_{b^*}(b^*)}{y^-_{b^*,n_k}-b^*}+\left(\lim_{k\to+\infty}\frac{\mu(y^-_{b^*,n_k})-\mu(b^*)}{y^-_{b^*,n_k}-b^*}-\lambda-\delta\right)\beta+\lambda\alpha\beta\\&\geq\left(\lim_{k\to+\infty}\frac{\mu(y^-_{b^*,n_k})-\mu(b^*)}{y^-_{b^*,n_k}-b^*}-\lambda-\delta\right)\beta+\lambda\alpha\beta\geq\lim_{k\to+\infty}\frac{h_{b^*}(y^+_{x,n_k})-h_{b^*}(x)}{y^+_{x,n_k}-x},\ x\geq b^*,\end{aligned} \tag{A.46}$$

where the second to the last inequality follows by (A.44) and the last inequality by (A.45). Thus, $\limsup_{y\downarrow x}\frac{h_{b^*}(y)-h_{b^*}(x)}{y-x}\leq 0$ for $x\geq b^*$, which, along with the continuity of h, implies $h_{b^*}(x)\leq h_{b^*}(b^*)=0$ for $x>b^*$.

(b) Suppose $b^*=0$. It follows by Definition 3.3 that $\inf\{b>0:\lim_{x\uparrow b}\frac{\mathrm{d}^2}{\mathrm{d}x^2}\mathcal{P}(x;L^b,L^b)\geq 0\}=0$. Therefore, we can find a strictly positive sequence $\{b_n\}$ with $\lim_{n\to+\infty}b_n=0$ such that for each n,

$$V''_{b_n}(b_n-)=\lim_{x\uparrow b_n}\frac{\mathrm{d}^2}{\mathrm{d}x^2}\mathcal{P}(x;L^{b_n},L^{b_n})\geq 0. \tag{A.47}$$

By noting (A.40), $V'_{b_n}(b_n)=\beta$ (by (3.9)) and $V''_{b_n}(b_n+)=0$ (by (3.9)), we can obtain

$$\begin{aligned}h_{b_n}(b_n+)&=\mu(b_n)\beta-(\lambda+\delta)V_{b_n}(b_n)+\lambda\alpha V^E_{b_n}(b_n)\\&\leq\frac{\sigma^2(b_n)}{2}V''_{b_n}(b_n-)+\mu(b_n)\beta-(\lambda+\delta)V_{b_n}(b_n)+\lambda\alpha V^E_{b_n}(b_n)=h_{b_n}(b_n-)=0,\end{aligned} \tag{A.48}$$

where the inequality in the third to the last step follows from (A.47), the second to the last equality follows by using (A.40) again, and the last equality is due to (A.41). Noticing from (3.9) that $V'_{b_n}(x)=\beta$

for $x \geq b_n$, $V''_{b_n}(x) = 0$ for $x > b_n$ and $V^{(3)}_{b_n}(x) = 0$ for $x \geq b_n$ and from (3.6) that $V^{E\prime}_{b_n}(x) = \beta$ for $x \geq b_n$, we can derive from (A.40) that $\limsup_{y\downarrow x} \frac{h_{b_n}(y)-h_{b_n}(x)}{y-x} = (\limsup_{y\downarrow x} \frac{\mu(y)-\mu(x)}{y-x} - \lambda - \delta)\beta + \lambda\alpha\beta = (\limsup_{y\downarrow x} \frac{\mu(y)-\mu(x)}{y-x} - \delta)\beta - \lambda(1-\alpha)\beta \leq 0$ for $x > b_n$, where the last inequality follows from the assumption $\frac{\mu(y)-\mu(x)}{y-x} \leq \lambda(1-\alpha) + \delta \leq \lambda + \delta$ for $y > x \geq 0$. Therefore,

$$h_{b_n}(x) \leq h_{b_n}(b_n+) \leq 0, \quad x > b_n, \tag{A.49}$$

where the last inequality follows from (A.48). By noting $V_{b_n}(x) = \beta(x - b_n) + V_{b_n}(b_n)$ for $x > b_n$ (see (3.9)), $\lim_{n\to\infty} b_n = 0$ and $V_0(x) = \beta x$ (see (3.2)), we can see

$$V_0(x) = \lim_{n\to+\infty} V_{b_n}(x), \quad V_0'(x) = \beta = \lim_{n\to+\infty} V'_{b_n}(x), \quad V_0''(x) = 0 = \lim_{n\to+\infty} V''_{b_n}(x). \tag{A.50}$$

Note $\lim_{n\to\infty} b_n = 0$. Then from (3.7) it follows that

$$\lim_{n\to\infty} V_{b_n}(b_n) = 0. \tag{A.51}$$

It follows by the assumption $b^* = 0$, (A.40), (A.50) and $\lim_{n\to\infty} V_{b_n}(x) = V_0^E(x)$ (see (3.5)) that $h_{b^*}(x) = h_0(x) = \lim_{n\to+\infty} h_{b_n}(x) \leq 0$ for $x \geq 0$, where the last inequality follows by (A.49).

