LOCAL BEHAVIOR OF ARITHMETICAL FUNCTIONS WITH APPLICATIONS TO AUTOMORPHIC L-FUNCTIONS

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Dedicated to Kai-Man Tsang on the occasion of his 60th birthday

ABSTRACT. We derive a Voronoi-type series approximation for the local weighted mean of an arithmetical function that is associated to Dirichlet series satisfying a functional equation with gamma factors. The series is exploited to study the oscillation frequency with a method of Heath-Brown and Tsang [7]. A by-product is another proof for the well-known result of no element in the Selberg class of degree $0 < d < 1$. Our major applications include the sign-change problem of the coefficients of automorphic L-functions for GL_m, which improves significantly some results of Liu and Wu [14]. The cases of modular forms of half-integral weight and Siegel eigenforms are also considered.

1. INTRODUCTION AND MAIN RESULTS

The functional equation satisfied by the Riemann zeta function is a prototype of the salient features of many interesting Dirichlet series $\sum_n a_n \lambda_n^{-s}$, including the Selberg class of L-functions whose theory were greatly advanced recently. This general class were studied quite long time ago, for instance, Chandrasekharan and Narasimhan [3] obtained nice Voronoi-type series approximation for the Riesz means of the coefficients a_n. Often the Voronoi series are effective for the study of many interesting properties such as the mean square formula, omega results and the occurrence of sign-changes in [10] and [12] respectively. The Riesz mean carries a weight function whose smoothing effect leads to permissible convergence of approximations. However this weight (of the Riesz mean) is not suitable for localizing a_n within a narrow range and hence not for local means. In practice the local mean can be applied to depict the oscillations.

The purpose of this paper is two-fold. Firstly we provide a Voronoi-type series to the local weighted mean of a_n, which is novel, and deduce the occurrence of oscillations over short intervals. A cute consequence is an alternative argument for the emptiness of the Selberg class of degree $0 < d < 1$. Secondly the oscillation result is applied to the sign-change problem of the Dirichlet series coefficients of automorphic L-functions. The sign problem drew good attentions (e.g. [11, 13, 14, 16, 17]) and the widely used approach, especially for somewhat general situations, is based on the first and second moments. This approach, however, may not be efficient for high rank groups due to the poor order estimate for the Rankin-Selberg L-function which is the key ingredient to evaluate the second moment. Here the oscillation is detected with the method in Heath-Brown and Tsang [7]. Our application in Section 2 shows not only the utility of the Voronoi series for local weighted means but also the effectiveness of the method in [7]. Theorems 1-3 of this section are for the general context, and the specific results for automorphic L-functions are in Section 2.

1.1. **Assumptions.**

We are concerned with the class of arithmetical functions considered as in [3, 2, 6] with a little variant. Let $\{a_n\}$ and $\{b_n\}$ be two sequences in $\mathbb{C}$, where a_1 and b_1 are nonzero, and $\{\lambda_n\}$ and $\{\mu_n\}$ be two strictly increasing positive number sequences tending to ∞. We assume the following conditions.

(A1) The two series

$$\phi(s) := \sum_{n \geqslant 1} a_n \lambda_n^{-s} \qquad \text{and} \qquad \psi(s) := \sum_{n \geqslant 1} b_n \mu_n^{-s}$$

converges absolutely in some half-plane $\Re e\, s \geqslant \sigma^*$ for some constant $\sigma^* > 0$.

(A2) Suppose $\alpha_\nu > 0$ and $\beta_\nu, \widetilde{\beta}_\nu \in \mathbb{C}$ for $\nu = 1, 2, \ldots, d$, and let

$$\Delta(s) := \prod_{1 \leqslant \nu \leqslant d} \Gamma(\alpha_\nu s + \beta_\nu) \quad \text{and} \quad \widetilde{\Delta}(s) := \prod_{1 \leqslant \nu \leqslant d} \Gamma(\alpha_\nu s + \widetilde{\beta}_\nu).$$

Then $\phi(s)$ and $\psi(s)$ satisfy the functional equation

$$\Delta(s)\phi(s) = \omega \widetilde{\Delta}(1 - s)\psi(1 - s)$$

for some constant $\omega \in \mathbb{C}$ of modulus 1.[†]

(A3) The function $\Delta(s)\phi(s)$ (and hence $\widetilde{\Delta}(s)\psi(s)$) extends meromorphically to the whole $\mathbb{C}$. All singularities of $\Delta(s)\phi(s)$ fall inside the disk $\mathcal{D}$ enclosed by the anti-clockwise circle $\mathcal{C}$: $|z| = R$ for some $R > 0$. As $\Delta(s)^{-1}$ is entire, all poles of $\phi(s)$ lie in $\mathcal{D}$. Moreover, $\Delta(\sigma + \mathrm{i}t)\phi(\sigma + \mathrm{i}t)$ tends to 0 uniformly in every vertical strip $b \leqslant \sigma \leqslant b'$ as $|t| \to \infty$.

Define

$$A := \alpha_1 + \alpha_2 + \cdots + \alpha_d,$$
$$B := \beta_1 + \beta_2 + \cdots + \beta_d,$$
$$\widetilde{B} := \widetilde{\beta}_1 + \widetilde{\beta}_2 + \cdots + \widetilde{\beta}_d,$$

and

$$h := \prod_{1 \leqslant \nu \leqslant d} \left(\frac{2A}{\alpha_\nu}\right)^{2\alpha_\nu},$$

$$a := \frac{1}{4A} - \frac{1}{2} - \frac{B - \widetilde{B}}{2A} = -\vartheta + \mathrm{i}\xi,$$

$$k := \frac{d}{2} - \frac{1}{4} - \frac{A + B + \widetilde{B}}{2} = \kappa + \mathrm{i}\eta.$$

Thus $A > 0$, $h > 0$ while $B, \widetilde{B}, a, k \in \mathbb{C}$ (with $\vartheta, \xi, \kappa, \eta \in \mathbb{R}$).

Remark 1. In [3, 2, 6], the gamma factors on both sides of the functional equation are equal, i.e. $\Delta(s) = \widetilde{\Delta}(s)$. However $\widetilde{\Delta}(s) := \overline{\Delta(\overline{s})}$ for the Selberg class.

[†]Here we allow different sets of the archimedean parameters β_ν on the two sides in order to cover the Selberg class and to specialize to the case $\delta = 1$ by replacing b_n with $b_n \mu_n^{1-\delta}$. So the generality is not lost.

1.2. **Set-up and results.**

We start with any fixed smooth function φ_0 supported on $[-1, 1]$ such that

(i) $0 \leqslant \varphi_0(u) \leqslant 1$ on $[-1, 1]$, and

(ii) $1 \leqslant \int_{\mathbb{R}} \varphi_0(u)\mathrm{u} \leqslant 2$.

