<table>
<thead>
<tr>
<th><strong>Title</strong></th>
<th>Incremental Displacement Collocation Method for the Evaluation of Tension Softening Curves of Quasi-brittle Materials</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Author(s)</strong></td>
<td>Su, KL; LIU, Q</td>
</tr>
<tr>
<td><strong>Citation</strong></td>
<td>Procedia Engineering, 2017, v. 172, p. 1059-1066</td>
</tr>
<tr>
<td><strong>Issued Date</strong></td>
<td>2017</td>
</tr>
<tr>
<td><strong>URL</strong></td>
<td><a href="http://hdl.handle.net/10722/243039">http://hdl.handle.net/10722/243039</a></td>
</tr>
<tr>
<td><strong>Rights</strong></td>
<td>This work is licensed under a Creative Commons Attribution-NonCommercial-NoDerivatives 4.0 International License.</td>
</tr>
</tbody>
</table>
Incremental Displacement Collocation Method for the Evaluation of Tension Softening Curves of Quasi-brittle Materials

R.K.L. Su\textsuperscript{a,\textdagger}, Q.F. Liu\textsuperscript{a}

\textsuperscript{a} Department of Civil Engineering, The University of Hong Kong, Pokfulam Road, Hong Kong, 999077, China

Abstract

In this paper, the use of incremental displacement collocation method (IDCM) for the evaluation of the tension softening curve of quasi-brittle materials is presented. By varying the residual tensile stress on the tension softening curve (TSC) for each loading step, the measured displacements are matched with the computed displacements at selected locations. By gradually increasing the loading step, the complete TSC can be determined in a step-by-step manner. As distinct global and local deformation responses are employed in the IDCM, the ill-posed problem commonly encountered in other inverse analyses is avoided. The effectiveness of the IDCM is demonstrated by obtaining TSCs of quasi-brittle materials, such as mortar, concrete and graphite, in the forms of bilinear, tri-linear and exponential curves.

Keywords: Tension softening curve; quasi-brittle; finite element; displacement field; inverse analysis.

1. Introduction

Concrete, mortar, graphite are all belongs to quasi-brittle materials. They are good at resisting compression but weak in tension. Linear elastic fracture mechanics (LEFM) is not applicable to quasi-brittle materials as stress singularity does not really exist at the crack tip. Nonlinear fracture mechanics (NLFM) should be used to analyse quasi-brittle materials.

In recent decades, many efforts have been made to the nonlinear fracture analysis of quasi-brittle materials [1-11]. Among these efforts, cohesive crack model (CCM) is extensively used to simulate the fracture behaviour of nonlinear fracture zone, which is so-called fracture process zone (FPZ). According to the CCM, all the nonlinear behaviours in the FPZ are represented by a cohesive crack, and the crack propagation is determined by the tension

\textsuperscript{\dagger} Corresponding author. Tel.: +852 2859 2648; fax: +852 2559 5337.

E-mail address: klsu@hkucc.hku.hk.
softening curve (TSC).

Some researchers [1-3] attempted to obtain the TSC of concrete from uniaxial tension tests. However, as the crack path may not be known a priori or the crack propagation may not be stable and symmetrical, it is difficult to accurately estimate the TSC from the test, and only the average cohesive stress and crack opening can be obtained [4]. Alternatively, inverse analyses were proposed to evaluate the TSC by minimizing the difference between the numerical results and experimental response using optimization algorithms [5-7]. The shapes of TSC are assumed to be linear [8], bilinear, tri-linear [9] or exponential [10,11]. Kitsutaka [12] and Kurihara et al. [13] proposed a polylinear softening model without a prior assumption on the shape of the TSC. It is noted that all these inverse analysis approaches only consider the global response such as load-deflection and load-CMOD curve but not the local response near the crack. Shen and Paulino [14] proposed a hybrid inverse technique and employed the local responses at one loading state in the post-peak stage of the beam. Škoček and Stang [15] used optical method to measure the displacements and performed wedge splitting tests to inversely estimated the fracture parameters, assuming the softening curve is piecewise and linear. Although optimization algorithms were utilized in the numerical analysis, ill-posed problems were commonly found in the aforementioned inverse analysis.