Let L be any admissible strategy and define $\bar{\pi}_s^{\eta_0,L,L^{b^*}} = L_s I\{s < \eta_0\} + L_s^{b^*} I\{s \geq \eta_0\}$. For convenience we use $\bar{\pi}_s$ to represent this throughout this proof. We can see that $\bar{\pi}$ is also admissible. Recall the definition of $\mathcal{G}$ in (A.10). By applying the Itô's formula, and using (2.1), we can obtain that for any $t > 0$,

$$\begin{aligned} & \mathrm{E}_x\left[e^{-(\lambda+\delta)T^{\bar{\pi}}\wedge t} V_{b^*}(R^{\bar{\pi}}_{T^{\bar{\pi}}\wedge t}) - V_{b^*}(R^{\bar{\pi}}_{0-})\right] \\ = \; & \mathrm{E}_x\Bigg[\int_0^{T^{\bar{\pi}}\wedge t} e^{-(\lambda+\delta)s}\mathcal{G}_{V_{b^*}}(R^{\bar{\pi}}_{s-})\mathrm{d}s + \int_0^{T^{\bar{\pi}}\wedge t} e^{-(\lambda+\delta)s}\sigma(R^{\bar{\pi}}_{s-})V_b'(R^{\bar{\pi}}_{s-})\mathrm{d}W_s \\ & - \int_0^{T^{\bar{\pi}}\wedge t} e^{-(\lambda+\delta)s}V'_{b^*}(R^{\bar{\pi}}_{s-})\mathrm{d}\bar{\pi}^c_s + \sum_{0\leq s\leq T^{\bar{\pi}}\wedge t} e^{-(\lambda+\delta)s}(V_{b^*}(R^{\bar{\pi}}_s) - V_{b^*}(R^{\bar{\pi}}_{s-}))\Bigg], \end{aligned} \tag{A.52}$$

where $\bar{\pi}^c_s$ is the continuous part of $\bar{\pi}$. It follows from the definition of $\mathcal{G}$ in (A.10) and (A.40) that

$$\mathcal{G}_{V_{b^*}}(R^{\bar{\pi}}_{s-}) = h_{b^*}(R^{\bar{\pi}}_{s-}) - \lambda\alpha V^E_{b^*}(R^{\bar{\pi}}_{s-}) \leq -\lambda\alpha V^E_{b^*}(R^{\bar{\pi}}_{s-}) \;\text{ for } 0 < s \leq T^{\bar{\pi}}. \tag{A.53}$$

where the last inequality is due to $h_{b^*}(x) \leq 0$ for $x \geq 0$ (by (A.42)). Note that the stochastic process, $\left\{\int_0^{T^{\bar{\pi}}\wedge t} e^{-(\lambda+\delta)s}\sigma(R^{\bar{\pi}}_{s-})V'_{b^*}(R^{\bar{\pi}}_{s-})\mathrm{d}W_s; t \geq 0\right\}$, is a P_x-local martingale. Therefore, there exists a positive increasing sequence of stopping times $\{t_n\}$ with $\lim_{n\to+\infty} t_n = +\infty$ such that $\mathrm{E}_x\left[\int_0^{T^{\bar{\pi}}\wedge t_n} e^{-(\lambda+\delta)s}\sigma(R^{\bar{\pi}}_{s-})V'_{b^*}(R^{\bar{\pi}}_{s-})\mathrm{d}W_s\right] = 0$. By noting that $V'_{b^*}(y) \geq \beta$ for $y \geq 0$ (see (3.1)), we can see $V'_{b^*}(R^{\bar{\pi}}_{s-}) \geq \beta$ for $s \in [0, T^{\bar{\pi}}]$. Then from Remark A.1, we know

$$\sum_{0\leq s\leq T^{\bar{\pi}}\wedge t_n} e^{-(\lambda+\delta)s}(V_{b^*}(R^{\bar{\pi}}_s) - V_{b^*}(R^{\bar{\pi}}_{s-})) - \int_0^{T^{\bar{\pi}}\wedge t_n} e^{-(\lambda+\delta)s}V'_{b^*}(R^{\bar{\pi}}_{s-})\mathrm{d}\bar{\pi}^c_s \leq -\beta\int_0^{T^{\bar{\pi}}\wedge t_n} e^{-(\lambda+\delta)s}\mathrm{d}\bar{\pi}^c_s,$$