Let $\delta > 0$ be any quantity and X a large number. Set

$$L := \delta^{-1} X^{1/(2A)} \qquad \text{and} \qquad \varphi(u) := \varphi_0((u-1)L).$$

Then φ is smooth and compactly supported on $(0, \infty)$ with $0 \leqslant \varphi(u) \leqslant 1$ on $[1 - L^{-1}, 1 + L^{-1}]$ and zero elsewhere. Its derivatives satisfy

$$(1.1) \qquad \|\varphi^{(r)}\|_\infty \ll_{\varphi_0, r} L^r \quad \text{for } r \geqslant 0$$

and, moreover,

$$(1.2) \qquad L^{-1} \leqslant \int_0^\infty \varphi(u)\mathrm{u} \leqslant 2L^{-1}.$$

Denote by

$$\widehat{\varphi}(s) := \int_0^\infty \varphi(u) u^{s-1} \mathrm{u}$$

its Mellin transform. Then $\widehat{\varphi}$ is entire and

$$(1.3) \qquad \widehat{\varphi}(s) \ll_r L^{r-1}(1 + |s|)^{-r} \quad \text{for all } r \geqslant 0.$$

Let $\sigma_1 > \max(\sigma^*, R)$. Then the disk $\mathcal{D}$ lies in the half-plane $\Re s < \sigma_1$. For $x \in [X, 4X]$, we have

$$\sum_{n \geqslant 1} a_n \varphi\left(\frac{\lambda_n}{x}\right) = \frac{1}{2\pi i} \int_{(\sigma_1)} \phi(s)\widehat{\varphi}(s) x^s \mathrm{s}.$$

Take a constant σ_2 such that

$$1 - \sigma_2 > \max\left\{\sigma^*, 1 + R, -\alpha_1^{-1}\widetilde{\beta}_1, \ldots, -\alpha_d^{-1}\widetilde{\beta}_d\right\}.$$

Then $\mathcal{D}$ lies in the right side of $\Re s = \sigma_2$ and all poles of $\widetilde{\Delta}(s)$ and $\widetilde{\Delta}(s)\psi(s)$ lie in $\Re s < 1 - \sigma_2$. We shift the line of integral to $\Re s = \sigma_2$ and apply the functional equation with a change of variable s into $1 - s$, and thus infer

$$
\begin{aligned}
S_\varphi(x) &:= \sum_{n \geqslant 1} a_n \varphi\left(\frac{\lambda_n}{x}\right) - M_\varphi(x) \\
(1.4) \qquad &= \frac{\omega}{2\pi i} \int_{(1-\sigma_2)} \frac{\widetilde{\Delta}(s)}{\Delta(1-s)} \psi(s)\widehat{\varphi}(1-s) x^{1-s} \mathrm{s} \\
&= \omega \sum_{n \geqslant 1} \frac{b_n}{\mu_n} I(\mu_n x),
\end{aligned}
$$

where

$$(1.5) \qquad M_\varphi(x) := \frac{1}{2\pi i} \int_{\mathfrak{e}} \phi(s)\widehat{\varphi}(s) x^s \mathrm{s}$$

and

$$(1.6) \qquad I(y) := \frac{1}{2\pi i} \int_{(1-\sigma_2)} \frac{\widetilde{\Delta}(s)}{\Delta(1-s)} \widehat{\varphi}(1-s) y^{1-s} \mathrm{s}.$$

Our first theorem is a Voronoi-type series approximation for the local weighted mean of a_n.

Theorem 1. *Let $\delta > 0$ be fixed and X any sufficiently large number. Under the above assumptions (A1)-(A3) and notation, the function $S_\varphi(x)$ defined in (1.4) satisfies the asymptotic expansion*

$$(1.7) \qquad S_\varphi(x) = \frac{\omega e_0}{2Ah}(hx)^{1-\vartheta+i\xi}S_{\varphi,0}(x) + O\left(L^{-1}x^{1-\vartheta-1/(2A)}\right)$$

for any $x \in [X, 4X]$, where $e_0 \in \mathbb{C}^\times$ is given by

$$(1.8) \qquad e_0 := \sqrt{\frac{2}{\pi}} \prod_{\nu=1}^{d}\left(\frac{2A}{\alpha_\nu}\right)^{\alpha_\nu+\beta_\nu-\widetilde{\beta}_\nu}$$

and

$$S_{\varphi,0}(x) := \sum_{n\geqslant 1}\frac{b_n}{\mu_n^{\vartheta-i\xi}}\int_0^\infty \frac{\varphi(u)}{u^\vartheta}u^{i\xi}\cos\left((h\mu_n xu)^{1/(2A)} + k\pi\right)u.$$

Besides we have

$$S_\varphi(x) \ll x^{1-\vartheta}L^{-1}$$

for all $x \in [X, 4X]$. The implied constants in the O-term and $\ll$-symbol depend only on φ_0, δ and the parameters in (A1) and (A2).

With the help of Heath-Brown and Tsang [7], we derive from Theorem 1 the following result concerning sign changes.

Theorem 2. *Under the assumptions (A1)-(A3), there exist small $\delta > 0$ and positive constants c_0, c_+, c_- which depend at most on the parameters in (A1) and (A2) such that for all sufficiently large $x \geqslant X_0(\delta)$, the two inequalities*

$$\pm\Re e\left(\varsigma^{-1}\frac{S_\varphi(x_\pm)}{(\mu_1 hx_\pm)^{i\xi}}\right) > c_\pm x^{1-\vartheta}L^{-1}$$

hold for some $x_+, x_- \in [x - c_0 x^{1-1/(2A)}, x + c_0 x^{1-1/(2A)}]$, where $\varsigma := \omega e_0 b_1/|b_1|$ and the parameter L in the set-up for φ takes the value $\delta^{-1}x^{1/(2A)}$.

Remark 2. (i) If $\eta \neq 0$, we can get the same result for the imaginary part as well.

(ii) Obviously the two inequalities imply a sign-change of $\Re e\left(\varsigma^{-1}S_\varphi(t)/(\mu_1 ht)^{i\xi}\right)$ in the short interval $[x - c_0 x^{1-1/(2A)}, x + c_0 x^{1-1/(2A)}]$. When $0 < 2A < 1$, the interval shrinks as $x \to \infty$, implying a highly oscillatory $x^{-i\xi}S_\varphi(x)$. This is unlikely to happen in many circumstances, see Theorem 3 below.

(iii) In case $\phi(s)$ has no pole in $\mathcal{D}$ (see (A3)), $M_\varphi(x) \equiv 0$ and thus the positivity (resp. negativity) of $S_\varphi(t)$ implies $a_n > 0$ (resp. $a_n < 0$) for some $|\lambda_n - x| \ll x^{1-1/(2A)}$.

The next is a consequence echoing the empty Selberg class for small degree $0 < d < 1$ proved in [5]. Note $d = 2A$ here.

Theorem 3. *Suppose $0 < 2A < 1$ and (A1)-(A3). If either of the following conditions:*

(a) *$\phi(s)$ is entire,*
(b) *$\xi = 0$ and all poles of $\phi(s)$ are real,*
(c) *all poles of $\phi(s)$ lie in the half-plane $\Re e\, s < \vartheta_0$ where $\vartheta_0 := 1 - \vartheta + 1/2A$,*

holds true, then for all large enough n, we have $\lambda_{n+1} - \lambda_n \ll \lambda_n^{1-1/(2A)}$, where the implied $\ll$-constant is independent of n.

Remark 3. Suppose $F(s)$ is belonged to the Selberg class $\mathcal{S}(\mathrm{d})$ of degree $0 < \mathrm{d} < 1$. Then $\lambda_n = n$ for all $n \geqslant 1$, and the corresponding ϑ_0 is $1/2 + 3/(2\mathrm{d}) > 1$. As $F(s)$ has at most one pole at $s = 1$, Condition (c) of Theorem 3 are fulfilled, implying $1 = \lambda_{n+1} - \lambda_n \ll n^{-(1-\mathrm{d})/\mathrm{d}}$ for all large n, an absurdity. Hence $\mathcal{S}(\mathrm{d}) = \emptyset$ for $0 < \mathrm{d} < 1$, cf. [5, 19].

2. APPLICATIONS

We consider the respective L-functions arising from self-contragredient representations for $GL_m(\mathbb{A}_\mathbb{Q})$, modular forms of half-integral weight and Siegel Hecke eigenforms. The latter two cases will be handled altogether.

