Control with Communications Constraints: Measuring the Instability in Parametric Linear Systems

Graziano Chesi
Department of Electrical and Electronic Engineering
The University of Hong Kong
Contact: http://www.eee.hku.hk/~chesi

Abstract

This paper investigates the instability measure of linear systems defined as the sum of the unstable eigenvalues in the continuous-time (CT) case and the product of the unstable eigenvalues in the discrete-time (DT) case. The problem consists of determining the largest instability measure in systems depending polynomially on parameters constrained in a semialgebraic set. It is shown that upper bounds of the sought measure can be established via linear matrix inequality (LMI) feasibility tests. Moreover, a priori and a posteriori conditions for establishing nonconservatism are proposed. Lastly, two special cases of the proposed methodology are investigated, the first one concerning systems with a single parameter, and the second one concerning the determination of the largest spectral abscissa and radius. Three applications in control with communications constraints are discussed.

I. Introduction

Measuring the instability, in particular the sum of the unstable eigenvalues (CT case) and the product of the unstable eigenvalues (DT case), is important for establishing whether a stabilizing controller can be designed in a number of frameworks in control with communications constraints. Indeed, [1] considers stochastic systems and derives that a stabilizing controller can be designed if and only if the data rate of the channel exceeds a certain function of the instability measure. Analogous results are proposed in [2] which considers the case of multiple sensors that partially observe the system, in [3] which addresses the design of controllers to achieve different control objectives, in [4] which describes a virtual system approach for digital finite communication bandwidth control, and in [5] where the channel is modeled as a finite logarithmic quantizer. Moreover, in [6], [7] it is shown that the lowest quantization density for stabilizability can be computed from the instability measure. See also [8] for a review of the instability measure.

As it always happens when dealing with physical systems, the mathematical model of the plant to be controlled is not exactly known in general. Indeed, the coefficients of such a model depend on some parameters, which represent physical quantities that cannot be measured exactly or that are subject to changes. These parameters are unknown, and typically one addresses the case where the available information is that the parameters belong to a set of interest. This means that the instability should be measured not just for one model but, instead, for a family of models. This problem is studied in [9] for the case of sets of parameters with known points by exploiting determinants, and in [10] for the case of DT polytopic systems by exploiting generalized eigenvalue problems.

This paper addresses the determination of the largest instability measure in linear systems depending polynomially on parameters constrained in a semialgebraic set. It is shown that a sufficient condition for establishing upper bounds of the sought measure can be obtained in terms of an LMI feasibility test based on polynomially parameter-dependent quadratic Lyapunov functions and sums-of-squares (SOS) matrix polynomials. Moreover, a sufficient condition is proposed for establishing nonconservatism of a computed upper bound. These conditions are also necessary under mild assumptions on the semialgebraic set. Hence, it is shown that a sufficient and necessary LMI condition with upper bounds on the degree of the Lyapunov functions can be obtained in the case of a single parameter constrained in an interval. The paper is concluded by explaining that the proposed methodology can also be used to determine the largest value of the spectral abscissa and radius. Three applications of the proposed methodology in control with communications constraints are discussed, namely signal-to-noise ratio (SNR) constrained feedback stabilization, quantized feedback stabilization, and stabilization with multirate sampling.

The paper is organized as follows. Section II provides the preliminaries. Section III describes the determination of the upper bounds. Section IV studies the nonconservatism of the upper bounds. Section V investigates the special cases. Section VI presents the examples. Lastly, Section VII concludes the paper with some final remarks.

II. PRELIMINARIES

A. Problem Formulation

Notation: $\mathbb{R}$, $\mathbb{C}$: sets of real and complex numbers; I: identity matrix; A': conjugate transpose; $A > 0$, $A \geq 0$: symmetric positive definite and semidefinite matrix; $\Re(A)$, $\Im(A)$: real and imaginary parts; $|a|$: magnitude; $(A)_{i,j}$: (i,j)-entry; $\det(A)$: determinant; $trace(A)$: trace; $adj(A)$: adjoint; $\ker(A)$: right null space; $spec(A)$: set of eigenvalues; $\lambda_i(A)$: i-th eigenvalue; $\lambda_{min}(A)$: minimum real eigenvalue; $diag(A, B, \ldots)$: block diagonal matrix with blocks A, B, $\ldots$; $E(A)$: statistical expectation; $\deg(A(p))$: degree of $A(p)$, i.e., largest degree among the entries of $A(p)$; Hurwitz/Schur matrix: a matrix whose eigenvalues have negative real parts/magnitude less than one.

Let us define the instability measure of $X \in \mathbb{R}^{n\times n}$ in the CT and DT cases as

$$\phi_a(X) = \begin{cases} \displaystyle\sum_{i=1}^{n} \max\left\{0, \Re\left(\lambda_i(X)\right)\right\} & \text{if } a = CT \\ \displaystyle\prod_{i=1}^{n} \max\left\{1, |\lambda_i(X)|\right\} & \text{if } a = DT. \end{cases} \tag{1}$$

where $\lambda_i(X)$ is the i-th eigenvalue of X. In the DT case, this measure is known as Mahler measure [11].

Let $A : \mathbb{R}^q \to \mathbb{R}^{n\times n}$ be a matrix polynomial (i.e., a matrix where each entry is a polynomial), and let us define the semialgebraic set

$$\mathcal{P} = \left\{p \in \mathbb{R}^q : \ r_i(p) \geq 0, \ \forall i = 1, \ldots, n_r\right\}, \tag{2}$$

where $r_1, \ldots, r_{n_r} : \mathbb{R}^q \to \mathbb{R}$ are polynomials.

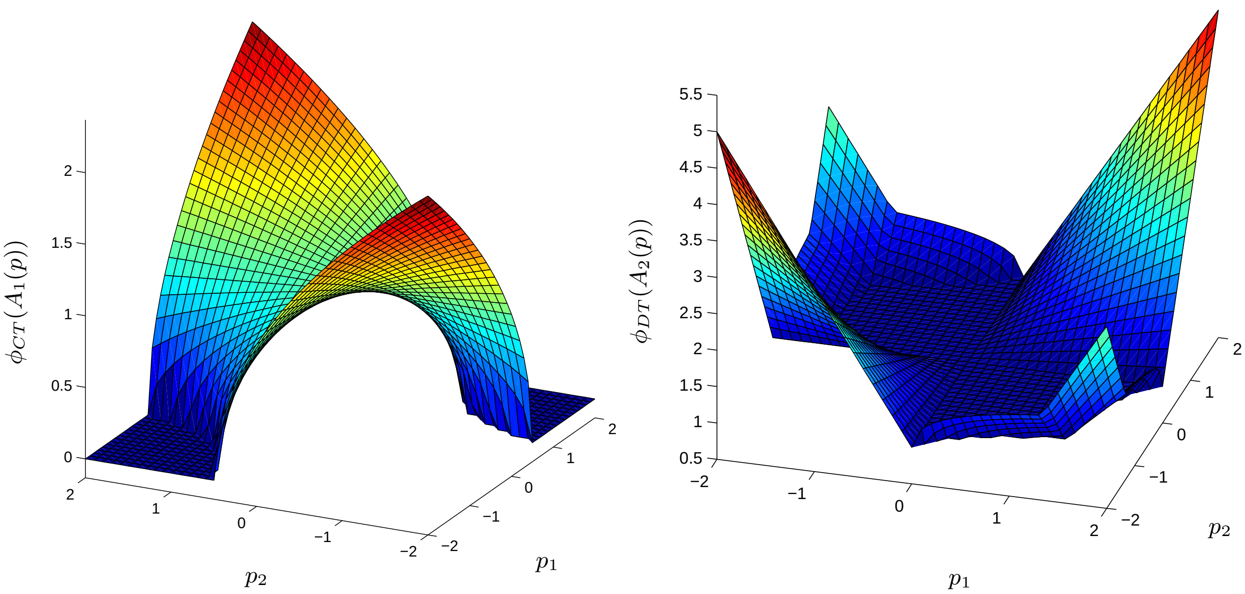
Problem 1. Determine the largest instability measure of $A(p)$ over $\mathcal{P}$, i.e.,

$$\phi_a^* = \sup_{p\in\mathcal{P}} \ \phi_a(A(p)), \quad a \in \{CT, DT\}. \tag{3}$$

□

Solving (3) is challenging since $\phi_{CT}(A(p))$ and $\phi_{DT}(A(p))$ can be non-concave functions of p even when $A(p)$ is linear. This is shown by Figure 1 for the matrices

$$A_1(p) = \begin{pmatrix} -1 & p_1 \\ p_2 & 1 \end{pmatrix}, \ A_2(p) = \begin{pmatrix} p_1 & 1 \\ -1 & p_2 \end{pmatrix}. \tag{4}$$



(a) (b)

Fig. 1. Instability measures $\phi_{CT}(A_1(p))$ (a) and $\phi_{DT}(A_2(p))$ (b).