which together with (A.52) and (A.53) implies $\mathrm{E}_x\left[e^{-(\lambda+\delta)(T^{\bar{\pi}}\wedge t_n)}V_{b^*}(R^{\bar{\pi}}_{T^{\bar{\pi}}\wedge t_n}) - V_{b^*}(R^{\bar{\pi}}_{0-})\right] \leq -\mathrm{E}_x\left[\beta\int_0^{T^{\bar{\pi}}\wedge t_n} e^{-(\lambda+\delta)s}\mathrm{d}\bar{\pi}_s\right] - \mathrm{E}_x\left[\int_0^{T^{\bar{\pi}}\wedge t_n} e^{-(\lambda+\delta)s}\lambda\alpha V^E_{b^*}(R^{\bar{\pi}}_{s-})\mathrm{d}s\right]$. By noting $\mathrm{E}_x\left[V_{b^*}(R^{\bar{\pi}}_{0-})\right] = V_{b^*}(x)$ and taking $\liminf_{n\to+\infty}$ on the above equation and then using the monotone convergence twice as well as Fatou's

Lemma, we obtain $V_{b^*}(x) \geq \mathrm{E}_x\left[\int_0^{T^{\bar{\pi}}} e^{-(\lambda+\delta)s}\lambda\alpha V_{b^*}^E(R_{s-}^{\bar{\pi}})\mathrm{d}s\right]+\mathrm{E}_x\left[\beta\int_0^{T^{\bar{\pi}}} e^{-(\lambda+\delta)s}\mathrm{d}\bar{\pi}_s\right]+\mathrm{E}_x\left[e^{-(\lambda+\delta)T^{\bar{\pi}}}V_{b^*}(R_{T^{\bar{\pi}}}^{\bar{\pi}})\right]$. Noting $V_{b^*}(R_{T^{\bar{\pi}}}^{\bar{\pi}}) = V_{b^*}(0) = 0$ (by (4.2)), we obtain that

$$V_{b^*}(x) \geq \mathrm{E}_x\left[\int_0^{T^{\bar{\pi}}} e^{-(\lambda+\delta)s}\lambda\alpha V_{b^*}^E(R_{s-}^{\bar{\pi}})\mathrm{d}s\right] + \mathrm{E}_x\left[\beta\int_0^{T^{\bar{\pi}}} e^{-(\lambda+\delta)s}\mathrm{d}\bar{\pi}_s\right]. \tag{A.54}$$

By (2.4) we have

$$\begin{aligned}\mathcal{P}(x;L,L^{b^*}) =&\mathrm{E}_x\Big[\int_0^{T^{\bar{\pi}}\wedge\eta_0} \beta e^{-\delta t}\mathrm{d}\bar{\pi}_t + I\{\eta_0 < T^{\bar{\pi}}\}\alpha\int_{\eta_0}^{T^{\bar{\pi}}} \beta e^{-\delta t}\mathrm{d}\bar{\pi}_t\Big]\\ =&\mathrm{E}_x\left[\beta\int_0^{T^{\bar{\pi}}} e^{-(\lambda+\delta)s}\mathrm{d}\bar{\pi}_s\right] + \mathrm{E}_x\left[\int_0^{T^{\bar{\pi}}} e^{-(\lambda+\delta)s}\lambda\alpha V_{b^*}^E(R_{s-}^{\bar{\pi}})\mathrm{d}s\right],\ x\geq 0,\end{aligned} \tag{A.55}$$

where the last equality follows by noting that the strategy $\bar{\pi} = \bar{\pi}^{\eta_0,L,L^{b^*}}$ coincides with L before η_0 and L^{b^*} after η_0, and using (A.1) and (A.3). Combining (A.54) and (A.55) yields $V_{b^*}(x) \geq \mathcal{P}(x;L,L^{b^*})$ for $x \geq 0$. Due to the arbitrariness of L, we can obtain $V_{b^*}(x) \geq \sum_{L\in\Pi}\mathcal{P}(x;L,L^{b^*})$ for $x \geq 0$. □

Proof of Lemma 4.3 Suppose $b^* = +\infty$. (i) We now use $V_U(\cdot)$ again to represent the same function defined in Equation (4.3) of (Shreve et al., 1984) and set some of the quantities in that reference as follows: $U = b$, $P = 0$, $\beta = \delta$ and $a(\cdot) \equiv \mu(\cdot)$. As noted in an earlier proof, the process ξ_U involved in the definition for V_U in (Shreve et al., 1984) coincides with the barrier strategy, L^U, in this paper, and therefore, from the definition for V_b^E in this paper we can observe that $V_b^E(\cdot) \equiv \beta V_b(\cdot)$. Note that U^* is defined to be the solution, if any, to $V_U''(U) = 0$ in the reference. From the definition for b^E in this paper we can observe that if $b^E = +\infty$, then U^* does not exist. Then from the paragraph following Equation (4.9) in (Shreve et al., 1984) we know $V_{b_1}^E(x) \leq V_{b_2}^E(x)$ for $x \geq 0$ and $0 < b_1 < b_2$. (ii) Define a modified performance index:

$$\mathcal{T}_g(L)(x) = \mathrm{E}_x\left[\int_0^{\eta_0\wedge T^L} \beta e^{-\delta t}\mathrm{d}L_t + I\{\eta_0 < T^L\}\alpha e^{-\delta\eta_0}g(R_t^L)\right]. \tag{A.56}$$