2.1. **Self-contragredient representations for $GL_m(\mathbb{A}_\mathbb{Q})$.**

Let $m \geqslant 2$ be an integer and let $\pi = \otimes \pi_p$ be an irreducible unitary cuspidal representation of $GL_m(\mathbb{A}_\mathbb{Q})$. We associate the local parameters $\{\alpha_\pi(p, j)\}_{j=1}^m \subset \mathbb{C}$ and $\{\mu_\pi(j)\}_{j=1}^m \subset \mathbb{C}$ respectively to π_p and π_∞ by the Langlands correspondence. The automorphic L-function $L(s, \pi)$ attached to π is entire and expands into

$$L(s, \pi) = \sum_{n \geqslant 1} \lambda_\pi(n) n^{-s}$$

for $\Re\, s > 1$, where

$$\lambda_\pi(n) := \prod_{p^\nu \| n} \sum_{\nu_1 + \cdots + \nu_m = \nu} \prod_{1 \leqslant j \leqslant m} \alpha_\pi(p, j)^{\nu_j}.$$

Set $\phi(s) = \pi^{-ms/2} q_\pi^{s/2} L(s, \pi)$ and $\psi(s) = \pi^{-ms/2} q_{\widetilde{\pi}}^{s/2} L(s, \widetilde{\pi})$, where $\widetilde{\pi}$ is the contragredient of π with $\{\overline{\mu_\pi(j)} : 1 \leqslant j \leqslant m\}$ as the set of parameters for $\widetilde{\pi}_\infty$ and q_π is the arithmetic conductor of π. The functional equation in (A2) is satisfied with $\omega = 1$,

$$\Delta(s) = \prod_{1 \leqslant j \leqslant m} \Gamma\left(\frac{s + \mu_\pi(j)}{2}\right) \quad \text{and} \quad \widetilde{\Delta}(s) = \prod_{1 \leqslant j \leqslant m} \Gamma\left(\frac{s + \overline{\mu_\pi(j)}}{2}\right).$$

It is known that from Kim & Sarnak [9] ($2 \leqslant m \leqslant 4$) and Luo, Rudnick & Sarnak [15] ($m \geqslant 5$) that

$$(2.1) \qquad |\alpha_\pi(p, j)| \leqslant p^{\theta_m} \qquad \text{and} \qquad |\Re\, \mu_\pi(j)| \leqslant \theta_m$$

for all primes p and $1 \leqslant j \leqslant m$, where

$$\theta_2 := \frac{7}{64}, \qquad \theta_3 := \frac{5}{14}, \qquad \theta_4 := \frac{9}{22}, \qquad \theta_m := \frac{1}{2} - \frac{1}{m^2 + 1} \quad (m \geqslant 5).$$

The Generalized Ramanujan Conjecture (GRC) asserts that the inequalities in (2.1) hold for all primes p and $1 \leqslant j \leqslant m$ with

$$\theta_m = 0.$$

Plainly (2.1) implies

$$|\lambda_\pi(n)| \leqslant n^{\theta_m} d_m(n)$$

for all $n \geqslant 1$, where $d_m(n) := \sum_{n_1 \cdots n_m = n} 1$.

When π is self-contragredient, we have

$$\{\mu_\pi(j) : 1 \leqslant j \leqslant m\} = \{\overline{\mu_\pi(j)} : 1 \leqslant j \leqslant m\}.$$

Thus $\widetilde{\Delta}(s) = \Delta(s)$ and $\lambda_\pi(n)$ is real for all $n \geqslant 1$. Further,

$$A = \tfrac{1}{2}m, \qquad B = \widetilde{B} \in \mathbb{R}, \qquad \xi = \eta = 0, \qquad e_0 = (2/\pi)^{1/2}(2m)^{m/2}.$$

Let

$$\mathscr{N}_\pi^\pm(x) := \sum_{\substack{n \leqslant x \\ \lambda_\pi(n) \gtrless 0}} 1.$$

Liu & Wu [14] proved that

$$(2.2) \qquad\qquad \mathscr{N}_\pi^\pm(x) \gg_\pi x^{1-2\theta_m}(\log x)^{2/m-2} \qquad (x \geqslant x_0(\pi))$$

holds unconditionally for $2 \leqslant m \leqslant 4$ and holds under Hypothesis H of Rudnick-Sarnak [21] for $m \geqslant 5$. Moreover, this result was applied to evaluate the number of sign changes $\mathrm{N}_\pi^*(x)$ of the sequence $\{\lambda_\pi(n)\}_{n \geqslant 1}$ in the interval $[1, x]$. It is shown in [14, Corollary 1] that

$$\mathrm{N}_\pi^*(x) \gg_\pi \log\log x \qquad (x \geqslant x_0(\pi))$$

unconditionally for $2 \leqslant m \leqslant 4$ and under Hypothesis H for $m \geqslant 5$.

Now by Theorem 2 with $A = m/2$ (and Remark 2 (iii)), the sequence $\{\lambda_\pi(n)\}$ has a sign-change over the short interval $\mathcal{I}_x := [x - cx^{1-1/m}, x + cx^{1-1/m}]$, i.e. $\lambda_\pi(n)\lambda_\pi(n') < 0$ where $n, n' \in \mathcal{I}_x$, for all sufficiently large x and for some absolute constant $c > 0$. Thus we obtain readily the following.

Theorem 4. *Let $m \geqslant 2$ and let π be a self-contragredient irreducible unitary cuspidal representation for $GL_m(\mathbb{A}_\mathbb{Q})$. Then we have*

$$\mathrm{N}_\pi^*(x) \gg_\pi x^{1/m}$$

for $x \geqslant x_0(\pi)$. In particular we have

$$(2.3) \qquad\qquad \mathscr{N}_\pi^\pm(x) \gg_\pi x^{1/m}$$

for $x \geqslant x_0(\pi)$.

Remark 4. (i) Neither the Rankin-Selberg L-function nor the bound towards Ramanujan conjecture were used, showing the robustness of the method in [7].

(ii) The inequality (2.3) improves Liu-Wu's (2.2) in two directions: the former is unconditional for all $m \geqslant 5$, and the exponent $1/m$ is better than $1 - 2\theta_m$ for $m \geqslant 3$.

2.2. Modular forms of half-integral weight and Siegel-Hecke eigenforms.
Let N be a positive integer and χ a Dirichlet character mod $4N$. For any odd integer $k > 1$, we define $\mathcal{S}_{k/2}^*(4N, \chi)$ to be the space of holomorphic cusp forms of half-integral weight $k/2$ and nebentypus χ for the congruence subgroup $\Gamma_0(4N)$ which are orthogonal to Θ where Θ is the span of unary theta series. Note that the space Θ is nonzero only when $k = 3$.

Let $f \in \mathcal{S}_{k/2}^*(4N, \chi)$ and write $f(z) = \sum_{n \geqslant 1} \lambda(n) n^{(k-2)/4} e(nz)$. Associated to f is an L-function defined as $L(s, f) = \sum_{n \geqslant 1} \lambda(n) n^{-s}$, which is entire and satisfies the functional equation

$$\Phi(s)L(s, f) = \omega\Phi(1-s)L(1-s, g)$$

where $\Phi(s) = \pi^{-s} N^{s/2} \Gamma(s + (k/2 - 1)/2)$ and $g \in \mathcal{S}^*_{k/2}(4N, \overline{\chi}\left(\frac{4N}{\cdot}\right))$. The character $\left(\frac{4N}{\cdot}\right)$ is defined as in [23, p.442]. We are interested in, for nonzero $f \in \mathcal{S}^*_{k/2}(4N, \chi)$, the sequence $\{\lambda_r(n)\}_{n \geqslant 1}$ defined by

$$(2.4) \qquad L(s, f)^r = \sum_{n \geqslant 1} \lambda_r(n) n^{-s}, \qquad \text{where } r \in \mathbb{N} \text{ is arbitrary.}$$

Next we turn to Siegel eigenforms. Let $1 \leqslant m \leqslant k$ be integers and $Sp(m, \mathbb{Z})$ be the symplectic group. Define $\mathcal{M}_k(Sp(m, \mathbb{Z}))$ (resp. $\mathcal{S}_k(Sp(m, \mathbb{Z}))$) to be the space of holomorphic modular (resp. cusp) forms of weight k for $Sp(m, \mathbb{Z})$. Hecke eigenforms mean the nonzero common eigenfunctions of the Hecke algebra. Attached to each Hecke eigenform $F \in \mathcal{M}_k(Sp(m, \mathbb{Z}))$, one defines, cf. [1], the *standard L-function*

$$L_1(s, F) := \zeta(s) \prod_p \prod_{1 \leqslant j \leqslant m} (1 - \alpha_j(p)p^{-s})^{-1}(1 - \alpha_j(p)^{-1}p^{-s})^{-1}$$

where $\{\alpha_j(p)\}_{1 \leqslant j \leqslant m}$ are the Satake parameters of F, and the *spinor L-function*

$$L_2(s, F) := \prod_p \prod_{0 \leqslant k \leqslant m} \prod_{1 \leqslant j_1 < \cdots < j_k \leqslant m} (1 - \alpha_0(p)\alpha_{j_1}(p) \cdots \alpha_{j_k}(p)p^{-s})^{-1}$$

where $\alpha_0(p)^2 \alpha_1(p) \cdots \alpha_m(p) = 1$. Here $\Re e \, s$ is taken to be sufficiently large.