Remark 1. Problem 1 considers systems depending polynomially on parameters constrained into a semialgebraic set. The first reason for considering such systems is that they include classical models typically adopted to describe parametric systems such as interval models and polytopic models. The second reason is that the methodology proposed in this paper can be applied to such classical models as well as to all the models where $A(p)$ and $r_i(p)$ are polynomial. $\square$

B. Motivation

Determining ϕ^*_{CT} and ϕ^*_{DT} is important for a number of problems related to control with communications constraints. Hereafter we provide three examples, namely 1) SNR constrained feedback stabilization, 2) quantized feedback stabilization, and 3) stabilization with multirate sampling.

1) **SNR Constrained Feedback Stabilization:** Consider

$$\begin{array}{rcl} \dot{x}(t) & = & Ax(t) + Bu_r(t) \\ u_r(t) & = & u_s(t) + h(t) \\ u_s(t) & = & -Kx(t), \end{array} \tag{5}$$

where $x \in \mathbb{R}^n$ is the state of the plant, $u_r \in \mathbb{R}$ is the received input, $u_s \in \mathbb{R}$ is the sent input, and $h \in \mathbb{R}$ is a zero-mean white Gaussian noise with power spectral density N.

Theorem 1 ([12], Theorem II.1): Suppose that (A, B) is stabilizable. There exists K such that $A - BK$ is Hurwitz and the sent input $u_s(t) = -Kx(t)$ satisfies the power constraint

$$\|u_s\|_{POW} = E\left(u_s(t)^2\right) < M, \tag{6}$$

for some $M \in \mathbb{R}$, if and only if

$$\frac{M}{N} > 2\phi_{CT}(A). \tag{7}$$

Whenever the matrices A and B are affected by parameters, Theorem 1 can be used to investigate stabilizability over the set of admissible parameters. Specifically, the following corollary provides a sufficient and necessary condition for the existence of a parameter-dependent controller (the proof is a direct consequence of Theorem 1 and the definition of ϕ^*_{CT}).

Corollary 1: Suppose that $(A(p), B(p))$ is stabilizable for all $p \in \mathcal{P}$. There exists $K(p)$ such that $A(p) - B(p)K(p)$ is Hurwitz and the sent input $u_s(t) = -K(p)x(t)$ satisfies the power constraint (6) for all $p \in \mathcal{P}$ if and only if

$$\frac{M}{N} > 2\phi^*_{CT}. \tag{8}$$

Corollary 1 provides a sufficient and necessary condition based only on ϕ^*_{CT} for the existence of a parameter-dependent controller that stabilizes the system and satisfies the power constraint. An interesting question is whether one can find a common controller with such properties. However, the existence of such a common controller cannot be established through ϕ^*_{CT} only as shown in the following example.

Example 1. For $\zeta_0, \zeta_1 \in \mathbb{R}$ let us consider

$$\begin{array}{rcl} A(p) & = & 2p^2 - 1 \\ B(p) & = & \zeta_0 + \zeta_1 p \\ p & \in & \mathcal{P} = [-1, 1]. \end{array}$$

The largest instability measure is independent on ζ_0 and ζ_1:

$$\phi_{CT}^* = 1.$$

Hereafter we consider two cases. The first case is

$$\zeta_0 = 1, \quad \zeta_1 = 0.$$

One has that $(A(p), B(p))$ is stabilizable for all $p \in \mathcal{P}$, and there exists a common controller K such that $A(p) - B(p)K$ is Hurwitz for all $p \in \mathcal{P}$ (indeed, any $K \in (-\infty, -1)$ satisfies this property). Next, let us consider the second case with

$$\zeta_0 = 0, \quad \zeta_1 = 1.$$

One has that $(A(p), B(p))$ is stabilizable for all $p \in \mathcal{P}$, however there does not exist any common controller K such that $A(p)-B(p)K$ is Hurwitz for all $p \in \mathcal{P}$. Indeed, for $p = -1$, $A(p)-B(p)K$ is Hurwitz if and only if $K > 1$, while, for $p = 1$, $A(p) - B(p)K$ is Hurwitz if and only if $K < -1$. □

Remark 2. Example 1 shows that the existence of a common controller K cannot be established through ϕ_{CT}^* only. Clearly, ϕ_{CT}^* can be used to provide a necessary condition for the existence of such a common controller, since a requirement for this is the existence of a parameter-dependent controller, which can be investigated with the sufficient and necessary condition provided by Corollary 1. □

2) **Quantized Feedback Stabilization:** Consider

$$\begin{array}{rcl} x(t+1) & = & Ax(t) + Bu(t) \\ u(t) & = & f(v(t)) \\ v(t) & = & Kx(t), \end{array} \tag{9}$$

where $x \in \mathbb{R}^n$ is the state of the plant, $u \in \mathbb{R}$ is the quantized input, $v \in \mathbb{R}$ is the unquantized input, $f(\cdot)$ is the logarithmic quantizer

$$f(v) = \begin{cases} u & \text{if } (1+\delta)^{-1}u < v \leq (1-\delta)^{-1}u,\ v > 0 \\ 0 & \text{if } v = 0 \\ -f(-v) & \text{if } v < 0, \end{cases} \tag{10}$$

where $\delta > 0$ defines the quantization level.

Theorem 2 ([7], Theorem 2.1): Suppose that (A, B) is stabilizable. There exists K such that (9) is stable if and only if

$$\delta \leq \frac{1}{\phi_{DT}(A)}. \tag{11}$$

Whenever the matrices A and B are affected by parameters, Theorem 2 can be used to provide conditions for the existence of a parameter-dependent controller similarly to Corollary 1.

3) **Stabilization with Multirate Sampling:** Consider

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ x_d(k) &= x(kT) \\ u(t) &= H_T(u_1(k), \ldots, u_m(k)), \end{aligned} \tag{12}$$

where $x \in \mathbb{R}^n$ is the state of the plant, $u \in \mathbb{R}$ is the input, $x_d(k) \in \mathbb{R}^n$ is the sampled state at step k, T is the sampling interval, $H_T(\cdot)$ is the zero-order hold with period T, and $u_i(k)$ is the output of the i-th channel. The channels are modeled as either signal-to-error ratio (SER) model

$$\begin{aligned} u_i(k) &= H_{K_i}(\tilde{u}_i(k)) \\ \tilde{u}_i(k) &= v_i(K_i k) + \Delta_i(v_i(K_i k)), \end{aligned} \tag{13}$$

or received signal-to-error ratio (R-SER), i.e.,

$$\begin{aligned} u_i(k) &= H_{K_i}(\tilde{u}_i(k)) \\ \tilde{u}_i(k) &= v_i(K_i k) + \Delta_i(\tilde{u}_i(k)), \end{aligned} \tag{14}$$

where $v_i(k)$ is the input, K_i is the downsampling rate, and $\Delta_i(\cdot)$ is an uncertain nonlinear, time-varying system with $\mathcal{L}_2$ gain δ_i. Let us define the total network capacity as

$$\mathcal{C} = \sum_{i=1}^{m} \mathcal{C}_i, \tag{15}$$

where $\mathcal{C}_i$ is the capacity of the i-th channel given by

$$\mathcal{C}_i = \frac{1}{K_i T} \ln \delta_i^{-1}. \tag{16}$$

Theorem 3 ([13], Theorems 4.1–4.2): Suppose that (A, B) is stabilizable. The multirate networked control system (12) with either SER channel model or R-SER channel model is stabilizable by state feedback if and only if

$$\mathcal{C} > \phi_{CT}(A). \tag{17}$$

Conditions for the existence of a parameter-dependent controller can be obtained similarly to Corollary 1 whenever the matrices A and B are affected by parameters.