We can see that

$$V_b(x) = \mathcal{P}(x;L^b,L^b) = \mathcal{T}_{V_b^E}(L^b)(x),\ x \geq 0. \tag{A.57}$$

Let Π_b denote the set of the admissible strategies so that the corresponding controlled surpluses are always below b. Then, $\Pi_b \in \Pi$. Define

$$M_{b,g}(x) = \sup_{L\in\Pi_b}\mathcal{T}_g(L)(x),\ x \geq 0. \tag{A.58}$$

We show in the following that

$$V_b(x) = M_{b,V_b^E}(x),\ x \geq 0. \tag{A.59}$$

It follows from (A.57) and (A.58) that $V_b(x) = \mathcal{T}_{V_b^E}(L^b)(x) \leq \sup_{L\in\Pi_b}\mathcal{T}_{V_b^E}(L)(x) = M_{V_b^E,b}(x)$ for $x \geq 0$. Thus, it is sufficient to show that $V_b(x) \geq M_{b,V_b^E}(x)$, $x \geq 0$. To this end, we apply Itô's formula to $e^{-(\lambda+\delta)(T^L\wedge t)}V_b(R_{T^L\wedge t}^L)$ for any $L \in \Pi^b$ and then take expectation under P_x, which leads to

$$\begin{aligned}&\mathrm{E}_x\left[e^{-(\lambda+\delta)(T^L\wedge t)}V_b(R_{T^L\wedge t}^L) - V_b(R_{0-}^L)\right]\\ =&\ \mathrm{E}_x\Big[\int_0^{T^L\wedge t} e^{-(\lambda+\delta)s}\mathcal{G}_{V_b}(R_{s-}^L)\mathrm{d}s + \int_0^{T^L\wedge t} e^{-(\lambda+\delta)s}\sigma(R_{s-}^L)V_b'(R_{s-}^L)\mathrm{d}W_s\\ &-\int_0^{T^{\bar{\pi}}\wedge t} e^{-(\lambda+\delta)s}V_b'(R_{s-}^{\bar{\pi}})\mathrm{d}L_s^c + \sum_{0\leq s\leq T^L\wedge t} e^{-(\lambda+\delta)s}(V_b(R_s^L) - V_b(R_{s-}^L))\Big],\end{aligned} \tag{A.60}$$

where $\mathcal{G}$ is defined in (A.10), $\{L_s^c\}$ is the continuous part of $\{L_s\}$. From the definition of Π^b we note

$$R_t^L \in [0,b] \text{ for any } t \in [0,T^L] \text{ and any } L \in \Pi^b. \tag{A.61}$$

By using (A.61) and (3.8) we can obtain that for any $L \in \Pi^b$,

$$\mathcal{G}_{V_b}(R_{s-}^L) = -\lambda\alpha V_b^E(R_{s-}^L) \ \text{ for } 0 < s \le T^L. \tag{A.62}$$

From now on in the proof of (i), we assume $L \in \Pi^b$. Noting the boundedness of $\{R_t^L : t \in [0,T^L]\}$ (see (A.61)), we can observe that $\left\{\int_0^{T^L\wedge t} e^{-(\lambda+\delta)s}\sigma(R_{s-}^L)V_b'(R_{s-}^L)\mathrm{d}W_s; t \ge 0\right\}$ is a P_x-martingale and hence,

$$\mathrm{E}_x\left[\int_0^{T^L} e^{-(\lambda+\delta)s}\sigma(R_{s-}^L)V_b'(R_{s-}^L)\mathrm{d}W_s\right] = 0. \tag{A.63}$$

Note that for any $b > 0$, $b < +\infty = b^*$. It follows by Definition 3.3(i) that $V_b''(b-) < 0$ for all $b \ge 0$. Noticing that $V_b'(b) = \beta$ (see (3.9)) and the concavity of V_b on $[0,b)$ (see Theorem4.1 (i)), it follows that $V_b'(x) > \beta$ for $x \in [0,b)$ and any $b > 0$. Since $R_t^L \in [0,b]$ for $t \in [0,T^L]$ (by (A.61)), we have $V_b'(R_{t-}^L) > \beta$ for $t \in [0,T^L]$. Thus, from Remark A.1, it follows that

$$\sum_{0\le s\le T^L} e^{-(\lambda+\delta)s}(V_b(R_s^L) - V_b(R_{s-}^L)) - \int_0^{T^L} e^{-(\lambda+\delta)s}V_b'(R_{s-}^L)\mathrm{d}L_s^c \le -\beta\int_0^{T^L} e^{-(\lambda+\delta)s}\mathrm{d}L_s, \tag{A.64}$$

where L^C is the continuous part of L. It follows from (A.60) and (A.62)-(A.64) that

$$\mathrm{E}_x\left[e^{-(\lambda+\delta)T^L}V_b(R_{T^L}^L) - V_b(R_{0-}^L)\right] \le -\mathrm{E}_x\left[\int_0^{T^L} e^{-(\lambda+\delta)s}\lambda\alpha V_b^E(R_{s-}^L)\mathrm{d}s + \beta\int_0^{T^L} e^{-(\lambda+\delta)s}\mathrm{d}L_s\right].$$