We know from [18] the following. When $m \equiv 0 \bmod 4$, the space $\mathcal{M}_k(Sp(m, \mathbb{Z}))$ may contain theta functions that are associated to symmetric positive definite even integral unimodular $m \times m$ matrices and polynomials satisfying some conditions. These theta functions generate a subspace $\mathcal{B}_k(Sp(m, \mathbb{Z}))$ in $\mathcal{M}_k(Sp(m, \mathbb{Z}))$ which is invariant under the action of Hecke algebra. Set $\mathcal{B}_k(Sp(m, \mathbb{Z})) = \{0\}$ if no such theta function exists. Define $\mathcal{H}^*_k(Sp(m, \mathbb{Z}))$ to be the set of Hecke eigenforms F in $\mathcal{S}_k(Sp(m, \mathbb{Z})) \setminus \mathcal{B}_k(Sp(m, \mathbb{Z}))$. The Euler product of $L_1(s, F)$ converges absolutely and uniformly for $\Re e \, s > m + 1$. Besides, the complete L-function

$$\Lambda_1(s, F) := \Gamma_{\mathbb{R}}(s + \delta_{2 \nmid m}) \prod_{1 \leqslant j \leqslant m} \Gamma_{\mathbb{C}}(s + k - j) L_1(s, F)$$

is holomorphic and satisfies the functional equation

$$\Lambda_1(s, F) = \Lambda_1(1 - s, F).$$

Here $\delta_* = 1$ if the condition $*$ holds, and 0 otherwise, $\Gamma_{\mathbb{R}}(s) = \pi^{-s/2}\Gamma(s/2)$ and $\Gamma_{\mathbb{C}}(s) = 2(2\pi)^{-s}\Gamma(s)$.

For $F \in \mathcal{H}^*_k(Sp(m, \mathbb{Z}))$, we define the sequence $\{\lambda_{1,F}(n)\}_{n \geqslant 1}$ by

$$(2.5) \qquad L_1(s, F) = \sum_{n \geqslant 1} \lambda_{1,F}(n) n^{-s} \quad \text{and} \quad \text{set } r := m + \tfrac{1}{2}.$$

The case of spinor L-functions is less understood, but the recipe in automorphic representation theory would suggest hypothetically a working ground, cf [22]. Let

$$\Lambda_2(s, F) = \Gamma_{\mathbb{C}}(s)^{N/2} \prod_{\substack{\varepsilon \in \{\pm 1\}^m \\ \alpha(\varepsilon) > 0}} \Gamma_{\mathbb{C}}\left(s + \tfrac{1}{2}|k\alpha(\varepsilon) - \beta(\varepsilon)|\right) \prod_{\substack{\varepsilon \in \{\pm 1\}^m \\ \alpha(\varepsilon) = 0, \beta(\varepsilon) > 0}} \Gamma_{\mathbb{C}}\left(s + \tfrac{1}{2}\beta(\varepsilon)\right)$$

where $N := \#\{\varepsilon = (\varepsilon_1, \ldots, \varepsilon_m) \in \{\pm 1\}^m : \alpha(\varepsilon) = \beta(\varepsilon) = 0\}$ with

$$\alpha(\varepsilon) := \varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_m \quad \text{and} \quad \beta(\varepsilon) := \varepsilon_1 + 2\varepsilon_2 + \cdots + m\varepsilon_m.$$

We *assume* the eigenform $F \in \mathcal{H}_k(Sp(n, \mathbb{Z}))$ satisfies the following hypothesis.

(EF) $\Lambda_2(s, F)$ extends to an entire function and satisfies the functional equation

$$\Lambda_2(s, F) = \omega\Lambda_2(1 - s, F) \qquad (|\omega| = 1).$$

Now let us introduce the sequence $\{\lambda_{2,F}(n)\}_{n \geqslant 1}$ for which

$$(2.6) \qquad\qquad L_2(s, F) = \sum_{n \geqslant 1} \lambda_{2,F}(n)n^{-s} \quad \text{and} \quad \text{set } r := 2^{m-1}.$$

Similarly to Theorem 4, we have the following result.

Theorem 5. *Let r and $\{a_r(n)\}_{n \geqslant 1}$ be the value and the sequence defined as in (2.4), (2.5) and (2.6), where the case (2.6) is particularly conditional on Hypothesis **(EF)**. Suppose $a_r(n) \in \mathbb{R}$ for all n, and x is any sufficiently large number. Then $\{a_r(n)\}$ has a sign-change as n runs over the short interval $[x - cx^{1-1/(2r)}, x + cx^{1-1/(2r)}]$. Moreover, the number of sign-changes in $\{a_r(n)\}_{1 \leqslant n \leqslant x}$ is $\gg x^{1/(2r)}$, so is the number of the terms in $\{a_r(n)\}_{1 \leqslant n \leqslant x}$ of the same sign.*

Remark 5. This extends the study in [20] on the spinor L-functions of Siegel eigenforms of genus 2, i.e. $m = 2$.

3. PREPARATION

In this section, we establish three preliminary lemmas for Theorems 1 and 2.

Lemma 3.1. *Let $J \geqslant 1$ be an integer. We have*

$$(3.1) \qquad\qquad \frac{\widetilde{\Delta}(s)}{\Delta(1 - s)} = \sum_{0 \leqslant j \leqslant J-1} e_j F_j(s) + F_0(s) \cdot O_J(|s|^{-J})$$

as $|s| \to \infty$, where $e_j \in \mathbb{C}$ are constants with e_0 given by (1.8) and

$$F_j(s) := h^{-s}\Gamma(2A(s + a) - j) \cos(\pi A(s + a) + k\pi).$$

The implied O-constant depends only on the parameters in (A2) and J.

Proof. This follows from the Stirling formula for $\log \Gamma(s)$: for any constant $c \in \mathbb{C}$ and any $J \in \mathbb{N}$,

$$\log \Gamma(s + c) = (s + c - \tfrac{1}{2}) \log s - s + \tfrac{1}{2} \log 2\pi + \sum_{1 \leqslant j \leqslant J-1} c_j s^{-j} + O(|s|^{-J})$$

as $|s| \to \infty$, uniformly for where $|\arg s| \leqslant \pi - \varepsilon < \pi$, where the constants c_j depend on c. Here the empty sum means 0 and the empty product means 1.

As in the proof of [4, Lemma 1], we obtain, for $\alpha > 0$, β and $\widetilde{\beta} \in \mathbb{C}$,

$$\log \frac{\Gamma(\alpha s + \widetilde{\beta})}{\Gamma(\alpha(1 - s) + \beta)} = (\alpha s + \widetilde{\beta} - \tfrac{1}{2}) \log s - (\alpha(1 - s) + \beta - \tfrac{1}{2}) \log(-s)$$

$$+ 2(\alpha \log \alpha - \alpha)s + (\widetilde{\beta} - \beta - \alpha) \log \alpha + \sum_{1 \leqslant j \leqslant J-1} c'_j s^{-j} + O(|s|^{-J})$$

for some constants c'_j depending on α, β and $\widetilde{\beta}$.