In this paper, a recently developed incremental displacement collocation method (IDCM) [16-18] is presented for the estimation of the TSCs of quasi-brittle materials in a step-by-step manner. At each numerical step, an assumed trial cohesive stress is used along with the measured global and local response, including the length of FPZ, to predict the TSC. The trial cohesive stress is determined by matching the displacements measured from experiment using electronic speckle pattern interferometry (ESPI) technique and that from the finite element (FE) analysis. Using the IDCM, the ill-posed problem is suppressed and the TSCs of mortar [16], concrete [17] and graphite [18] are determined.

2. Theoretical background of IDCM

2.1. Basic assumptions

To determine the TSC with IDCM [16], which is facilitated by incorporating CCM into the finite element method (FEM), the following assumptions are adopted:

(1) A two-dimensional crack model is employed.
(2) The nonlinear fracture behaviour is represented by a cohesive crack ahead of the notch tip
(3) The remaining parts of the specimen are assumed to be isotropic and linear elastic.
(4) The cohesive crack starts to propagate when the maximum tensile stress equals to the tensile strength.
(5) The trial cohesive stress follows a decreasing trend as the crack opening increases.
(6) The cohesive stresses in the FPZ are determined by the extended TSC and the measured crack opening displacement (COD).

2.2. The CCM

As shown in Fig. 1, the nonlinear behaviour of all the crack regions in the CCM is represented by TSC which relates the cohesive stress \( \sigma \) with the crack opening \( w \) in the FPZ. From Fig. 1b and c, the following boundary conditions for \( \sigma = f(w) \) should be satisfied:

\[
w = w_c, \quad \sigma = 0; \quad w = 0, \quad \sigma = f_t. \tag{1}\n\]

For a cohesive crack with its front and rear ends at the locations \( x=x_2 \) and \( x=x_1 \), respectively, the length of the cohesive crack is

\[
l_p = x_2 - x_1. \tag{2}\n\]

Because the crack opening \( w \) at all locations can be obtained directly from the experimental COD profile, the length of the FPZ can be determined with IDCM. Once the piecewise-linear relationship of the TSC has been
defined, the cohesive stress distribution within the FPZ can be evaluated.

![Figure 1](image1.png)

**Fig. 1.** (a) Sketch of a crack configuration. (b) Sketch of a cohesive crack. (c) Cohesive law [16].

### 3. IDCM methodology

#### 3.1. FEM

The three point bend test was simulated using the FEM. According to the symmetry of the specimen, only one half of the beam was analyzed. The specimen configuration and FE meshes are illustrated in Fig. 2 as an example. In the IDCM, the Young’s modulus $E$ is determined using FEM by comparing the simulated and experimental displacements. The Poisson’s ratio $\nu$ was taken as 0.2.

![Figure 2](image2.png)

**Fig. 2.** Specimen configuration and FE meshes (unit: mm) [17].

#### 2.1.2. Procedure of the IDCM

Both the global responses, such as the load-deflection and load-CMOD curves, and the local responses, including the complete COD profile, the location of crack front ($x_2$) and hence $l_p$ at each of the loading steps, are determined by the ESPI technique. By matching displacements from the FE analysis and experiments at various collocation points, the piecewise-linear relationship of the TSC and the Young’s modulus $E$ of the material can be evaluated. In this method, the TSC is determined in a step-by-step manner as shown in Fig. 3, and the details are described herein.

First, the experimental displacements, including mid-span deflection $\delta$, CMOD, NTOD and the COD profile, are extracted with the ESPI technique and used in the numerical analysis.