By noting $\mathrm{E}_x\left[V_b(R_{0-}^L)\right] = V_b(x)$ and $V_b(R_{T^L}^L) = V_b(0) = 0$ (by (4.2)), we obtain that

$$V_b(x) \ge \mathrm{E}_x\left[\int_0^{T^L} e^{-(\lambda+\delta)s}\lambda\alpha V_b^E(R_{s-}^L)\mathrm{d}s\right] + \mathrm{E}_x\left[\beta\int_0^{T^L} e^{-(\lambda+\delta)s}\mathrm{d}L_s\right]. \tag{A.65}$$

According to the definition of the modified performance index in (A.56) we have

$$\begin{aligned}\mathcal{T}_{V_b^E}(L)(x) =& \mathrm{E}_x\left[\int_0^{T^L\wedge\eta_0} \beta e^{-\delta t}\mathrm{d}L_t + I\{\eta_0 < T^L\}\alpha e^{-\delta\eta_0}V_b^E(R_{\eta_0}^L)\mathrm{d}s\right]\\ =& \mathrm{E}_x\left[\beta\int_0^{T^L} e^{-(\lambda+\delta)s}\mathrm{d}L_s\right] + \mathrm{E}_x\left[\int_0^{T^L} e^{-(\lambda+\delta)s}\lambda\alpha V_b^E(R_{s-}^L)\mathrm{d}s\right],\ x \ge 0,\end{aligned} \tag{A.66}$$

where the last equality follows by (A.1) and (A.2). Combining (A.65) and (A.66) yields $V_b(x) \ge \mathcal{T}_{V_b^E}(L)(x)$. Due to the arbitrariness of L in Π^b, we have $V_b(x) \ge \sup_{L\in\Pi^b}\mathcal{T}_{V_b^E}(L)(x) = M_{b,V_b^E}(x)$.

Furthermore, noticing that $\Pi_{b_1} \subset \Pi_{b_2}$ for any b_1 and any b_2 with $0 \le b_1 \le b_2$, we have $\sup_{L\in\Pi_{b_1}}\mathcal{T}_g(L)(x) \le \sup_{L\in\Pi_{b_2}}\mathcal{T}_g(L)(x)$, $x \ge 0$. Therefore, for any b_1 and any b_2 with $0 \le b_1 \le b_2$,

$$\mathcal{M}_{b_1,g}(x) = \sup_{L\in\Pi_{b_1}}\mathcal{T}_g(L)(x) \le \sup_{L\in\Pi_{b_2}}\mathcal{T}_g(x) = \mathcal{M}_{b_2,g}(x),\ x \ge 0. \tag{A.67}$$

It follows by (A.56) that for any L and any $g_1 \le g_2$, $\mathcal{T}_{V_{b_1}^E}(L)(x) = \mathrm{E}_x\left[\int_0^{\eta_0\wedge T^L}\beta e^{-\delta t}\mathrm{d}L_t\right] + \mathrm{E}_x\left[I\{\eta_0 < T^L\}\alpha e^{-\delta\eta}V_{b_1}^E(R_t^L)\right] \le \mathrm{E}_x\left[\int_0^{\eta_0\wedge T^L}\beta e^{-\delta t}\mathrm{d}L_t\right] + \mathrm{E}_x\left[I\{\eta_0 < T^L\}\alpha e^{-\delta\eta_0}V_{b_2}^E(R_t^L)\right] = \mathcal{T}_{V_{b_2}^E}(L)(x)$, and as a result, $\mathcal{M}_{b,V_{b_1}^E}(x) \le \mathcal{M}_{b,V_{b_2}^E}(x)$ for $x \ge 0$. Combining this inequality with (A.59) and (A.67), we obtain $V_{b_1}(x) = \mathcal{M}_{b_1,V_{b_1}^E}(x) \le \mathcal{M}_{b_2,V_{b_1}^E}(x) \le \mathcal{M}_{b_2,V_{b_2}^E}(x) = V_{b_2}(x)$ for $x \ge 0$. □