Consequently,

$$
\begin{aligned}
\log \frac{\widetilde{\Delta}(s)}{\Delta(1-s)} &= \left(As + \widetilde{B} - \tfrac{1}{2}d\right)\log s - \left(A(1-s) + B - \tfrac{1}{2}d\right)\log(-s) \\
&\quad + 2\left(\sum_{1\leqslant\nu\leqslant d} \alpha_\nu \log \alpha_\nu - A\right)s + f + \sum_{1\leqslant j\leqslant J-1} d_j^{(1)} s^{-j} + O\!\left(|s|^{-J}\right)
\end{aligned}
$$

(3.2)

where and throughout this proof, $d_j^{(1)}$, $d_j^{(2)}$, ... denote some constants, and

$$
f := \sum_{1\leqslant\nu\leqslant d} (\widetilde{\beta}_\nu - \beta_\nu - \alpha_\nu)\log \alpha_\nu.
$$

On the other hand, we have

$$
\Gamma(2A(s+a)) \cos(\pi A(s+a) + k\pi) = \frac{\pi\Gamma(2A(s+a))}{\Gamma(\tfrac{1}{2} - A(s+a) - k)\Gamma(\tfrac{1}{2} + A(s+a) + k)}
$$

and thus

$$
\begin{aligned}
\log &\left(\Gamma(2A(s+a)) \cos(\pi A(s+a) + k\pi)\right) \\
&= \left(A(s+a) - k - \tfrac{1}{2}\right)\log s + \left(A(s+a) + k\right)\log(-s) \\
&\quad + 2\left(A\log(2A) - A\right)s + f' + \sum_{1\leqslant j\leqslant J-1} d_j^{(2)} s^{-j} + O\!\left(|s|^{-J}\right),
\end{aligned}
$$

(3.3)

where

$$
f' := \left(2Aa - \tfrac{1}{2}\right)\log(2A) + \tfrac{1}{2}\log \tfrac{\pi}{2} = (\widetilde{B} - B - A)\log(2A) + \tfrac{1}{2}\log \tfrac{\pi}{2}.
$$

In view of the values of a, h and k, the difference between (3.2) and (3.3) equals

$$
f - f' - s\log h + \sum_{1\leqslant j\leqslant J-1} d_j^{(3)} s^{-j} + O\!\left(|s|^{-J}\right).
$$

Clearly for some constants $d_j^{(4)}$, we can write

$$
\begin{aligned}
\exp &\left(f - f' - s\log h + \sum_{1\leqslant j\leqslant J-1} d_j^{(3)} s^{-j} + O\!\left(|s|^{-J}\right)\right) \\
&= e_0 h^{-s}\left(1 + \sum_{1\leqslant j\leqslant J-1} d_j^{(4)} s^{-j} + O\!\left(|s|^{-J}\right)\right)
\end{aligned}
$$

with $e_0 = e^{f-f'}$ (as given by (1.8)) and the empty sum means 0.
    Besides, for each $1 \leqslant j \leqslant J-1$, we have

$$
\frac{1}{s^j} - \frac{(2A)^j}{\prod_{\ell=1}^{j}(2A(s+a) - \ell)} = \sum_{j+1\leqslant j'\leqslant J-1} d_{j'}^{(5)} s^{-j} + O\!\left(|s|^{-J}\right).
$$

Consequently, a successive application of this formula gives

$$\sum_{1\leqslant j\leqslant J-1} d_j^{(4)} s^{-j} = \frac{d_1^{(5)}}{(2A(s+a)-1)} + \sum_{2\leqslant j\leqslant J-1} d_j^{(5)} s^{-j} + O(|s|^{-J})$$

$$= \sum_{1\leqslant j\leqslant 2} \frac{d_j^{(6)}}{\prod_{\ell=1}^{j}(2A(s+a)-\ell)} + \sum_{3\leqslant j\leqslant J-1} d_j^{(6)} s^{-j} + O(|s|^{-J})$$

$$\cdots$$

$$= \sum_{1\leqslant j\leqslant J-1} \frac{d_j^{(J+3)}}{\prod_{\ell=1}^{j}(2A(s+a)-\ell)} + O(|s|^{-J}).$$

For the difference between (3.2) and (3.3), the left-hand side is

$$\log \frac{\widetilde{\Delta}(s)}{\Delta(1-s)} - \log \left(\Gamma(2A(s+a))\cos(\pi A(s+a)+k\pi) \right),$$

and the right-hand side is

$$\log \left(h^{-s} \sum_{0\leqslant j\leqslant J-1} \frac{e_j}{\prod_{\ell=1}^{j}(2A(s+a)-\ell)} + h^{-s} O(|s|^{-J}) \right).$$

Thus,

$$\frac{\widetilde{\Delta}(s)}{\Delta(1-s)} = \frac{\cos(\pi A(s+a)+k\pi)}{h^s} \sum_{0\leqslant j\leqslant J-1} \frac{e_j \Gamma(2A(s+a))}{\prod_{\ell=1}^{j}(2A(s+a)-\ell)} + F_0(s) O(|s|^{-J}).$$

Using the formula $\Gamma(s+1) = s\Gamma(s)$, it is easy to see that

$$\frac{\Gamma(2A(s+a))}{\prod_{\ell=1}^{j}(2A(s+a)-\ell)} = \frac{\Gamma(2A(s+a)-1)}{\prod_{\ell=2}^{j}(2A(s+a)-\ell)} = \cdots = \Gamma(2A(s+a)-j).$$

Inserting into the preceding formula, we obtain the required (3.1). $\qquad\square$

Lemma 3.2. *Let $\varepsilon > 0$ be arbitrarily small. There exists $J' \in \mathbb{N}$ such that for all $J \geqslant J'$,*

$$I(y) = \frac{y}{2A} \sum_{0\leqslant j\leqslant J-1} e_j \int_0^\infty (hyu)^{a-j/(2A)} \varphi(u) \cos \left((hyu)^{1/(2A)} + (k+\tfrac{1}{2}j)\pi \right) u$$

$$+ O_{\varepsilon,J} \left(L^{-1} y^{1-\vartheta-(J-\frac{1}{2})/(2A)+\varepsilon} \right),$$

where the implied O-constant depends only on ε, J, φ_0 and the parameters in (A1) *and* (A2).

Proof. Let $0 < \varepsilon < 1/(4A)$ and write $\theta_J(\varepsilon) := \vartheta - \varepsilon + (J - \tfrac{1}{2})/(2A)$. Choose $J' \in \mathbb{N}$ such that $\theta_{J'}(\varepsilon) > 1 - \sigma_2$ and consider $J \geqslant J'$. As $\widehat{\varphi}(s)$ decays rapidly (thanks to (1.3)), we can move the line of integration of $I(y)$ in (1.6) from $\Re s = 1 - \sigma_2$ to the right line $\Re s = \theta_J(\varepsilon)$. Then we insert (3.1) into (1.6) and integrate term by term.

Let $s = \sigma + it$. We have $F_0(s) \ll |t|^{2A(\sigma-\vartheta)-1/2}$, as $|t| \to \infty$, in any vertical strip $b < \sigma < b'$. By (1.3) with $r = 1$, the rightmost term $F_0(s) O_J(|s|^{-J})$ of (3.1) contributes at most $O_\varepsilon(y^{1-\theta_J(\varepsilon)})$ to the integral $I(y)$.