C. SOS Matrix Polynomials

A symmetric matrix polynomial $V : \mathbb{R}^q \to \mathbb{R}^{u\times u}$ is said to be SOS if there exist matrix polynomials $V_1, \ldots, V_k : \mathbb{R}^q \to \mathbb{R}^{u\times u}$ such that

$$V(p) = \sum_{i=1}^{k} V_i(p)' V_i(p). \tag{18}$$

One can establish whether $V(p)$ is SOS via an LMI feasibility test, see [14]–[19]. Specifically, let d be a nonnegative integer such that $\deg(V(p)) \leq 2d$. Then, $V(p)$ can be expressed via the square matricial representation (SMR) (also known as Gram matrix method in the case $u = 1$) as

$$V(p) = (b(p) \otimes I)' \left(W + L(\alpha)\right) \left(b(p) \otimes I\right), \tag{19}$$

where $b : \mathbb{R}^q \to \mathbb{R}^{\sigma(q,d)}$ is a vector of monomials of degree not greater than d in p and

$$\sigma(q, d) = \frac{(q + d)!}{q!d!}, \tag{20}$$

$W \in \mathbb{R}^{u\sigma(q,d)\times u\sigma(q,d)}$ is a symmetric matrix satisfying

$$V(p) = (b(p) \otimes I)' W \left(b(p) \otimes I\right), \tag{21}$$

$L : \mathbb{R}^{\tau(q,2d,u)} \to \mathbb{R}^{u\sigma(q,d)\times u\sigma(q,d)}$ is a linear parametrization of the linear subspace

$$\mathcal{L} = \left\{ \tilde{L} = \tilde{L}' : \;\; (b(q) \otimes I)' \tilde{L} \left(b(q) \otimes I\right) = 0 \right\}, \tag{22}$$

where

$$\tau(q, 2d, u) = \frac{u}{2} \left(\sigma(q, d) \left(u\sigma(q, d) + 1\right) - (u + 1)\sigma(q, 2d)\right), \tag{23}$$

and $\alpha \in \mathbb{R}^{\tau(q,2d,u)}$ is a free vector. It follows that $V(p)$ is SOS if and only if there exists α satisfying the LMI

$$W + L(\alpha) \geq 0. \tag{24}$$

SOS matrix polynomials are useful in order to investigate positive semidefiniteness of matrix polynomials. Indeed, for the unconstrained case, $V(p)$ SOS ensures that

$$V(p) \geq 0 \quad \forall p \in \mathbb{R}^q. \tag{25}$$

The conservatism of this sufficient condition can be decreased by multiplying $V(p)$ times a SOS polynomial, indeed it is known from Artin's theorem [20], [21] that any nonnegative polynomial is the ratio of two SOS polynomials. For the constrained case, SOS matrix polynomials can be useful to establish whether

$$V(p) \geq 0 \quad \forall p \in \mathcal{P}, \tag{26}$$

where $\mathcal{P}$ is the semialgebraic set in (2). In the case $u = 1$, this can be done by exploiting the Positivstellensatz [22], [23], which consists of introducing SOS polynomial multipliers in order to take into account the constraint $p \in \mathcal{P}$. In the case $u \geq 1$, SOS matrix polynomials can be used by adopting an extension of this technique as proposed in [24].

There have been numerous applications of SOS matrix polynomials in control systems, see for instance [19]. In particular, SOS matrix polynomials have been used in the context of nonlinear systems for establishing whether an equilibrium point is stable and for estimating its domain of attraction, in the context of uncertain systems for establishing whether an equilibrium point is robustly stable, and in several other contexts including hybrid systems, game theory, and systems biology.

III. Establishing Upper Bounds

Given $X \in \mathbb{R}^{n\times n}$ and $k \in \{1, \ldots, n\}$, let $\Omega_{CT,k}(X)$ and $\Omega_{DT,k}(X)$ be square matrices with the property that their eigenvalues are all the sums and products of k distinct eigenvalues of X, i.e.,

$$spec(\Omega_a(X)) = \begin{cases} \left\{ \sum_{i\in z} \lambda_i(X),\ z \in \mathcal{I}_k \right\} & \text{if } a = CT \\ \left\{ \prod_{i\in z} \lambda_i(X),\ z \in \mathcal{I}_k \right\} & \text{if } a = DT, \end{cases} \tag{27}$$

where $\mathcal{I}_k$ is the set of k-tuples in $\{1,\ldots,n\}$, i.e.,

$$\begin{aligned} \mathcal{I}_k \;\; = \;\; & \{(z_1,\ldots,z_k): \; z_i \in \{1,\ldots,n\}, \; z_i < z_{i+1} \\ & \forall i = 1,\ldots,k-1\}. \end{aligned} \tag{28}$$

The matrix functions $\Omega_{CT,k}(X)$ and $\Omega_{DT,k}(X)$ can be built as follows. Let us define the number of k-tuples in $\mathcal{I}_k$ as

$$c_k = \frac{n!}{(n-k)!k!}, \tag{29}$$

and denote the k-tuples in $\mathcal{I}_k$ as $z(1),\ldots,z(c_k)$, where the numeration is made according to the lexicographical order. Then, $\Omega_{CT,k}(X)$ is the $c_k \times c_k$ matrix whose (i,j)-th entry is

$$(\Omega_{CT,k}(X))_{i,j} = \begin{cases} \text{if } i = j, & trace(Y_1) \\ \text{else if } Y_2 \in \mathbb{R}, & (-1)^{y_3} Y_2 \\ \text{else}, & 0, \end{cases} \tag{30}$$

where

- $Y_1 \in \mathbb{R}^{k\times k}$ is the submatrix of X built with the rows indexed by $z(i)$ and the columns indexed by $z(j)$;
- Y_2 is the submatrix of X built similarly to Y_1 by removing from $z(i)$ and $z(j)$ the common entries;
- y_3 is the difference between the sum of the indexes of the common entries in $z(j)$ and the same sum in $z(i)$.

Also, $\Omega_{DT,k}(X)$ is the $c_k \times c_k$ matrix whose (i,j)-th entry is

$$(\Omega_{DT,k}(X))_{i,j} = \det(Y_1). \tag{31}$$

Example 2. In order to clarify the construction of $\Omega_{CT,k}(X)$ and $\Omega_{DT,k}(X)$, let us consider $n = 3$ and

$$X = \begin{pmatrix} x_1 & x_4 & x_7 \\ x_2 & x_5 & x_8 \\ x_3 & x_6 & x_9 \end{pmatrix}.$$

One has

$$\begin{aligned} \mathcal{I}_1 &= \{z(1) = 1, z(2) = 2, z(3) = 3\} \\ \mathcal{I}_2 &= \{z(1) = (1,2), z(2) = (1,3), z(3) = (2,3)\} \\ \mathcal{I}_3 &= \{z(1) = (1,2,3)\}. \end{aligned}$$

Consequently, from (30) it follows that

$$\begin{aligned}\Omega_{CT,1}(X) &= X \\ \Omega_{CT,2}(X) &= \begin{pmatrix} x_1+x_5 & x_8 & -x_7 \\ x_6 & x_1+x_9 & x_4 \\ -x_3 & x_2 & x_5+x_9 \end{pmatrix} \\ \Omega_{CT,3}(X) &= trace(X),\end{aligned}$$

and, from (31),

$$\Omega_{DT,1}(X) = X$$

$$\Omega_{DT,2}(X) = \begin{pmatrix} x_1x_5-x_2x_4 & x_1x_8-x_2x_7 & x_4x_8-x_5x_7 \\ x_1x_6-x_3x_4 & x_1x_9-x_3x_7 & x_4x_9-x_6x_7 \\ x_2x_6-x_3x_5 & x_2x_9-x_3x_8 & x_5x_9-x_6x_8 \end{pmatrix}$$

$$\Omega_{DT,3}(X) = \det(X).$$

□

Let us define the quantities

$$\gamma_a = \begin{cases} 0 & \text{if } a = CT \\ 1 & \text{if } a = DT \end{cases} \tag{32}$$

and

$$\mathcal{K} = \{1, \ldots, n\}. \tag{33}$$

Theorem 4: Let $a \in \{CT, DT\}$ and $X \in \mathbb{R}^{n\times n}$. Then,

$$\phi_a(X) = \max_{k\in\mathcal{K}} \ \max\left\{\gamma_a, \psi_a\left(\Omega_{a,k}(X)\right)\right\}, \tag{34}$$

where $\Omega_{a,k}(X)$ is defined by (27), and

$$\psi_a(Y) = \begin{cases} \max\limits_{i=1,\ldots,m} \Re(\lambda_i(Y)) & \text{if } a = CT \\ \max\limits_{i=1,\ldots,m} |\lambda_i(Y)| & \text{if } a = DT \end{cases} \tag{35}$$

is the spectral abscissa (CT case) or radius (DT case) of $Y \in \mathbb{R}^{m\times m}$.

Proof. Consider $a = CT$. If the number of eigenvalues of X with nonnegative real part is different from k, from (27) it follows that $\max\{0, \psi_{CT}(\Omega_{a,k}(X))\} \leq \phi_{CT}(X)$. Moreover, if the number of eigenvalues of X with nonnegative real part is equal to k, one has $\max\{0, \psi_{CT}(\Omega_{a,k}(X))\} =$

$\phi_{CT}(X)$. Hence, (34) holds with $a = CT$, and similarly one proves that (34) holds with $a = DT$. □

Theorem 4 states that the instability measure (1) can be expressed through the spectral abscissa (CT case) and radius (DT case) of a family of matrices.