Proof of Theorem 4.4 Suppose $b^* < +\infty$. We use proof by contradiction. Suppose there exists a $b_0 > 0$ such that L^{b_0} is a MPE. That is,

$$\mathcal{P}(x; L^{b_0}, L^{b_0}) = \sup_{L \in \Pi} \mathcal{P}(x; L, L^{b_0}). \tag{A.68}$$

From the definition of b^*, we can see that

$$V_b''(b-) < 0, \ \ b > 0. \tag{A.69}$$

Note that $V_{b_0}(x)$ is continuously differentiable on $[0, +\infty)$ and twice continuously differentiable on $[0, b_0) \cup (b_0, +\infty)$. Define

$$h_{b_0}(x) = \frac{\sigma^2(x)}{2} V_{b_0}''(x) + \mu(x) V_{b_0}'(x) - (\lambda + \delta) V_{b_0}(x) + \lambda \alpha V_{b_0}^E(x), \ \ x \in [0, b_0) \cup (b_0, +\infty). \tag{A.70}$$

It follows by (3.8) that

$$h_{b_0}(x) = 0, \ \ x \in [0, b), \tag{A.71}$$

which implies $h_{b_0}(b_0-) = 0$. From (A.70), we can see $h_{b_0}(b_0+) = h_{b_0}(b_0-) + \frac{\sigma^2(b_0)}{2}(V_{b_0}''(b_0+) - V_{b_0}''(b_0-)) > 0$, where the last inequality follows by noting $V_{b_0}''(b_0+) = 0$ and $V_{b_0}''(b_0-) < 0$, which is due to (A.69). Hence, there exists an $\epsilon_0 > 0$ such that

$$h_{b_0}(x) > 0, \ \ x \in (b_0, b_0 + \epsilon_0). \tag{A.72}$$

We apply Itô's formula to $V_{b_0}(R^{L^{b_0+\epsilon_0}})$ and take expectation under P_x. This gives

$$\begin{aligned}
&\mathrm{E}_x \left[e^{-(\lambda+\delta)(T^{L^{b_0+\epsilon_0}} \wedge t)} V_{b_0}(R^{L^{b_0+\epsilon_0}}_{T^{L^{b_0+\epsilon_0}} \wedge t}) - V_{b_0}(R^{L^{b_0+\epsilon_0}}_{0-}) \right] \\
= \ &\mathrm{E}_x \Bigg[\int_0^{T^{L^{b_0+\epsilon_0}} \wedge t} e^{-(\lambda+\delta)s} \mathcal{G} V_{b_0}(R^{L^{b_0+\epsilon_0}}_{s-}) \mathrm{d}s + \int_0^{T^{L^{b_0+\epsilon_0}} \wedge t} e^{-(\lambda+\delta)s} \sigma(R^{L^{b_0+\epsilon_0}}_{s-}) V_b'(R^{L^{b_0+\epsilon_0}}_{s-}) \mathrm{d}W_s \\
&- \int_0^{T^{L^{b_0+\epsilon_0}} \wedge t} e^{-(\lambda+\delta)s} V_{b_0}'(R^{L^{b_0+\epsilon_0}}_{s-}) \mathrm{d}L^{b_0+\epsilon_0, c}_s + \sum_{0 \le s \le T^{L^{b_0+\epsilon_0}} \wedge t} e^{-(\lambda+\delta)s} (V_{b_0}(R^{L^{b_0+\epsilon_0}}_s) - V_{b_0}(R^{L^{b_0+\epsilon_0}}_{s-})) \Bigg].
\end{aligned} \tag{A.73}$$

It follows from (A.10), (A.70), (A.71) and (A.72) that

$$\mathcal{G} V_{b_0}(R^{L^{b_0+\epsilon_0}}_{s-}) = h_{b_0}(R^{L^{b_0+\epsilon_0}}_{s-}) - \lambda \alpha V_{b_0}^E(R^{L^{b_0+\epsilon_0}}_{s-}) \begin{cases} = -\lambda \alpha V_{b_0}^E(R^{L^{b_0+\epsilon_0}}_{s-}) & \text{if } R^{L^{b_0+\epsilon_0}}_{s-} \in [0, b_0] \\ > -\lambda \alpha V_{b_0}^E(R^{L^{b_0+\epsilon_0}}_{s-}) & \text{if } R^{L^{b_0+\epsilon_0}}_{s-} \in (b_0, b_0 + \epsilon_0). \end{cases} \tag{A.74}$$

Note that $R^{L^{b_0+\epsilon_0}}_{s-} \in [0, b_0 + \epsilon_0]$ is bounded for $s \in (0, T^{L^{b_0+\epsilon_0}}]$. Hence, using the martingale property and a standard stopping argument shows

$$\mathrm{E}_x \left[\int_0^{T^{L^{b_0+\epsilon_0}} \wedge t} e^{-(\lambda+\delta)s} \sigma(R^{L^{b_0+\epsilon_0}}_{s-}) V_{b_0}'(R^{L^{b_0+\epsilon_0}}_{s-}) \mathrm{d}W_s \right] = 0. \tag{A.75}$$