Next we evaluate the integral of every summand in the sum $\sum_{j=0}^{J-1}$, which can be expressed as $e_j y I_j$ where

$$I_j := \frac{1}{2\pi i} \int_{(\theta_J(\varepsilon))} \Gamma(2A(s+a) - j) \cos(\pi A(s+a) + k\pi) \widehat{\varphi}(1-s)(hy)^{-s} \mathrm{s}.$$

Note that $\Re e\,(2A(s+a) - j) > 0$ on $\Re e\,s = \theta_J(\varepsilon)$ for $1 \leqslant j \leqslant J-1$. After a change of variable $z = 2A(s+a) - j$ and moving the line of integration to $\Re e\,z = \frac{1}{4}$, we derive that

$$I_j = \frac{1}{2A} \cdot \frac{1}{2\pi i} \int_{(\frac{1}{4})} \Gamma(z) \cos\left(\tfrac{1}{2}\pi z + (k + \tfrac{1}{2}j)\pi\right) \widehat{\varphi}\left(1 + a - \frac{z+j}{2A}\right) (hy)^{a-(z+j)/(2A)} \mathrm{z}.$$

In virtue of the formula: for $0 < c < \frac{1}{2}$ and $\alpha \in \mathbb{C}$,

$$\frac{1}{2\pi i} \int_{(c)} \Gamma(s) \cos\left(\tfrac{1}{2}\pi s + \alpha\right) y^{-s} \mathrm{s} = \cos(y + \alpha),$$

we replace $\widehat{\varphi}$ with its inverse Mellin transform and interchange the integrals, and consequently obtain

$$I_j = \frac{1}{2A} \int_0^\infty (hyu)^{a-j/(2A)} \varphi(u) \cos\left((hyu)^{1/(2A)} + (k + \tfrac{1}{2}j)\pi\right) \mathrm{u}.$$

This completes the proof. $\qquad\square$

Lemma 3.3. *Let* $\tau, \rho, \theta \in \mathbb{R}$, *and*

$$K_{\tau,\rho}(t) = (1 - |t|)(1 + \tau \cos(2\rho t + \theta)), \qquad \forall\, t \in [-1, 1].$$

Then for any real υ,

$$\int_{-1}^1 K_{\tau,\rho}(t) e^{\mathrm{i}2\upsilon t} t = \left(\frac{\sin \upsilon}{\upsilon}\right)^2 + \frac{\tau e^{\mathrm{i}\theta}}{2}\left(\frac{\sin(\upsilon + \rho)}{\upsilon + \rho}\right)^2 + \frac{\tau e^{-\mathrm{i}\theta}}{2}\left(\frac{\sin(\upsilon - \rho)}{\upsilon - \rho}\right)^2.$$

This follows from

$$\int_{-1}^1 (1 - |t|) e^{\mathrm{i}2\upsilon t} t = \left(\frac{\sin \upsilon}{\upsilon}\right)^2.$$

4. Proof of Theorem 1

Here and in the sequel, for all the implied constants in the O- or $\ll$-symbols, we shall not indicate their dependence on the parameters in (A1) and (A2) for simplicity. For $x \in \mathbb{R}$, we denote by the symbol $\lceil x \rceil_+$ the *smallest positive* integer greater than x.

Let J' be defined as in Lemma 3.2 and set

$$J_0 = J' + \lceil 2A(\sigma^* - \vartheta) + \tfrac{1}{2}\rceil_+ \geqslant 2.$$

Write

$$\vartheta(j) := \vartheta + \frac{j}{2A} \qquad (\text{thus } \frac{j}{2A} - a = \vartheta(j) - \mathrm{i}\xi)$$

and

$$S_{\varphi,j}(x) := \sum_{n \geqslant 1} \frac{b_n}{\mu_n^{\vartheta(j) - \mathrm{i}\xi}} \int_0^\infty \frac{\varphi(u)}{u^{\vartheta(j)}} u^{\mathrm{i}\xi} \cos\left((h\mu_n x u)^{1/(2A)} + (k + \tfrac{1}{2}j)\pi\right) \mathrm{u}.$$

We apply Lemma 3.2 to (1.4) with J_0 in place of J, so together with (A1),

$$(4.1) \qquad S_\varphi(x) = \frac{\omega}{2Ah} \sum_{0 \leqslant j \leqslant J_0} e_j(hx)^{1-\vartheta(j)+\mathrm{i}\xi} S_{\varphi,j}(x) + O_\varepsilon\left(L^{-1} x^{1-\vartheta-1/(2A)}\right)$$

for any $x \in [X/2, X]$.

It remains to estimate $S_{\varphi,j}(x)$ in (4.1) for $1 \leqslant j \leqslant J_0$. Replacing $u^{1/(2A)}$ by w, the integral in $S_{\varphi,j}(x)$ equals the real part of a scalar multiple of

$$\int_0^\infty \frac{\varphi(w^{2A})}{w^{2A\vartheta(j)}} w^{2A-1+\mathrm{i}2A\xi} e^{\mathrm{i}(h\mu_n x)^{1/(2A)}w} \mathrm{w} = \int_0^\infty F(w)e^{\mathrm{i}Yw} \mathrm{w} \quad \text{(say)}.$$

Clearly, for all $r \geqslant 0$ the support of $F^{(r)}(w)$ is contained in the interval $|w-1| \ll L^{-1}$ due to the support of φ, and $F^{(r)}(w) \ll_r \sum_{j=0}^r \|\varphi^{(j)}\|_\infty \ll_r L^r$, thanks to (1.1). Thus a successive integration by parts shows that for any $r \geqslant 0$, the right-side is $\ll Y^{-r} \int_0^\infty |F^{(r)}(w)|\mathrm{w} \ll_r Y^{-r} L^{r-1} \ll L^{-1}(L/Y)^r$, that is,

$$(4.2) \qquad \int_0^\infty \frac{\varphi(u)}{u^{\vartheta(j)}} u^{\mathrm{i}\xi} \cos\left((h\mu_n xu)^{1/(2A)} + (k+\tfrac{1}{2}j)\pi\right) \mathrm{u} \ll L^{-1}\left(\frac{L}{(\mu_n x)^{1/(2A)}}\right)^r$$

$$\ll_r L^{-1}\delta^{-r}\mu_n^{-r/(2A)}.$$

Choose $r(j) = \lceil 2A(\sigma^* - \vartheta(j)) \rceil_+$. It follows immediately that for $j \leqslant J_0$,

$$S_{\varphi,j}(x) \ll L^{-1}\delta^{-r(j)} \sum_{n \geqslant 1} \frac{|b_n|}{\mu_n^{\vartheta(j)+r(j)/(2A)}} \ll_\delta L^{-1}.$$

Inserting into (4.1), we get immediately the desired results.

5. Proof of Theorem 2

Let $\tau = \pm 1$ be selected up to our disposal. In view of (1.7), we are led to consider

$$(5.1) \qquad \tau \frac{|b_1|}{b_1 \mu_1^{\mathrm{i}\xi}} S_{\varphi,0}(x) + O\left(L^{-1} X^{-1/(2A)}\right)$$

and find x such that its real part is bounded below by cL^{-1} for some constant $c > 0$. We divide $S_{\varphi,0}$ into two subsums over $1 \leqslant n \leqslant N$ and $n > N$ respectively,

$$(5.2) \qquad S_{\varphi,0} = S_{\varphi,0}^{\leqslant N} + S_{\varphi,0}^{>N}.$$

For $S_{\varphi,0}^{>N}$, we repeat the above argument with (4.2) and a choice of $r = \lceil 2A(\sigma^* - \vartheta) \rceil_+$ in order that

$$(5.3) \qquad S_{\varphi,0}^{>N}(x) \ll L^{-1}\delta^{-r} \sum_{n > N} \frac{|b_n|}{\mu_n^{\vartheta+r/(2A)}} \leqslant \delta L^{-1}$$

for some suitably large $N = N(\delta)$.