Theorem 5: Let $a \in \{CT, DT\}$ and $w \in (\gamma_a, \infty)$. Then,

$$\phi_a^* < w \tag{36}$$

if and only if, for all $k \in \mathcal{K}$, there exists a symmetric matrix polynomial $F_k : \mathbb{R}^q \to \mathbb{R}^{c_k \times c_k}$ such that

$$\forall p \in \mathcal{P} \begin{cases} F_k(p) & > & 0 \\ G_{a,k}(p) & > & 0 \end{cases} \tag{37}$$

and

$$\deg(F_k(p)) \leq d_{a,k}^*, \tag{38}$$

where

$$d_{a,k}^* = \begin{cases} 2^{-1}\left(c_k^2 + c_k - 2\right)\deg(A(p)) & \text{if } a = CT \\ k\left(c_k^2 + c_k - 2\right)\deg(A(p)) & \text{if } a = DT, \end{cases} \tag{39}$$

and $G_{a,k} : \mathbb{R}^q \to \mathbb{R}^{c_k \times c_k}$ is the matrix polynomial

$$G_{a,k}(p) = \begin{cases} 2wF_k(p) - F_k(p)B_{a,k}(p) - B_{a,k}(p)'F_k(p) & \text{if } a = CT \\ w^2 F_k(p) - B_{a,k}(p)'F_k(p)B_{a,k}(p) & \text{if } a = DT, \end{cases} \tag{40}$$

with $B_{a,k} : \mathbb{R}^q \to \mathbb{R}^{c_k \times c_k}$ given by

$$B_{a,k}(p) = \Omega_{a,k}(A(p)). \tag{41}$$

Proof. "⇒" Consider $a = CT$. Suppose that $\phi_a^* < w$. From the definition of ϕ_{CT}^* in (3) and by exploiting Theorem 4, it follows that

$$\begin{array}{rcl} w & > & \sup_{p \in \mathcal{P}} \phi_{CT}(A(p)) \\ & = & \sup_{p \in \mathcal{P}} \max_{k=1,\ldots,n} \max\{0, \psi_{CT,k}(A(p))\}, \end{array}$$

which means that

$$w > \sup_{p\in\mathcal{P}} \max_{k=1,\dots,n} \psi_{CT,k}(A(p)),$$

and, hence,

$$B_{CT,k}(p) - wI \text{ is Hurwitz } \forall k = 1,\dots,n \ \forall p \in \mathcal{P}.$$

From the Lyapunov stability theory, it follows that, for all $k \in \mathcal{K}$, the equation

$$F_k(p)\left(B_{CT,k}(p) - wI\right) + \left(B_{CT,k}(p) - wI\right)' F_k(p) = -I$$

admits a unique solution $F_k(p)$ satisfying

$$F_k(p) > 0 \quad \forall p \in \mathcal{P}.$$

This equation can be rewritten as

$$E_{CT,k}(p) f_k(p) = g,$$

where the vectors $f_k(p)$ and g gather the free coefficients of $F_k(p)$ and $-I$, in number equal to $c_k(c_k+1)/2$, and $E_{CT,k}(p)$ is a square matrix polynomial. Since the solution for $F_k(p)$ is unique, it follows that

$$\det(E_{CT,k}(p)) \neq 0 \quad \forall p \in \mathcal{P},$$

and, hence, that $f_k(p)$ is a rational function given by

$$f_k(p) = \frac{adj(E_{CT,k}(p))}{\det(E_{CT,k}(p))} g.$$

Let $F_k(p)$ be the matrix function corresponding to the found $f_k(p)$. Let us observe that $F_k(p)$ is a matrix rational function, and can be transformed into a matrix polynomial by multiplying it times $\det(E_{CT,k}(p))$, which is the denominator in the previous equation. Hence, we redefine $F_k(p)$ as

$$F_k(p) \leftarrow F_k(p) \det(E_{CT,k}(p)) \det(E_{CT,k}(p_0)),$$

where p_0 is any vector in $\mathcal{P}$. It follows that the new $F_k(p)$ is a matrix polynomial satisfying (37) since

$$\det(E_{CT,k}(p)) \det(E_{CT,k}(p_0)) > 0 \quad \forall p \in \mathcal{P},$$

and

$$\begin{aligned} G_{CT,k}(p) &= -F_k(p)\left(B_{CT,k}(p) - wI\right) \\ &\quad - \left(B_{CT,k}(p) - wI\right)' F_k(p) \\ &= \det(E_{CT,k}(p)) \det(E_{CT,k}(p_0)) I. \end{aligned}$$

Moreover, since $\deg(E_{CT,k}(p)) = \deg(A(p))$, it follows that $\deg(F_k(p)) = \deg(adj(E_{CT,k}(p))g)$. Since the size of $E_{CT,k}(p)$ is $c_k(c_k+1)/2 \times c_k(c_k+1)/2$, and since the entries of the adjoint of a $c_k(c_k+1)/2 \times c_k(c_k+1)/2$ matrix are sums of products of $c_k(c_k+1)/2-1$ entries of that matrix, it follows that $\deg(F_k(p)) = \deg(adj(E_{CT,k}(p))g) \leq \deg(adj(E_{CT,k}(p))) \leq (c_k(c_k+1)/2-1)\deg(A(p)) = d^*_{CT,k}$.

Similarly, in the case $a = DT$, one has

$$w > \sup_{p\in\mathcal{P}} \max_{k=1,\ldots,n} \psi_{DT,k}(A(p)),$$

i.e.,

$$w^{-1}B_{DT,k}(p) \text{ is Schur } \forall k \in \mathcal{K}\ \forall p \in \mathcal{P}.$$

Hence, for all $k = 1,\ldots,n$ the equation

$$w^{-1}B_{DT,k}(p)'F_k(p)w^{-1}B_{DT,k}(p) - F_k(p) = -I$$

admits a unique matrix rational function $F_k(p)$ which is positive definite over $\mathcal{P}$. Let $E_{DT,k}(p)$ be the matrix analogous to $E_{CT,k}(p)$ for the DT case. Multiplying this matrix rational function times $\det(E_{DT,k}(p))\det(E_{DT,k}(p_0))$, a matrix polynomial $F_k(p)$ satisfying (37) can be obtained since

$$G_{DT,k}(p) = -w^2\left(w^{-2}B_{DT,k}(p)'F_k(p)B_{DT,k}(p) - F_k(p)\right).$$

Moreover, since $\deg(E_{DT,k}(p)) = 2k\deg(A(p))$, one has $\deg(F_k(p)) = \deg(adj(E_{DT,k}(p))g) \leq \deg(adj(E_{DT,k}(p))) \leq 2k(c_k(c_k+1)/2-1)\deg(A(p)) = d^*_{DT,k}$.

"$\Leftarrow$" Suppose that, for all $k = 1,\ldots,n$, there exists a matrix polynomial $F_k(p)$ satisfying (37). For the case $a = CT$ this implies that

$$B_{CT,k}(p) - wI \text{ is Hurwitz } \forall k \in \mathcal{K}\ \forall p \in \mathcal{P},$$

and, hence,

$$w > \sup_{p\in\mathcal{P}} \max_{k\in\mathcal{K}} \psi_{CT,k}(A(p)).$$

Since $w \geq 0$, it follows that

$$\begin{aligned} w &> \sup_{p\in\mathcal{P}} \max_{k\in\mathcal{K}} \max\{0, \psi_{CT,k}(A(p))\} \\ &= \phi^*_{CT}. \end{aligned}$$

Similarly, for the case $a = DT$, (37) implies that

$$w^{-1}B_{DT,k}(p) \text{ is Schur } \forall k \in \mathcal{K}\ \forall p \in \mathcal{P},$$

and, hence,

$$w > \sup_{p\in\mathcal{P}} \max_{k\in\mathcal{K}} \psi_{DT,k}(A(p)).$$

Since $w \geq 1$, it follows that

$$\begin{aligned} w &> \sup_{p\in\mathcal{P}} \max_{k\in\mathcal{K}} \max\{1, \psi_{DT,k}(A(p))\} \\ &= \phi^*_{DT}. \end{aligned}$$

□

Theorem 5 provides a sufficient and necessary condition for establishing whether w is an upper bound of ϕ^*_{CT} and ϕ^*_{DT} based on the existence of a symmetric matrix polynomial $F_k(p)$ satisfying the inequalities in (37). An upper bound on the degree of $F_k(p)$ is also provided in Theorem 5.