Note that

$$V_{b_0}'(x) = \beta \text{ for } x \geq b_0, \quad \text{and hence } V_{b_0}(x) = V_{b_0}(b_0) + \beta(x - b_0) \text{ for } x \geq b_0. \tag{A.76}$$

Further notice that according to the definition of the barrier strategy in Definition 3.1, we know that

$$R_{s-}^{L^{b_0+\epsilon_0}} = R_s^{L^{b_0+\epsilon_0}} \quad \text{for } s > 0, \quad R_0^{L^{b_0+\epsilon_0}} I\{R_{0-}^{L^{b_0+\epsilon_0}} < b_0 + \epsilon_0\} = R_{0-}^{L^{b_0+\epsilon_0}} I\{R_{0-}^{L^{b_0+\epsilon_0}} < b_0 + \epsilon_0\}, \tag{A.77}$$

$$R_0^{L^{b_0+\epsilon_0}} I\{R_{0-}^{L^{b_0+\epsilon_0}} \geq b_0 + \epsilon_0\} = (b_0 + \epsilon) I\{R_{0-}^{L^{b_0+\epsilon_0}} \geq b_0 + \epsilon_0\}. \tag{A.78}$$

It follows by using (A.77) and (A.78) that

$$\begin{aligned}
&\sum_{0 \leq s \leq T^{L^{b_0+\epsilon_0}} \wedge t} e^{-(\lambda+\delta)s}(V_{b_0}(R_s^{L^{b_0+\epsilon_0}}) - V_{b_0}(R_{s-}^{L^{b_0+\epsilon_0}})) = (V_{b_0}(R_0^{L^{b_0+\epsilon_0}}) - V_{b_0}(R_{0-}^{L^{b_0+\epsilon_0}})) I\{R_{0-}^{L^{b_0+\epsilon_0}} \geq b_0 + \epsilon_0\} \\
&= \left(V_{b_0}(b_0) + \beta\epsilon_0 - (V_{b_0}(b_0) + \beta(R_{0-}^{L^{b_0+\epsilon_0}} - b_0))\right) I\{R_{0-}^{L^{b_0+\epsilon_0}} \geq b_0 + \epsilon_0\} \\
&= -\beta(R_{0-}^{L^{b_0+\epsilon_0}} - (b_0 + \epsilon_0)) I\{R_{0-}^{L^{b_0+\epsilon_0}} \geq b_0 + \epsilon_0\} = -\beta \sum_{0 \leq s \leq T^{L^{b_0+\epsilon_0}} \wedge t} e^{-(\lambda+\delta)s}\beta(R_{s-}^{L^{b_0+\epsilon_0}} - R_s^{L^{b_0+\epsilon_0}}) \\
&= -\beta \sum_{0 \leq s \leq T^{L^{b_0+\epsilon_0}} \wedge t} e^{-(\lambda+\delta)s}(L_s^{b_0+\epsilon_0} - L_{s-}^{b_0+\epsilon_0}),
\end{aligned} \tag{A.79}$$

where the second equality follows by (A.76), and the second to the last equality by (A.77) and (A.78). By noticing $\mathrm{d}L_s^{b_0+\epsilon,c} = I\{R_{s-}^{L^{b_0+\epsilon_0}} = b_0 + \epsilon_0\}\mathrm{d}L_s^{b_0+\epsilon,c}$ and $V_{b_0}'(b_0 + \epsilon_0) = \beta$ (see (A.76)), we can obtain

$$\int_0^{T^{L^{b_0+\epsilon_0}} \wedge t} e^{-(\lambda+\delta)s} V_{b^*}'(R_{s-}^{L^{b_0+\epsilon_0}}) \mathrm{d}L_s^{b_0+\epsilon,c} = \int_0^{T^{L^{b_0+\epsilon_0}} \wedge t} e^{-(\lambda+\delta)s} \beta \mathrm{d}L_s^{b_0+\epsilon,c}. \tag{A.80}$$