Second, the ultimate tension strength $f_t$ are determined inversely at the early loading stages or obtained from the
splitting tension test. At the early loading stage, the crack is small and the length of FPZ is short. The cohesive stress in the FPZ can be assumed to be uniformly distributed and equal to \( f_t \). Then the Young’s modulus \( E \) can be determined with FEM by matching the calculated and measured displacements in the linear elastic deformation stage.

Third, in the COD profile, the positions of the front of the crack and the rear end of the cohesive crack (e.g., the initial notch tip) are identified; thus, the length of the FPZ can be determined. In the FPZ, cohesive stresses will be assigned to the interfacial elements along the crack line.

Fourth, from the COD profile, the crack opening \( w(y) \) can be determined at various interfacial nodes in the FPZ of the FEM data set. The \( y \) axis is defined along the crack line with the origin at the notch mouth. As shown in Fig. 3, at the \( i^{th} \) loading step, all of the nodal points on the TSC in the previous \( i-1^{th} \) loading steps would have been defined; only the last step \( (w_i, \sigma_i) \) needs to be determined using the IDCM. The cohesive stresses at all the interfacial nodes with \( w \) less than or equal to \( w_{i-1} \) can then be established. Because the TSC must be a decreasing function, the unknown stress \( \sigma_i \) should satisfy the following requirement:

\[
\sigma_i \leq \sigma_{i-1}
\]

where \( \sigma_{i-1} \) is the cohesive stress determined at the \( i-1^{th} \) loading step. For \( i = 1 \), \( \sigma_0 \) is equal to \( f_t \), which can be determined by the splitting tension test.

Using linear interpolation, the nodal cohesive stress \( \sigma(y) \) at the \( j^{th} \) segment of the TSC can be expressed in terms of the crack opening:

\[
\sigma(y) = \sigma_{j-1} + \frac{\sigma_j - \sigma_{j-1}}{w_j - w_{j-1}}(w(y) - w_{j-1})
\]

where \( w(y) \) is the crack opening of the node considered; \( (w_j, \sigma_j) \) and \( (w_{j-1}, \sigma_{j-1}) \) are the end coordinates of the \( j^{th} \) line segment; \( (w(y), \sigma(y)) \) is a point on the line segment; and \( j = 1, 2, \ldots, i \). By assigning a certain trial value for \( \sigma \) that satisfies Equation (2), and using Equation (3), all of the nodal stresses along the FPZ can be obtained.

Lastly, the nodal cohesive stress \( \sigma_i \) is accepted only when two additional requirements, displacement and stress,
are satisfied as follows. The displacement requirement is
\[
|d_n - d_e| < \text{Tolerance}
\]  
(5)

where \(d_n\) and \(d_e\) represent the calculated and measured displacements, respectively, in terms of \(\delta\), CMOD and NTOD (Fig. 3b).

For the stress requirement, the calculated stresses in the relevant domain should not be greater than \(f_n\), and the numerical stress profile in the FPZ should be a smooth curve. Prior to the formation of a fully developed FPZ, all nodal cohesive stresses should be greater than zero. It can then proceed to the next loading step. The above procedure is repeated until the crack opening at the initial notch tip reaches \(w_c\) and \(\sigma=0\).

4. Results and discussion

4.1. Specimens

Two central-notched mortar beams (Unit 1 and Unit 2), five batches of concrete samples (C40 to C90) and two types of graphites (IG11 Series and NG-CT-01 Series) were tested to evaluate their TSCs. Parameters of the central-notched specimen are shown in Table 1. After being cast and cured, the mortar and concrete specimens were placed in air (temperature: 20±2°C; relative humidity: 75-85%) until the date of testing. The cube compressive strength \(f_{cu}\), the static modulus of elasticity in compression \(E_c\) and the splitting tensile strength \(f_{st}\) were obtained according to the Hong Kong construction standard.