Let us observe that $F_k(p)$ defines a polynomially parameter-dependent quadratic Lyapunov function candidate of the form

$$\tilde{v}_k(x) = \tilde{x}'_k F_k(p)\tilde{x}_k, \tag{42}$$

where $\tilde{x}_k \in \mathbb{R}^{c_k}$, for the CT system

$$\dot{\tilde{x}}_k(t) = \left(B_{CT,k}(p) - \frac{w}{2}I\right)\tilde{x}_k(t), \tag{43}$$

or for the DT system

$$\tilde{x}_k(t+1) = \frac{1}{\sqrt{w}}B_{DT,k}(p)\tilde{x}_k(t). \tag{44}$$

Theorem 6: Let $a \in \{CT, DT\}$ and $w \in (\gamma_a, \infty)$. Then,

$$\phi^*_a < w \tag{45}$$

if, for all $k \in \mathcal{K}$, there exist symmetric matrix polynomials $F_k, Q_{i,k}, S_{i,k} : \mathbb{R}^q \to \mathbb{R}^{c_k\times c_k}$, $i = 1,\dots,n_r$, and $\varepsilon \in \mathbb{R}$ satisfying the LMI condition

$$\begin{array}{l} H_k(p) - I \text{ is SOS} \\ J_{a,k}(p) - \varepsilon I \text{ is SOS} \\ Q_{i,k}(p) \text{ is SOS } \forall i = 1,\dots,n_r \\ S_{i,k}(p) \text{ is SOS } \forall i = 1,\dots,n_r \\ \varepsilon > 0, \end{array} \tag{46}$$

where

$$\begin{array}{rcl} H_k(p) & = & F_k(p) - \sum_{i=1}^{n_r} r_i(p) Q_{i,k}(p) \\ J_{a,k}(p) & = & G_{a,k}(p) - \sum_{i=1}^{n_r} r_i(p) S_{i,k}(p), \end{array} \tag{47}$$

with $G_{a,k}(p)$ defined as in Theorem 5.

Proof. Suppose that (46) holds. The first and third conditions in (46) imply that

$$\forall p \in \mathbb{R}^q \left\{ \begin{array}{rcl} H_k(p) & \geq & I \\ Q_{i,k}(p) & \geq & 0. \end{array} \right.$$

Let $p \in \mathcal{P}$. Since $r_i(p) \geq 0$, it follows that

$$\begin{array}{rcl} I & < & F_k(p) - \sum_{i=1}^{n_r} r_i(p) Q_{i,k}(p) \\ & \leq & F_k(p). \end{array}$$

Similarly, one gets that $G_{a,k}(p) \geq \varepsilon I$ by exploiting the second and fourth conditions in (46). Since $\varepsilon > 0$ from the fifth condition in (46), it follows that (37) holds. Moreover, from Theorem 5 we conclude that $\phi_a^* < w$. □

Theorem 6 provides a sufficient condition for establishing upper bounds of ϕ_{CT}^* and ϕ_{DT}^*. The matrix polynomials $F_k(p)$, $Q_{i,k}(p)$, $S_{i,k}(p)$ and the scalar ε are the decision variables of this condition.

The condition provided by Theorem 6 is based on SOS matrix polynomials and is equivalent to an LMI feasibility test as explained in Section II-C since $H_k(p) - I$ and $J_{a,k}(p) - \varepsilon I$ depend linearly on the decision variables $F_k(p)$, $Q_{i,k}(p)$, $S_{i,k}(p)$ and ε.

The search for the decision variables $F_k(p)$, $Q_{i,k}(p)$, $S_{i,k}(p)$ and ε satisfying (46) can be directly performed with software for SOS programming such as SOSTOOLS [25], which converts the SOS conditions into LMIs and then exploit software for semidefinite programming such as SeDuMi [26] in order to solve the obtained LMIs.

The condition provided by Theorem 6 is sufficient for any degree of the matrix polynomials $F_k(p)$, $Q_{i,k}(p)$ and $S_{i,k}(p)$. As it will be shown in Theorem 8, this condition is also necessary by using matrix polynomials with degree sufficiently large under mild assumptions on $\mathcal{P}$.

Let us define the upper bound of ϕ_a^*, $a \in \{CT, DT\}$, provided by Theorem 6 as

$$\hat{\phi}_a = \max_{k \in \mathcal{K}} \hat{\phi}_{a,k}, \tag{48}$$

where

$$\begin{array}{rcl} \hat{\phi}_{a,k} & = & \inf\limits_{w} \; w \\ & & \text{s.t.} \left\{ \begin{array}{l} w > \gamma_a \\ \exists F_k(p), Q_{i,k}(p), S_{i,k}(p), \varepsilon : \text{ (46) holds} \\ \deg(F_k(p)) \leq d_k \\ \deg(Q_{i,k}(p)) \leq e_{i,k} \\ \deg(S_{i,k}(p)) \leq f_{i,k} \end{array} \right. \end{array} \tag{49}$$

and d_k, $e_{i,k}$ and $f_{i,k}$ are any chosen bounds on the degrees of the matrix polynomials $F_k(p)$, $Q_{i,k}(p)$ and $S_{i,k}(p)$. Let us observe that, whenever w is variable, the condition (46) involves either bilinear matrix inequalities (BMIs) in the CT case or nonlinear matrix inequalities (NMIs) in the DT case because either w or w^2 multiplies $F_k(p)$. Nevertheless, (49) can be simply solved through a bisection algorithm on w where the LMI condition (46) is tested for fixed values of w at each step.

Remark 3. The computational burden of the LMI condition (46) grows with the dimension of the problem and the degree of the decision variables. Unfortunately, this growth is faster than linear due to its combinatorial nature. □

IV. ESTABLISHING NONCONSERVATISM

This section investigates the nonconservatism of the methodology introduced in Section III.

Theorem 7: Let $a \in \{CT, DT\}$. Without loss of generality, suppose that $\hat{\phi}_a > \gamma_a$. Then,

$$\hat{\phi}_a = \phi_a^* \tag{50}$$

if there exists $k^* \in \mathcal{K}$ and $p^* \in \mathbb{R}^q$ such that

$$\begin{array}{l} \lambda_{min}\left(J^*_{a,k^*}(p^*)\right) = 0 \\ \phi_a(A(p^*)) = \hat{\phi}_a \\ p^* \in \mathcal{P}, \end{array} \tag{51}$$

where $\lambda_{min}(\cdot)$ denotes the minimum eigenvalue, and $J_{a,k}^*(p)$ is $J_{a,k}(p)$ evaluated for the optimal values of w, $F_k(p)$, $S_{i,k}(p)$ and ε in (49). Moreover, if $\mathcal{P}$ is compact, this condition is not only sufficient but also necessary.

Proof. "$\Leftarrow$" Suppose that (51) holds for some $k^* \in \mathcal{K}$ and $p^* \in \mathbb{R}^q$. Then, since Theorem 6 ensures that $\hat{\phi}_a \geq \phi_a^*$, and since the second condition in (51) ensures that $\hat{\phi}_a \leq \phi_a^*$, it follows that $\hat{\phi}_a = \phi_a^*$.

"$\Rightarrow$" Suppose that $\hat{\phi}_a = \phi_a^*$ and that $\mathcal{P}$ is compact. Since $\phi_a(A(p))$ is continuous, it follows that there exists a global maximizer in the optimization problem (3), i.e., $p^* \in \mathcal{P}$ such that $\phi(A(p^*)) = \phi_a^*$. Since $\hat{\phi}_a > \gamma_a$, it follows from Theorem 4 that there exists $k^* \in \{1, \ldots, n\}$ such that

$$\psi_{k^*}(A(p^*)) = \phi_a^*.$$

Let us consider $B_{a,k^*}(p^*)$. This matrix has an eigenvalue $\lambda^* \in \mathbb{C}$ such that $\Re(\lambda^*) = \phi_a^*$ in the CT case, and $|\lambda^*| = \phi_a^*$ in the DT case. Let $\tilde{x}^* \in \mathbb{C}^{c_k^*}$ be an eigenvector of $B_{a,k^*}(p^*)$ corresponding to λ^*. Let us denote with w^*, $F_k^*(p)$, $G_{a,k}^*(p)$, $S_{i,k}^*(p)$ and ε^* the optimal values of w, $F_k(p)$, $G_{a,k}(p)$, $S_{i,k}(p)$ and ε in (49). Observe that $w^* = \phi_a^*$. Moreover, if $a = CT$,

$$\begin{array}{rl} & \tilde{x}^{*\prime} G_{a,k^*}^*(p^*)\tilde{x}^* \\ = & \tilde{x}^{*\prime}\left(2w^* F_{k^*}(p^*) - F_{k^*}(p^*)B_{CT,k^*}(p^*)\right. \\ & \left. -B_{CT,k^*}(p^*)' F_{k^*}(p^*)\right)\tilde{x}^* \\ = & 2\phi_a^*\tilde{x}^{*\prime} F_{k^*}(p^*)\tilde{x}^* - 2\phi_a^*\tilde{x}^{*\prime} F_{k^*}(p^*)\tilde{x}^* \\ = & 0, \end{array}$$

and, if $a = DT$,

$$\begin{array}{rl} & \tilde{x}^{*\prime} G_{a,k^*}^*(p^*)\tilde{x}^* \\ = & \tilde{x}^{*\prime}\left((w^*)^2 F_{k^*}(p^*) - B_{DT,k^*}(p^*)' F_{k^*}(p^*) B_{DT,k^*}(p^*)\right)\tilde{x}^* \\ = & (\phi_a^*)^2\tilde{x}^{*\prime} F_{k^*}(p^*)\tilde{x}^* - (\phi_a^*)^2\tilde{x}^{*\prime} F_{k^*}(p^*)\tilde{x}^* \\ = & 0. \end{array}$$