Combining (A.73), (A.74), (A.75), (A.79) and (A.80) yields
$\mathrm{E}_x\left[e^{-(\lambda+\delta)(T^{L^{b_0+\epsilon_0}} \wedge t)} V_{b_0}(R_{T^{L^{b_0+\epsilon_0}} \wedge t}^{L^{b_0+\epsilon_0}}) - V_{b_0}(R_{0-}^{L^{b_0+\epsilon_0}})\right] > -\mathrm{E}_x\left[\int_0^{T^{L^{b_0+\epsilon_0}} \wedge t} e^{-(\lambda+\delta)s} \lambda\alpha V_{b_0+\epsilon_0}^E(R_{s-}^{L^{b_0+\epsilon_0}})\mathrm{d}s\right.$
$\left.+\beta \int_0^{T^{L^{b_0+\epsilon_0}} \wedge t} e^{-(\lambda+\delta)s} \mathrm{d}L_s^{b_0+\epsilon_0}\right]$. By noting $\mathrm{E}_x\left[V_{b_0}(R_{0-}^{L^{b_0+\epsilon_0}})\right] = V_{b_0}(x)$, $R_{s-}^{L^{b_0+\epsilon_0}} \in [0, x \vee (b_0 + \epsilon_0)]$ given $R_{0-} = x$, and taking $\lim_{n\to+\infty}$ on both sides of the above inequality and using the dominated convergence we obtain
$V_{b_0}(x) < \mathrm{E}_x\left[\int_0^{T^{L^{b_0+\epsilon_0}}} e^{-(\lambda+\delta)s} \lambda\alpha V_{b_0}^E(R_{s-}^{L^{b_0+\epsilon_0}})\mathrm{d}s\right] + \mathrm{E}_x\left[\beta \int_0^{T^{L^{b_0+\epsilon_0}}} e^{-(\lambda+\delta)s} \mathrm{d}L_s^{b_0+\epsilon_0}\right]$
$+\mathrm{E}_x\left[e^{-(\lambda+\delta)T^{L^{b_0+\epsilon_0}}} V_{b_0}(R_{T^{L^{b_0+\epsilon_0}}}^{L^{b_0+\epsilon_0}})\right]$. Noting $V_{b_0}(R_{T^{L^{b_0+\epsilon_0}}}^{L^{b_0+\epsilon_0}}) = V_{b_0}(0) = 0$ (by (4.2)), we arrive that

$$V_{b_0}(x) < \mathrm{E}_x\left[\beta \int_0^{T^{L^{b_0+\epsilon_0}}} e^{-(\lambda+\delta)s} \mathrm{d}L_s^{b_0+\epsilon_0}\right] + \mathrm{E}_x\left[\int_0^{T^{L^{b_0+\epsilon_0}}} e^{-(\lambda+\delta)s} \lambda\alpha V_{b_0}^E(R_{s-}^{L^{b_0+\epsilon_0}})\mathrm{d}s\right]. \tag{A.81}$$

By (2.4), it follows that

$$\begin{aligned}
\mathcal{P}(x; L^{b_0+\epsilon_0}, L^{b_0+\epsilon_0}) &= \mathrm{E}_x\left[\int_0^{T^{L^{b_0+\epsilon_0}} \wedge \eta_0} \beta e^{-\delta t} \mathrm{d}L_t^{b_0+\epsilon_0} + I\{\eta_0 < T^{L^{b_0+\epsilon_0}}\}\alpha \int_{\eta_0}^{T^{L^{b_0+\epsilon_0}}} \beta e^{-\delta t} \mathrm{d}L_t^{b_0+\epsilon_0}\right] \\
&= \mathrm{E}_x\left[\beta \int_0^{T^{L^{b_0+\epsilon_0}}} e^{-(\lambda+\delta)s} \mathrm{d}L_s^{b_0+\epsilon_0}\right] + \mathrm{E}_x\left[\int_0^{T^{L^{b_0+\epsilon_0}}} e^{-(\lambda+\delta)s} \lambda\alpha V_{b_0+\epsilon_0}^E(R_{s-}^{L^{b_0+\epsilon_0}})\mathrm{d}s\right], \\
&> \mathrm{E}_x\left[\beta \int_0^{T^{L^{b_0+\epsilon_0}}} e^{-(\lambda+\delta)s} \mathrm{d}L_s^{b_0+\epsilon_0}\right] + \mathrm{E}_x\left[\int_0^{T^{L^{b_0+\epsilon_0}}} e^{-(\lambda+\delta)s} \lambda\alpha V_{b_0}^E(R_{s-}^{L^{b_0+\epsilon_0}})\mathrm{d}s\right], \quad x > 0,
\end{aligned} \tag{A.82}$$

where the equality in the second to the last step follows by noticing $T^{L^{b_0+\epsilon_0}} = T^{\bar{\pi}^{\eta_0, L^{b_0+\epsilon_0}, L^{b_0+\epsilon_0}}}$, (A.1) and (A.3), the equality in the last step follows according to the definition of V_b^E in Definition 3.2 and the last inequality follows by noting $V_{b_0}^E(y) < V_{b_0+\epsilon_0}^E(y)$ for $y > 0$ (see Lemma 4.3(ii)). By combining (A.81) and (A.82), we obtain $\mathcal{P}(x; L^{b_0}, L^{b_0}) = V_{b_0}(x) < \mathcal{P}(x; L^{b_0+\epsilon_0}, L^{b_0+\epsilon_0})$ for $x > 0$, which is a contradiction to (A.68). This completes the proof. □

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