<table>
<thead>
<tr>
<th>Materials</th>
<th>Maximum size of aggregate (mm)</th>
<th>Initial notch depth (mm)</th>
<th>Width of the notch (mm)</th>
<th>Specimen size (mm)</th>
<th>Span of the beam (mm)</th>
<th>(f_{cu}) (MPa)</th>
<th>(f_{st}) (MPa)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mortar</td>
<td>5</td>
<td>30</td>
<td>3</td>
<td>500 × 100 × 40</td>
<td>400</td>
<td>29.5</td>
<td>2.25</td>
</tr>
<tr>
<td>Concrete</td>
<td>10</td>
<td>45</td>
<td>3</td>
<td>710 × 150 × 80</td>
<td>600</td>
<td>39.7-86.7</td>
<td>2.5-5.4</td>
</tr>
<tr>
<td>IG11</td>
<td>20</td>
<td>0.5</td>
<td>220 × 50 × 25</td>
<td>200</td>
<td>78</td>
<td>76</td>
<td>24</td>
</tr>
<tr>
<td>NG-CT-01</td>
<td>20</td>
<td>0.5</td>
<td>220 × 50 × 25</td>
<td>200</td>
<td>78</td>
<td>76</td>
<td>24</td>
</tr>
</tbody>
</table>

4.2. The TSC of mortar

Fig. 4. TSC of mortar obtained from the IDCM [16].
Using IDCM, the estimated TSCs for both specimens are shown in Fig. 4. The characteristic crack openings \( w_c \) for Units 1 and 2 are 72.9 \( \mu m \) and 87.5 \( \mu m \), respectively. The ratio of \( w_c/f_t \) to the fracture energy \( G_F \) (81.78 N/m) is approximately 3.24.

It is found that, at the peak load, the cohesive stresses at the initial notch tip are about 1.65 MPa (0.5 ft) and the corresponding crack opening \( w \) is 15 \( \mu m \). When \( w \) increases from zero to 25 \( \mu m \), the cohesive stresses drop quickly from \( f_t \) to 0.8 MPa for Unit 1 and 1 MPa for Unit 2, which correspond to 0.25\( f_t \) and 0.33\( f_t \) respectively. From that point, the stress decreases gradually until \( w \) increases to 75 \( \mu m \) for Unit 1 and 60 \( \mu m \) for Unit 2 and the stress decreases rapidly to zero. Referring to the aforementioned assumptions on the shape of the TSC, such as linear, bilinear, tri-linear and exponential curves, a tri-linear curve was found to be the best approximation of mortar.

4.3. The TSC of concrete

To facilitate future simulations of concrete fractures using commercial finite element packages, the TSCs identified in the current study were simplified to bilinear and exponential curves using regression analysis, as shown in Fig. 5. The basic parameters used to define the TSC, including the total fracture energy \( G_F \) and the critical crack opening displacement \( w_c \), were obtained using inverse analysis. The tensile strength was obtained from the splitting tension test or the inverse analysis. In addition to these parameters, parameters are determined for each individual curve. The critical parameter for a bilinear curve is the location of the kink point \((w_1, f_1)\), see Fig. 5a. The exponential curve was derived empirically by Hordijk [19], see Fig. 5b. The function is expressed as

\[
\sigma / f_t = \left(1 + (c_1 w / w_c)^3\right) \exp\left(-c_2 w / w_c\right) - w / w_c \left(1 + c_1^3\right) \exp(-c_2)
\]

(6)

\[
w_c = c_3 G_F / f_t
\]

(7)

where \( c_1 \), \( c_2 \) and \( c_3 \) are the parameters of the exponential curves to be determined for concrete.

A complete set of parameters for defining the bilinear and exponential curves are listed in Table 2. To compare the modelling effect of the idealised curves, the root-mean-square deviation (RMSD) was evaluated, as presented in Table 2. For concrete with compressive strength \( f_{cu} \leq 60 \text{ MPa} \), the bilinear curve can model the TSCs fairly accurately, while for C80 and C90 concrete, the RMSD of bilinear fitting is statistically significant. For bilinear fitting, the stress ratio \( f_1/f_t \) at the kink point varies within 0.21-0.28. Exponential curves can provide satisfactory approximations for the TSCs of all concrete strengths.