Now we apply the method in [7]. Recall $k = \kappa + \mathrm{i}\eta$ and put

$$K_{\tau,\rho}(t) = (1 - |t|)\{1 + \tau \cos\left(2\rho t + \kappa\pi\right)\}, \qquad \forall\, t \in [-1, 1],$$

where $\rho := (h\mu_1)^{1/(2A)}\alpha$ and the parameter α will be specified later. Then $K_{\tau,\rho}(t) \geqslant 0$ for all $t \in [-1,1]$ and

$$\int_{-1}^{1} K_{\tau,\rho}(t)\,\mathrm{d}t \leqslant 2.$$

Consider any $T \in [(2X)^{1/(2A)}, (3X)^{1/(2A)}]$. We have

$$(5.4) \qquad \int_{-1}^{1} S_{\varphi,0}\big((T+2\alpha t)^{2A}\big)K_{\tau,\rho}(t)\,\mathrm{d}t = \int_{-1}^{1} S_{\varphi,0}^{\leqslant N}\big((T+2\alpha t)^{2A}\big)K_{\tau,\rho}(t)\,\mathrm{d}t + \mathrm{RT},$$

where the remainder term RT satisfies $|\mathrm{RT}| \leqslant 2\delta L^{-1}$. Besides,

$$(5.5) \qquad \int_{-1}^{1} S_{\varphi,0}^{\leqslant N}\big((T+2\alpha t)^{2A}\big)K_{\tau,\rho}(t)\,\mathrm{d}t = \sum_{1\leqslant n\leqslant N} \frac{b_n}{\mu_n^{\vartheta-\mathrm{i}\xi}}I_n$$

where

$$I_n := \int_0^\infty \frac{\varphi(u)}{u^\vartheta}u^{\mathrm{i}\xi}\int_{-1}^{1} K_{\tau,\rho}(t)\cos\big((h\mu_n u)^{1/(2A)}(T+2\alpha t)+k\pi\big)\,\mathrm{d}t\,\mathrm{d}u.$$

The inner integral inside I_n equals

$$\frac{1}{2}\left(\mathrm{e}^{\mathrm{i}((h\mu_n u)^{1/(2A)}T+k\pi)}\mathscr{I}_n^+(u) + \mathrm{e}^{-\mathrm{i}((h\mu_n u)^{1/(2A)}T+k\pi)}\mathscr{I}_n^-(u)\right)$$

with

$$\mathscr{I}_n^\pm(u) := \int_{-1}^{1} K_{\tau,\rho}(t)\mathrm{e}^{\pm\mathrm{i}2vt}\,\mathrm{d}t$$

where $v = (h\mu_n u)^{1/(2A)}\alpha$. Let $\varpi_n^\pm(u) := \alpha\big((h\mu_n u)^{1/(2A)} \pm (h\mu_1)^{1/(2A)}\big) = v \pm \rho$ and recall $\delta_* = 1$ if $*$ holds and 0 otherwise. By Lemma 3.3, we have

$$\mathscr{I}_n^\pm(u) = \left(\frac{\sin v}{v}\right)^2 + \frac{\tau\mathrm{e}^{\mathrm{i}\kappa\pi}}{2}\left(\frac{\sin\varpi_n^\pm(u)}{\varpi_n^\pm(u)}\right)^2 + \frac{\tau\mathrm{e}^{-\mathrm{i}\kappa\pi}}{2}\left(\frac{\sin\varpi_n^\mp(u)}{\varpi_n^\mp(u)}\right)^2$$

$$= \delta_{n=1}\frac{\tau}{2}\left(\frac{\sin\varpi_1^-(u)}{\varpi_1^-(u)}\right)^2\mathrm{e}^{\mp\mathrm{i}\kappa\pi} + O\left(\frac{1}{\alpha^2\mu_n^{1/A}}\right)$$

where we have tacitly used $\big|(\mu_1 u)^{1/(2A)}-\mu_n^{1/(2A)}\big| \gg \mu_n^{1/(2A)}$ for $n \geqslant 2$ and $|u-1| \ll L^{-1}$ whenever X is sufficiently large, leading to $|\varpi_n^\pm(u)| \gg |\alpha|\mu_n^{1/(2A)}$ for all $n \geqslant 2$ and $|\varpi_1^+(u)| \gg |\alpha|\mu_1^{1/(2A)}$. Thus the inner integral inside I_n equals

$$\delta_{n=1}\frac{\tau}{2}\left(\frac{\sin\varpi_1^-(u)}{\varpi_1^-(u)}\right)^2\cos\big((h\mu_n u)^{1/(2A)}T + \mathrm{i}\eta\pi\big) + O\left(\frac{1}{\alpha^2\mu_n^{1/A}}\right).$$

With (1.2), it follows that

$$I_n = \delta_{n=1}\frac{\tau}{2}I_1^* + O\left(\frac{1}{\alpha^2 L\mu_n^{1/A}}\right) \qquad (n \geqslant 1),$$

where

$$I_1^* := \int_0^\infty \frac{\varphi(u)}{u^\vartheta}\left(\frac{\sin\varpi_1^-(u)}{\varpi_1^-(u)}\right)^2 u^{\mathrm{i}\xi}\cos\big((h\mu_1 u)^{1/(2A)}T + \mathrm{i}\eta\pi\big)\,\mathrm{d}u.$$

Thus for (5.5), we obtain with $\tau^2 = 1$ that

$$(5.6) \qquad \tau \frac{|b_1|}{b_1 \mu_1^{\mathrm{i}\xi}} \int_{-1}^1 S_{\varphi,0}^{\leqslant N}\big((T + 2\alpha t)^{2A}\big) K_{\tau,\rho}(t)\mathrm{t} = \frac{|b_1|}{2\mu_1^\vartheta} I_1^* + O\bigg(\frac{1}{\alpha^2 L} \sum_{n=1}^N \frac{|b_n|}{\mu_n^{\vartheta + 1/A}}\bigg).$$

Again using the fact that φ is supported on $[1 - 1/L, 1 + 1/L]$ and $L = \delta^{-1} X^{1/(2A)}$, we easily infer $\varpi_1^-(u) \ll \alpha L^{-1}$ and $u^{1/(2A)}T = T + O(\delta)$, and hence

$$(5.7) \qquad I_1^* = \big\{\cos\big((h\mu_1)^{1/(2A)}T + \mathrm{i}\eta\pi\big) + O\big(\delta + \alpha L^{-1}\big)\big\} \int_0^\infty \varphi(u)\mathrm{u}.$$

Recall $N = N(\delta)$ is determined in (5.3). Next we choose a sufficiently large $\alpha = \alpha(\delta)$ so that the O-term in (5.6) is $\ll \delta L^{-1}$, and then $\alpha L^{-1} \leqslant \delta$ for $X \geqslant \alpha(\delta)^{2A}$. Let $X_0(\delta) \geqslant \alpha(\delta)^{2A}$ be specified later. Following from (5.2)–(5.7) (and (1.2)), we obtain

$$(5.8) \qquad \begin{aligned} &\tau \frac{|b_1|}{b_1 \mu_1^{\mathrm{i}\xi}} \int_{-1}^1 S_{\varphi,0}\big((T + 2\alpha t)^{2A}\big) K_{\tau,\rho}(t)\mathrm{t} \\ &= \frac{|b_1|}{2\mu_1^\vartheta} \cos\big((h\mu_1)^{1/(2A)}T + \mathrm{i}\eta\pi\big) \int_0^\infty \varphi(u)\mathrm{u} + O(\delta L^{-1}) \end{aligned}$$

provided $X \geqslant X_0(\delta)$.

We choose δ small enough so that the O-term in (5.8) is less than $\frac{1}{4}|b_1|\mu_1^{-\vartheta}L^{-1}$ in size. When $T = 2n\pi/(h\mu_1)^{1/(2A)}$ with $n \in \mathbb{N}$, the main term in (5.8) is equal to

$$\frac{|b_1|}{2\mu_1^\vartheta} \cosh(\eta\pi) \int_0^\infty \varphi(u)\mathrm{u} \geqslant \frac{|b_1|}{2\mu_1^\vartheta} L^{-1},$$

by the first inequality of (1.2). Thus for all $X \geqslant X_0(\delta)$ and any $T = 2n\pi/(h\mu_1)^{1/(2A)} \in [(2X)^{1/(2A)}, (3X)^{1/(2A)}]$ ($n \in \mathbb{N}$), there exists $T_\tau \in [T - 2\alpha, T + 2\alpha]$ such that

$$\Re\bigg(\tau \frac{2|b_1|}{b_1 \mu_1^{\mathrm{i}\xi}} S_{\varphi,0}(T_\tau^{2A})\bigg) \geqslant \int_{-1}^1 \Re\bigg(\tau \frac{|b_1|}{b_1 \mu_1^{\mathrm{i}\xi}} S_{\varphi,0}\big((T + 2\alpha t)^{2A}\big)\bigg) K_{\tau,\rho}(t)\mathrm{t}$$

$$\geqslant \frac{1}{4} \frac{|b_1|}{\mu_1^\vartheta} L^{-1}$$

To its end, we further enlarge $X_0(\delta)$ such that the O-term in (5.1) does not exceed $\frac{1}{6}|b_1|\mu_1^{-\vartheta}L^{-1}$ for $X \geqslant X_0(\delta)$, and $X_0(\delta) \geqslant (2\alpha(\delta))^{2A}$ (so that $T > 2\alpha$). A little manipulation gives the desired inequalities, completing the proof.