Hence,

$$\begin{array}{rcl} 0 & \leq & \tilde{x}^{*\prime} J_{a,k^*}^*(p^*)\tilde{x}^* \\ & = & \tilde{x}^{*\prime}\left(G_{a,k^*}^*(p^*) - \sum_{i=1}^{n_r} r_i(p^*) S_{i,k^*}^*(p^*)\right)\tilde{x}^* \\ & = & -\tilde{x}^{*\prime}\left(\sum_{i=1}^{n_r} r_i(p^*) S_{i,k^*}^*(p^*)\right)\tilde{x}^*. \end{array}$$

Since $J^*_{a,k^*}(p^*) \geq 0$ and $S^*_{i,k^*}(p^*) \geq 0$, this implies that $\tilde{x}^{*\prime} J^*_{a,k^*}(p^*)\tilde{x}^* = 0$, and, hence,

$$\lambda_{min}\left(J^*_{a,k^*}(p^*)\right) = 0,$$

i.e., (51) holds. □

Theorem 7 provides a sufficient condition for establishing whether the computed upper bounds $\hat{\phi}_{CT}$ and $\hat{\phi}_{DT}$ are tight. This is important because, if one can establish that the computed upper bounds are tight, then the search for less conservative upper bounds of ϕ^*_{CT} and ϕ^*_{DT} can be terminated.

Observe that there is no loss of generality in supposing that, for $a \in \{CT, DT\}$, one has $\hat{\phi}_a > \gamma_a$. Indeed, since $\phi^*_a \geq \gamma_a$ for definition, it follows that

$$\hat{\phi}_a = \gamma_a \quad \Rightarrow \quad \hat{\phi}_a = \phi^*_a. \tag{52}$$

The condition provided by Theorem 7 is also necessary whenever $\mathcal{P}$ is compact. Let us observe that this assumption on $\mathcal{P}$ is a mild one since the parameters are typically bounded when representing physical quantities.

The condition (51) consists of a numerical test, namely establishing whether $\phi_a(A(p^*)) = \hat{\phi}_a$ for some $p^* \in \mathcal{P}$ satisfying $\lambda_{min}\left(J^*_{a,k^*}(p^*)\right) = 0$ for some $k^* \in \mathcal{K}$. Since $J^*_{a,k^*}(p)$ is a SOS matrix polynomial due to the second condition in (46), computing such a point p^* amounts to looking for vectors $p^* \in \mathbb{R}^q$ and $x^* \in \mathbb{C}^{c^*_k}$ such that

$$p^* \otimes x^* \in \ker\left(K_{a,k^*}\right), \tag{53}$$

where K_{a,k^*} is the matrix $W + L(\alpha)$ in (19), obtained by replacing $V(p)$ with $J^*_{a,k^*}(p)$, and evaluated for any α such that $W + L(\alpha) \geq 0$. As explained for instance in [19], the vectors $p^* \in \mathbb{R}^q$ and $x^* \in \mathbb{C}^{c_k}$ satisfying (53) can be searched for through linear algebra operations.

In order to present the next result, let us introduce the following definition.

Definition 1. The semialgebraic set $\mathcal{P}$ in (2) is said to be strongly compact if $\mathcal{P}$ is compact and the polynomials $r_i(p)$, $i = 1, \ldots, n_r$, have even degree and their highest degree forms have no common roots except zero.

Theorem 8: Let $a \in \{CT, DT\}$. Suppose that $\mathcal{P}$ is strongly compact. Then:

- the sufficient condition provided by Theorem 6 is also necessary;
- there exist sufficiently large degree bounds d_k, $e_{i,k}$ and $f_{i,k}$ such that the upper bound $\hat{\phi}_a$ is tight, i.e., $\hat{\phi}_a = \phi_a^*$.

Proof. Suppose that $\phi_a^* < w$. Then, from Theorem 5 it follows that there exists a matrix polynomial $F_k(p)$ with $\deg(F_k(p)) \leq d_{a,k}^*$ satisfying (37). Such a matrix polynomial can be scaled in order to satisfy

$$\forall p \in \mathcal{P} \left\{ \begin{array}{rcl} F_k(p) - I & \geq & 0 \\ G_{a,k}(p) - \varepsilon I & \geq & 0 \\ \varepsilon & > & 0. \end{array} \right.$$

This can be rewritten as

$$\forall (\tilde{x}_k, p) \in \mathcal{S}_k \times \mathcal{P} \left\{ \begin{array}{rcl} f_k(\tilde{x}_k, p) - \|\tilde{x}_k\|^2 & \geq & 0 \\ g_{a,k}(\tilde{x}_k, p) - \varepsilon \|\tilde{x}_k\|^2 & \geq & 0 \\ \varepsilon & > & 0, \end{array} \right.$$

where $\tilde{x}_k \in \mathbb{R}^{c_k}$,

$$\begin{array}{rcl} f_k(\tilde{x}_k, p) & = & \tilde{x}_k' F_k(p) \tilde{x}_k \\ g_{a,k}(\tilde{x}_k, p) & = & \tilde{x}_k' G_{a,k}(p) \tilde{x}_k, \end{array}$$

and

$$\mathcal{S}_k = \left\{ \tilde{x}_k \in \mathbb{R}^{c_k} : \ \|\tilde{x}_k\| = 1 \right\}.$$

Since $\mathcal{P}$ is strongly compact, it follows from [23] that there exist polynomials $q_{i,k}(\tilde{x}_k, p)$ and $s_{i,k}(\tilde{x}_k, p)$, $i = 1, \ldots, n_r$, such that

$$\begin{array}{l} h_k(\tilde{x}_k, p) - \|\tilde{x}_k\|^2 \text{ is SOS} \\ j_{a,k}(\tilde{x}_k, p) - \varepsilon \|\tilde{x}_k\|^2 \text{ is SOS} \\ q_{i,k}(\tilde{x}_k, p) \text{ is SOS } \forall i = 1, \ldots, n_r \\ s_{i,k}(\tilde{x}_k, p) \text{ is SOS } \forall i = 1, \ldots, n_r \\ \varepsilon > 0, \end{array}$$

where

$$\begin{array}{rcl} h_k(\tilde{x}_k, p) & = & f_k(\tilde{x}_k, p) - \displaystyle\sum_{i=1}^{n_r} r_i(p) q_{i,k}(\tilde{x}_k, p) \\ j_{a,k}(\tilde{x}_k, p) & = & g_{a,k}(\tilde{x}_k, p) - \displaystyle\sum_{i=1}^{n_r} r_i(p) s_{i,k}(\tilde{x}_k, p). \end{array}$$

Since $f_k(\tilde{x}_k, p)$ and $g_{a,k}(\tilde{x}_k, p)$ are homogeneous quadratic in $\tilde{x}_k$, and $\tilde{x}_k$ is constrained on the unitary sphere, the polynomials $q_{i,k}(\tilde{x}_k, p)$ and $s_{i,k}(\tilde{x}_k, p)$ can be chosen homogeneous quadratic in $\tilde{x}_k$ as well. Hence, such polynomials can be expressed similarly to $f_k(\tilde{x}_k, p)$ and $g_{a,k}(\tilde{x}_k, p)$ for symmetric matrix polynomials $Q_{i,k}(p)$ and $S_{i,k}(p)$. This implies that the previous condition coincides with (46) by observing that also $h_k(\tilde{x}_k, p)$ and $j_{a,k}(\tilde{x}_k, p)$ are homogeneous quadratic in $\tilde{x}_k$ and can be expressed similarly to $f_k(\tilde{x}_k, p)$ and $g_{a,k}(\tilde{x}_k, p)$ for the symmetric matrix polynomials $H_k(p)$ and $J_{a,k}(p)$ in (47). □

Theorem 8 guarantees that the upper bounds ϕ^*_{CT} and ϕ^*_{DT} are tight for sufficiently large degree of the matrix polynomials $F_k(p)$, $Q_{i,k}(p)$ and $S_{i,k}(p)$ under the assumption that $\mathcal{P}$ is strongly compact. This is a mild assumption. Indeed, the parameters are typically bounded when representing physical quantities. Moreover, the constraints on the polynomials $r_i(p)$ are typically satisfied whenever $\mathcal{P}$ is bounded, for instance this is the case of hyper-cubes and hyper-spheres.

V. Special Cases

This section investigates two special cases of the proposed methodology, the first one concerning systems with a single parameter, and the second one concerning the determination of the largest spectral abscissa and radius.