Table 2. Parameters of the idealised TSC curves [17].

<table>
<thead>
<tr>
<th>Series</th>
<th>Fracture parameters</th>
<th>Bilinear curve</th>
<th>Exponential curve</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( G_F ) (N/m)</td>
<td>( w_c ) (( \mu m ))</td>
<td>( f_t ) (MPa)</td>
</tr>
<tr>
<td>C40</td>
<td>96.4</td>
<td>214.0</td>
<td>2.5</td>
</tr>
<tr>
<td>C50</td>
<td>120.1</td>
<td>184.3</td>
<td>2.9</td>
</tr>
<tr>
<td>C60</td>
<td>129.6</td>
<td>238.3</td>
<td>3.3</td>
</tr>
<tr>
<td>C80</td>
<td>124.2</td>
<td>188.7</td>
<td>4.5</td>
</tr>
<tr>
<td>C90</td>
<td>114.6</td>
<td>165.0</td>
<td>5.4</td>
</tr>
</tbody>
</table>

To ease the estimation of the fracture properties of concrete, the parameters for defining an exponential TSC were related to the compressive strength \( f_{cu} \). Empirical equations for \( c_1 \), \( c_2 \) and \( c_3 \) were expressed in terms of \( f_{cu} \) that ranges from 40 MPa to 90 MPa and the typo-errors originally occurred in Ref [17] have been corrected.

\[ c_1 = 0.431 \exp(0.0304 f_{cu}) \]

(8)
From the empirical formulas, the parameters of the exponential TSC seem only rely on the compressive strength. However, it should be noted that the tensile properties of concrete are dependent on the physical properties of the material, e.g. aggregate size, and the size of the tested specimen. With comparable aggregate size and specimen size, current empirical formulas are feasible to estimate the TSC from the compressive strength even for concrete with compressive strengths outside the range of 40-90 MPa.

\[
c_2 = 2.965 \exp(0.0165 f_{cu})
\]

\[
c_3 = 4.486 \exp(0.0053 f_{cu})
\]

4.4. The TSC of graphite

Similar to concrete, the TSC of graphite can also be simplified to be bilinear, tri-linear and exponential curves. A complete set of parameters for defining the bilinear, tri-linear and exponential curves is listed in Table 3. All the idealised curves could provide a satisfactory approximation for the TSCs if the initial descent of the TSC was not too sharp. However, when the tensile strength is high and the cohesive stress drops suddenly in the initial region of the TSC, only the tri-linear curve can accurately capture the abrupt change. Thus, the tri-linear curve is the best for fitting the TSC of nuclear graphite.

Conclusions

The assumptions and formulation of IDC are have been described in detail. The effectiveness of the method is revealed by determining the TSCs of various quasi-brittle materials in the forms of bilinear, tri-linear and
exponential functions.

Table 3. Parameters of the idealised TSC curves of graphite [18].

<table>
<thead>
<tr>
<th>Series</th>
<th>Fracture parameters</th>
<th>Bilinear curve</th>
<th>Tri-linear curve</th>
<th>Exponential curve</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$G_F$ (N/m)</td>
<td>$f_1$ (MPa)</td>
<td>$w_1$ (μm)</td>
<td>$f_1$ (MPa)</td>
</tr>
<tr>
<td>IG11</td>
<td>192.1</td>
<td>18.2</td>
<td>5.1</td>
<td>7.0</td>
</tr>
<tr>
<td>NG-CT-01</td>
<td>180.2</td>
<td>19.4</td>
<td>3.7</td>
<td>6.8</td>
</tr>
</tbody>
</table>

References