6. Proof of Theorem 3

Let us begin with an evaluation for $M_\varphi(x)$ defined in (1.5).

Lemma 6.1. *Let Υ be the set of all poles of $\phi(s)$, so $\Upsilon \subset \mathcal{D}$ is finite. With the notation in Theorem 2, there exists a positive integer M such that for any large $x \geqslant X_0(\delta)$ and any y with $|y - x| \leqslant c_0 x^{1 - 1/(2A)}$, we have*

$$y^{-\mathrm{i}\xi} M_\varphi(y) = \sum_{v \in \Upsilon} \sum_{0 \leqslant j < \mathrm{ord}_v} \sum_{1 \leqslant m \leqslant M} c_{j,m}(v) y^{v - \mathrm{i}\xi} (\log y)^j L^{-m} + O\big(y^{1 - \vartheta} L^{-2}\big),$$

where ord_v is the order of ϕ at the pole v, the coefficients $c_{j,m}(v) \in \mathbb{C}$ are independent of x and y, and $L := \delta^{-1} x^{1/(2A)}$.

Proof. Applying the residue theorem to the integral in (1.5), we obtain

$$M_\varphi(y) = \sum_{v \in \Upsilon} \sum_{\substack{i \geq 0,\, j \geq 0 \\ i+j < \mathrm{ord}_v}} d_{i,j}(v) y^v \widehat{\varphi}^{(i)}(v) (\log y)^j$$

for some coefficients $d_{i,j}(v) \in \mathbb{C}$. By Taylor's theorem, we have the expansion

(6.1)
$$\begin{aligned}
\widehat{\varphi}^{(i)}(v) &= \frac{1}{L} \int_{-1}^{1} \varphi_0(u)(1 + uL^{-1})^{v-1} \log^i(1 + uL^{-1}) \mathbb{u} \\
&= \sum_{1 \leq m \leq M} d'_{i,m}(v) L^{-m} + O_{v,i,M}(L^{-M-1}).
\end{aligned}$$

Since $L \asymp x^{1/(2A)} \asymp y^{1/(2A)}$, we take M sufficiently large in (6.1) and insert into the preceding formula. The lemma follows. $\square$

Next we prove the claim: Either condition of (a)-(c) in Theorem 3 implies that

(6.2)
$$\lambda_{n+1} - \lambda_n \geq (4\delta + 2c_0)\lambda_n^{1-1/(2A)}$$

holds only for finitely many n's, where δ and c_0 are the constants in Theorem 2.

Consider any sufficiently large $\lambda_n \geq X_0(\delta)$ and take $x = (\lambda_n + \lambda_{n+1})/2$, the midpoint of λ_n and λ_{n+1}. Since $\lambda_n < x$ and $0 < 2A < 1$, $\lambda_n^{1-1/(2A)} > x^{1-1/(2A)}$. If (6.2) holds, then

$$\lambda_n < x - (2\delta + c_0)x^{1-1/(2A)} < x + (2\delta + c_0)x^{1-1/(2A)} < \lambda_{n+1}.$$

By Theorem 2 with this choice of x and observing

$$\lambda_n < x_\pm - 2\delta x^{1-1/(2A)} < x_\pm + 2\delta x^{1-1/(2A)} < \lambda_{n+1},$$

we infer that $S_\varphi(x_\pm) = -M_\varphi(x_\pm)$ because $\varphi(\lambda_m/x_\pm) = 0$ for all m, and that

(6.3)
$$\mp \Re e \left(\varsigma^{-1} \frac{M_\varphi(x_\pm)}{(\mu_1 h x_\pm)^{\mathrm{i}\xi}} \right) > c_1 x^{1-\vartheta} L^{-1}$$

for some constant $c_1 > 0$.

Now let us invoke individually the conditions in Theorem 3.

(a) Assume $\phi(s)$ has no pole. Then $M_\varphi(x) \equiv 0$ and thus (6.3) cannot happen.

(b) Suppose $\xi = 0$ and $\Upsilon \subset \mathbb{R}$. Lemma 6.1 gives

$$\Re e \left(\frac{\varsigma^{-1} M_\varphi(y)}{(\mu_1 hy)^{\mathrm{i}\xi}} \right) = \sum_{v \in \Upsilon} \sum_{0 \leq j < \mathrm{ord}_v} \sum_{1 \leq m \leq M} c'_{j,m}(v)\, y^v (\log y)^j L^{-m} + O(y^{1-\vartheta} L^{-2}),$$

where $c'_{j,m}(v) \in \mathbb{R}$, $L = \delta^{-1} x^{1/(2A)}$ and $|y - x| \ll x^{1-1/(2A)}$. For such y and large x, the multiple sum on the right can be expressed as

$$Q(x) := \{1 + O(x^{-1/(2A)})\} \sum_{v \in \Upsilon} \sum_{0 \leq j < \mathrm{ord}_v} \sum_{1 \leq m \leq M} c'_{j,m}(v)\, x^v (\log x)^j L^{-m}.$$

As $y^{1-\vartheta} L^{-2} = o(x^{1-\vartheta} L^{-1})$, we infer by (6.3) that $\mp Q(x) > \frac{1}{2} c_1 x^{1-\vartheta} L^{-1}$ occurs simultaneously for the same x, which is impossible.

(c) Suppose $\Re e\, v < \vartheta_0 = 1 - \vartheta + 1/(2A)$ for all $v \in \Upsilon$. In this case, Lemma 6.1 implies

$$\Re e \left(\frac{\varsigma^{-1} M_\varphi(y)}{(\mu_1 hy)^{\mathrm{i}\xi}} \right) = \widetilde{Q}(y) + O(y^{1-\vartheta} L^{-2}),$$

where

$$\widetilde{Q}(y) = \sum_{\upsilon \in \Upsilon} \sum_{0 \leqslant j < \mathrm{ord}_\upsilon} \sum_{1 \leqslant m \leqslant M} c''_{j,m}(\upsilon)\, y^{\Re e\,\upsilon} (\log y)^j L^{-m} \cos\left((\Im m\,\upsilon - \xi)\log y + \theta_{j,m}(\upsilon)\right)$$

for some real constants $c''_{j,m}(\upsilon)$ and $\theta_{j,m}(\upsilon)$. As $|y - x| \ll x^{1-1/(2A)}$, it follows that

$$\widetilde{Q}(y) = \widetilde{Q}(x) + O\left(x^{\upsilon_* - 1/(2A)}(\log x)^M L^{-1}\right)$$

where $\upsilon_* := \max\{\Re e\,\upsilon : \upsilon \in \Upsilon\} < \vartheta_0$. As $\upsilon_* - 1/(2A) < 1 - \vartheta$, from (6.3) we deduce concurrently the two inequalities $\mp\widetilde{Q}(x) > \frac{1}{2}c_1 x^{1-\vartheta} L^{-1}$ once x is large enough, a contradiction.

In summary, we have shown that (6.2) cannot hold for any sufficiently large λ_n; in other words, for some n_0 and for all $n \geqslant n_0$,

$$\lambda_{n+1} - \lambda_n \ll \lambda_n^{1-1/(2A)}$$

with the implied constant independent of n.

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