A. Single Parameter

Here we investigate the case of a single parameter (i.e., $q = 1$). In particular, we consider

$$\mathcal{P} = [0, 1]. \tag{54}$$

Theorem 9: Let $a \in \{CT, DT\}$ and $w \in (\gamma_a, \infty)$. Let $\mathcal{P}$ be defined as in (54). Then,

$$\phi^*_a < w \tag{55}$$

if and only if, for all $k \in \mathcal{K}$, there exist a symmetric matrix polynomial $F_k : \mathbb{R} \to \mathbb{R}^{c_k \times c_k}$ and $\varepsilon \in \mathbb{R}$ satisfying the LMI condition

$$\begin{cases} T_k(p^2) - (1+p^2)^{\deg(F_k(p))} I \text{ is SOS} \\ U_{a,k}(p^2) - \varepsilon(1+p^2)^{\deg(G_{a,k}(p))} I \text{ is SOS} \\ \varepsilon > 0 \end{cases} \tag{56}$$

and

$$\deg(F_k(p)) \leq d^*_{a,k}, \tag{57}$$

where $d^*_{a,k}$ is given by (39), $T_k, U_{a,k} : \mathbb{R} \to \mathbb{R}^{c_k \times c_k}$ are the matrix polynomials

$$\left\{ \begin{array}{rcl} T_k(p^2) & = & (1+p^2)^{\deg(F_k(p))} F_k(\theta(p^2)) \\ U_{a,k}(p^2) & = & (1+p^2)^{\deg(G_{a,k}(p))} G_{a,k}(\theta(p^2)), \end{array} \right. \tag{58}$$

and $\theta : \mathbb{R} \to \mathbb{R}$ is the rational function defined by

$$\theta(p^2) = \frac{p^2}{1+p^2}. \tag{59}$$

Proof. "$\Rightarrow$" Suppose that (56) holds. By proceeding analogously to the proof of Theorem 6 and exploiting the definition of $T_k(p^2)$ and $U_{a,k}(p^2)$ in (58), one gets that

$$\forall p \in \mathbb{R} \left\{ \begin{array}{rcl} F_k(\theta(p^2)) & \geq & I \\ G_{a,k}(\theta(p^2)) & \geq & \varepsilon I. \end{array} \right.$$

Let us observe that

$$\mathcal{P} = \left\{ \theta(p^2),\ p \in \mathbb{R} \right\}.$$

By expressing the previous inequalities in terms of $\tilde{p} = \theta(p^2)$, one can write

$$\forall \tilde{p} \in \mathcal{P} \left\{ \begin{array}{rcl} F_k(\tilde{p}) & \geq & I \\ G_{a,k}(\tilde{p}) & \geq & \varepsilon I. \end{array} \right.$$

Therefore, (37) holds, and from Theorem 5 we conclude that $\phi^*_a < w$.

"$\Leftarrow$" Suppose that $\phi^*_a < w$ holds. From Theorem 5, this implies that, for all $k \in \mathcal{K}$, there exists a matrix polynomial $F_k : \mathbb{R} \to \mathbb{R}^{c_k \times c_k}$ with $\deg(F_k(p)) \leq d^*_{a,k}$ such that (37) holds. Since $\mathcal{P}$ is bounded, it follows that $F_k(p)$ can be scaled in order to satisfy

$$\forall p \in \mathcal{P} \left\{ \begin{array}{rcl} F_k(p) & \geq & I \\ G_{a,k}(p) & \geq & \varepsilon I, \end{array} \right.$$

for some $\varepsilon > 0$. Since the image of $\mathbb{R}$ through the function $\theta(p^2)$ is $\mathcal{P}$, one can also write that

$$\forall p \in \mathbb{R} \left\{ \begin{array}{rcl} F_k(\theta(p^2)) & \geq & I \\ G_k(\theta(p^2)) & \geq & \varepsilon I. \end{array} \right.$$

Moreover, since $1 + p^2$ is positive, this implies that

$$\forall p \in \mathbb{R} \left\{ \begin{array}{rcl} T_k(p^2) & \geq & (1+p^2)^{\deg(F_k(p))} I \\ U_{a,k}(p^2) & \geq & \varepsilon(1+p^2)^{\deg(G_{a,k}(p))} I. \end{array} \right.$$

The proof is concluded by observing that univariate matrix polynomials are positive semidefinite if and only if they are SOS [19]. $\square$

Theorem 9 provides a sufficient and necessary LMI condition with an upper bound on the degree of the Lyapunov function for establishing upper bounds of ϕ_{CT}^* and ϕ_{DT}^* in the case of a single parameter.

Let us observe that the condition of Theorem 9 does not require the introduction of multipliers as done in Theorems 6 and 8. Also, the matrix polynomials in the LMI condition (56) contain only even powers of p and, hence, the number of LMI scalar variables required for testing this condition can be reduced as explained in [27].

Let us define the upper bound of ϕ_a^*, $a \in \{CT, DT\}$, provided by Theorem 9 as

$$\tilde{\phi}_a = \max_{k \in \mathcal{K}} \tilde{\phi}_{a,k}, \tag{60}$$

where

$$\begin{array}{rcl} \tilde{\phi}_{a,k} & = & \inf\limits_{w} \; w \\ & & \text{s.t.} \left\{ \begin{array}{l} w > \gamma_a \\ \exists F_k(p), \varepsilon : \text{ (56) holds} \\ \deg(F_k(p)) \leq d_k \end{array} \right. \end{array} \tag{61}$$

and d_k is any chosen bound on the degree of the matrix polynomial $F_k(p)$. The nonconservatism of these upper bounds can be established as done in Theorem 7 for the upper bounds $\hat{\phi}_{CT}$ and $\hat{\phi}_{DT}$, and the details are omitted for brevity.

B. Largest Spectral Abscisa and Radius

The methodology proposed in this paper can also be used to determine the largest spectral abscissa and radius of $A(p)$ over $\mathcal{P}$, i.e.,

$$\psi_a^* = \sup_{p \in \mathcal{P}} \; \psi_a(A(p)), \quad a \in \{CT, DT\} \tag{62}$$

where ψ_a is given by (35). Indeed, since $\Omega_{a,1}(X) = X$ from (27), it follows that ψ_a^* can be studied with Theorems 5–9 by simply redefining γ_a and $\mathcal{K}$ in (32)–(33) as

$$\gamma_a = \left\{ \begin{array}{rl} -\infty & \text{if } a = CT \\ 0 & \text{if } a = DT \end{array} \right. \tag{63}$$

and

$$\mathcal{K} = \{1\}. \tag{64}$$

VI. EXAMPLES

In this section we present some illustrative examples. The LMI feasibility tests (46) and (56) are solved in Matlab on a personal computer with Windows 8, Intel Core i7, 3.4 GHz, 8 GB RAM. We denote with NuVa the number of LMI scalar variables minus one (since these tests are defined up to a positive scale factor). Moreover, we denote with CoTi the average computational time in the bisection algorithm.

A. Example 3 (SNR Constrained Feedback Stabilization)

Consider an example in the framework of the SNR constrained feedback stabilization introduced in Section II-B1. The matrices A and B in (5) are chosen as

$$\begin{cases} A(p) &= \begin{pmatrix} 0 & 1+p_1 & -1 \\ 2-p_2 & 0 & 1 \\ -1 & 1 & p_1+p_2 \end{pmatrix} \\ B(p) &= \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}' \\ p &\in \mathcal{P} = \{p \in \mathbb{R}^2 : \ p_1^2 + p_2^2 \leq 1\}. \end{cases}$$

The power spectral density of the zero-mean white Gaussian noise is chosen as $N = 1$, while the bound in the power constraint is chosen as $M = 8$.

Let us compute the upper bound $\hat{\phi}_{CT}$ in (48)–(49) of ϕ^*_{CT}. We express $\mathcal{P}$ as in (2) by defining $r_1(p) = 1 - p_1^2 - p_2^2$. We choose the degree bounds $d_k = e_{i,k} = f_{i,k} = 0$ for all $k = 1, 2, 3$. We find

$$\begin{cases} \hat{\phi}_{CT,1} = 2.154 & \text{(NuVa=21, CoTi=0.2 s)} \\ \hat{\phi}_{CT,2} = 3.628 & \text{(NuVa=21, CoTi=0.4 s)} \\ \hat{\phi}_{CT,3} = 1.414 & \text{(NuVa=2, CoTi=0.1 s)}, \end{cases}$$

which provide $\hat{\phi}_{CT} = 3.628$.

Let us establish whether the found upper bound is tight by using Theorem 7. We have that $\hat{\phi}_{CT} = \hat{\phi}_{CT,k^*}$ for $k^* = 1$. By using (53) we find that (51) holds with $p^* = (0.953, 0.303)'$, hence implying that $\hat{\phi}_{CT}$ is tight, i.e., $\phi^*_{CT} = 3.628$.

Therefore, since $M/N > 2\phi^*_{CT}$, from Corollary 1 it follows that, for all $p \in \mathcal{P}$, there exists a controller $K(p)$ such that $A(p) - B(p)K(p)$ is Hurwitz and the sent input $u_s(t) = -K(p)x(t)$ satisfies the power constraint (6).

B. Example 4 (Stabilization with Multirate Sampling)

Here we consider an example in the framework of the stabilization with multirate sampling introduced in Section II-B3. The matrices A and B in (12) are chosen as

$$\begin{cases} A(p) &= \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -3 & 2+3p & -1 & 2 & -3 & 2+p \end{pmatrix} \\ B(p) &= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}' \\ p &\in \mathcal{P} = [-1, 1]. \end{cases}$$

The sampling interval T, downsampling rates K_1, K_2, and $\mathcal{L}_2$ gains δ_1, δ_2 are chosen as $T = 0.5$, $K_1 = 1$, $K_2 = 2$, $\delta_1 = 0.3$ and $\delta_2 = 0.2$.

Since we are in the case of a single parameter, let us compute the upper bound $\tilde{\phi}_{CT}$ in (60)–(61) of ϕ^*_{CT}. To this end, we replace p with $(p + 1)/2$ in order to have $\mathcal{P}$ as in (54). Moreover, we reduce the number of LMI scalar variables in (56) as explained in [27]. We choose the degree bound $d_k = 0$ for all $k = 1, \ldots, 6$. We find

$$\begin{cases} \tilde{\phi}_{CT,1} = 2.101 & \text{(NuVa=21 CoTi=0.2 s)} \\ \tilde{\phi}_{CT,2} = 2.916 & \text{(NuVa=120, CoTi=0.4 s)} \\ \tilde{\phi}_{CT,3} = 3.730 & \text{(NuVa=210, CoTi=0.4 s)} \\ \tilde{\phi}_{CT,4} = 4.357 & \text{(NuVa=120, CoTi=0.3 s)} \\ \tilde{\phi}_{CT,5} = 3.679 & \text{(NuVa=21, CoTi=0.3 s)} \\ \tilde{\phi}_{CT,6} = 3.000 & \text{(NuVa=1, CoTi=0.1 s)}, \end{cases}$$

which provide $\tilde{\phi}_{CT} = 4.357$.

Let us establish whether the found upper bound is tight. Similarly to the previous example, we find that $\phi_{CT}(A(p^*)) = \tilde{\phi}_{CT}$ for $p^* = 1.000$, hence implying that $\tilde{\phi}_{CT}$ is tight, i.e., $\phi^*_{CT} = 4.357$.

Therefore, since $\mathcal{C} \not> \phi^*_{CT}$ where $\mathcal{C} = 4.017$ is the total network capacity given by (15), from Theorem 3 it follows that for some $p \in \mathcal{P}$ there does not exist a stabilizing state feedback $K(p)$ for the networked control system with multirate sampling (12) with either SER channel model or R-SER channel model.

VII. Conclusions

It has been shown that upper bounds of the largest instability measure in linear systems depending polynomially on parameters constrained in a semialgebraic set can be established via LMI feasibility tests. Moreover, the conservatism of these upper bounds has been investigated by proposing a priori and a posteriori conditions. Lastly, two special cases of the proposed methodology have been investigated, the first one concerning systems with a single parameter, and the second one concerning the determination of the largest spectral abscissa and radius. Future work will explore the possibility of using the proposed methodology in the design of linear systems for reducing the instability.

References

[1] G. Nair and R. J. Evans, "Stabilizability of stochastic linear systems with finite feedback data rates," *SIAM Journal on Control and Optimization*, vol. 43, no. 2, pp. 413–436, 2004.

[2] A. Matveev and A. Savkin, "Multirate stabilization of multiple sensor systems via limited capacity communication channels," *SIAM Journal on Control and Optimization*, vol. 44, no. 2, pp. 584–617, 2005.

[3] S. Tatikonda and S. K. Mitter, "Control under communication constraints," *IEEE Transactions on Automatic Control*, vol. 49, no. 7, pp. 1056–1068, 2004.

[4] K. Li and J. Baillieul, "Robust and efficient quantization and coding for control of multidimensional linear systems under data rate constraints," *International Journal of Robust and Nonlinear Control*, vol. 17, no. 10-11, pp. 898–920, 2007.

[5] S. Wan and L. Qiu, "Stabilization of networked control systems with finite data rate," in *Asian Control Conference*, Hong Kong, China, 2009, pp. 1067–1073.

[6] N. Elia and S. K. Mitter, "Stabilization of linear systems with limited information," *IEEE Transactions on Automatic Control*, vol. 46, no. 9, pp. 1384–1400, 2001.

[7] M. Fu and L. Xie, "The sector bound approach to quantized feedback control," *IEEE Transactions on Automatic Control*, vol. 50, no. 11, pp. 1698–1711, 2005.

[8] L. Qiu, "Quantify the unstable (semiplenary lecture)," in *International Symposium on Mathematical Theory of Networks and Systems*, Budapest, Hungary, 2010.

[9] G. Chesi, "Instability analysis of uncertain systems via determinants and LMIs," *IEEE Transactions on Automatic Control*, vol. 60, no. 9, pp. 2548–2563, 2015.

[10] ——, "Worst-case Mahler measure in polytopic uncertain systems," *IEEE Transactions on Automatic Control*, vol. 57, no. 12, pp. 3208–3213, 2012.

[11] K. Mahler, "An application of Jensen's formula to polynomials," *Mathematika*, vol. 7, pp. 98–100, 1960.

[12] J. H. Braslavsky, R. H. Middleton, and J. S. Freudenberg, "Feedback stabilization over signal-to-noise ratio constrained channels," *IEEE Transactions on Automatic Control*, vol. 52, no. 8, pp. 1391–1403, 2007.

[13] W. Chen and L. Qiu, "Stabilization of networked control systems with multirate sampling," *Automatica*, vol. 49, no. 6, pp. 1528–1537, 2013.

[14] G. Chesi, A. Tesi, A. Vicino, and R. Genesio, "On convexification of some minimum distance problems," in *European Control Conference*, Karlsruhe, Germany, 1999.

[15] P. A. Parrilo, “Structured semidefinite programs and semialgebraic geometry methods in robustness and optimization,” Ph.D. dissertation, California Institute of Technology, 2000.

[16] J.-B. Lasserre, “Global optimization with polynomials and the problem of moments,” *SIAM Journal of Optimization*, vol. 11, no. 3, pp. 796–817, 2001.

[17] S. Prajna, A. Papachristodoulou, and F. Wu, “Nonlinear control synthesis by sum of squares optimization: a Lyapunov-based approach,” in *Asian Control Conference*, 2004.

[18] C. W. J. Hol and C. W. Scherer, “Computing optimal fixed order $\mathcal{H}_\infty$-synthesis values by matrix sum of squares relaxations,” in *IEEE Conference on Decision and Control*, Paradise Island, Bahamas, 2004, pp. 3147–3153.

[19] G. Chesi, “LMI techniques for optimization over polynomials in control: a survey,” *IEEE Transactions on Automatic Control*, vol. 55, no. 11, pp. 2500–2510, 2010.

[20] B. Reznick, “Some concrete aspects of Hilbert’s 17th problem,” *Contemporary Mathematics*, vol. 253, pp. 251–272, 2000.

[21] G. Chesi, “On the gap between positive polynomials and SOS of polynomials,” *IEEE Transactions on Automatic Control*, vol. 52, no. 6, pp. 1066–1072, 2007.

[22] G. Stengle, “A nullstellensatz and a positivstellensatz in semialgebraic geometry,” *Math. Ann.*, vol. 207, pp. 87–97, 1974.

[23] M. Putinar, “Positive polynomials on compact semi-algebraic sets,” *Indian Univeristy Mathematics Journal*, vol. 42, no. 3, pp. 969–984, 1993.

[24] C. W. Scherer and C. W. J. Hol, “Matrix sum-of-squares relaxations for robust semi-definite programs,” *Mathematical Programming Series B*, vol. 107, no. 1-2, pp. 189–211, 2006.

[25] S. Prajna, A. Papachristodoulou, and P. A. Parrilo, “SOSTOOLS: a general purpose sum of squares programming solver,” in *IEEE Conference on Decision and Control*, Las Vegas, Nevada, 2002.

[26] J. F. Sturm, “Using SeDuMi 1.02, a MATLAB toolbox for optimization over symmetric cones,” *Optimization Methods and Software*, vol. 11-12, pp. 625–653, 1999.

[27] G. Chesi, “Establishing robust stability of discrete-time systems with time-varying uncertainty: the Gram-SOS approach,” *Automatica*, vol. 50, no. 11, pp. 2813–2821, 2014.