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Adaptive Parameter-Dependent Output Feedback Controllers Synthesis Through LMI-Based Optimization

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***Abstract*—A fundamental problem in engineering consists of designing adaptive output feedback controllers for stabilizing plants affected by parameters. This paper addresses this problem by proposing a novel approach for designing fixed-order fixed-degree adaptive parameter-dependent output feedback controllers. The proposed approach requires the solution of convex optimization problems with linear matrix inequalities, and provides a sufficient condition based on the construction of a function that quantifies a stability margin of the closed-loop system depending on the controller. This condition is nonconservative under some mild assumptions by increasing the size of the linear matrix inequalities.**

***Index Terms*—Adaptive Controller; Parameter-dependent; Stability; Linear matrix inequality.**

I. Introduction

A fundamental problem in engineering consists of stabilizing a plant. This is generally achieved by designing a stabilizing output feedback controller, i.e., a controller that elaborates the output of the plant in order to provide an input for the plant that makes the closed-loop system is stable. The design of such a controller is based on the model of the plant, and several techniques can be used.

Real plants are often affected by parameters. These can happen due to various reasons. One reason is that such parameters can represent quantities that the user can modify, such as the gain of an amplifier, in order to achieve a different performance. Another reason is that such parameters can represent quantities that are unknown or subject to changes, such as the mass, resistance, temperature, etc.

Whenever the plant is affected by parameters, the output feedback controller should be able to ensure stability for all admissible values of the parameters. For this, the controller should be dependent on the parameters in general, i.e., should be able to adapt to different plants corresponding to different values of the parameters. Such a controller would be, hence, adaptive, in particular parameter-dependent.

This paper addresses this problem, specifically, the design of adaptive output feedback controllers for stabilizing plants affected by time-invariant parameters. A preliminary conference version of this paper appeared as reported in [1].

It turns out that this is a difficult problem. Indeed, several conditions do exist in the literature for establishing stability of systems affected by parameters, in particular conditions based on convex optimization constrained by Linear Matrix Inequalities (LMIs); see for instance [2] [3] [4] [5] [6] [7] [8] [9]. However, such conditions lead to nonconvex optimization whenever a controller is searched for, due to the product of the Lyapunov function and the controller that generates Bilinear Matrix Inequalities (BMIs). See also [10] [11] for related studies. Also, several non-LMI strategies are available for the design of stabilizing feedback controllers for plants that are not affected by parameters, however, for plants affected by parameters, such strategies cannot be easily used in general.

In order to deal with this problem, a novel approach is proposed in this paper, which allows one to design a fixed-order fixed-degree adaptive parameter-dependent output feedback controller by solving convex optimization problems with LMIs. The proposed approach requires the solution of convex optimization problems with LMIs, and provides a sufficient condition based on the construction of a function that quantifies a stability margin of the closed-loop system depending on the controller. This function is searched for by exploiting polynomials that can be written as Sums Of Squares (SOS) of polynomials. The sufficient condition provided in this paper is nonconservative under some mild assumptions by increasing the size of the LMIs. Some numerical examples illustrate the proposed approach. This paper extends the technique for the design of robust static output feedback controllers proposed in our previous work [12].

The paper is organized as follows. Section II introduces the preliminaries. Section III discusses the motivation. Section IV describes the proposed approach. Section V present some illustrative examples. Lastly, Section VI concludes the paper with some final remarks.

II. Preliminaries

This section provides the preliminaries. Specifically, Section II-A introduces the problem formulation, and Section II-B reviews the class of SOS polynomials.

A. Problem Formulation

The notation adopted in this paper is as follows:

- $\mathbb{N}$: set of nonnegative integer numbers;
- $\mathbb{R}$: set of real numbers;
- $\mathbb{C}$: set of complex numbers;
- I: $n \times n$ identity matrix (of size specified by the context);
- A': transpose of matrix A;
- $adj(A)$: adjoint of matrix A;
- $\det(A)$: determinant of matrix A;
- $spec(A)$: set of eigenvalues of matrix A;
- $A > 0$: symmetric positive definite matrix A;
- $A \geq 0$: symmetric positive semidefinite matrix A;
- $\deg(a(x))$: degree of polynomial $a(x)$;
- s.t.: subject to.

Let us consider the plant

$$\left\{ \begin{array}{rcl} \dot{x}(t) & = & A_{pla}(p)x_{pla}(t) + B_{pla}(p)u(t) \\ y(t) & = & C_{pla}(p)x_{pla}(t) + D_{pla}(p)u(t) \end{array} \right. \quad (1)$$

where $t \in \mathbb{R}$ is the time, $x_{pla}(t) \in \mathbb{R}^{n_{pla}}$ is the state, $u(t) \in \mathbb{R}^{n_u}$ is the input, $y(t) \in \mathbb{R}^{n_y}$ is the output, $p \in \mathbb{R}^q$ is the vector of time-invariant parameters, and the matrices $A_{pla}(p)$, $B_{pla}(p)$, $C_{pla}(p)$ and $D_{pla}(p)$ are given matrix polynomials.

It is supposed that the vector of parameters is constrained into a semi-algebraic set, in particular

$$p \in \mathcal{P} \quad (2)$$

where

$$\mathcal{P} = \{p \in \mathbb{R}^q : \ a_i(p) \geq 0, \ i = 1 \ldots, n_a\} \quad (3)$$

and $a_i(p)$, $i = 1, \ldots, n_a$, are polynomials.

The plant (1) is controlled by the parameter-dependent output feedback controller

$$\left\{ \begin{array}{rcl} \dot{x_{con}}(t) & = & A_{con}(p)x_{con}(t) + B_{con}(p)y(t) \\ u(t) & = & C_{con}(p)x_{con}(t) + D_{con}(p)y(t) \end{array} \right. \quad (4)$$

where $x_{con}(t) \in \mathbb{R}^{n_{con}}$ is the state of chosen order $n_{con} \in \mathbb{N}$, and the matrices $A_{con}(p)$, $B_{con}(p)$, $C_{con}(p)$ and $D_{con}(p)$ are matrix polynomials to determine of chosen degree. For computation purpose, these matrix polynomials are expressed as

$$\left\{ \begin{array}{rcl} A_{con}(p) & = & \Phi_A(p, v) \\ B_{con}(p) & = & \Phi_B(p, v) \\ C_{con}(p) & = & \Phi_C(p, v) \\ D_{con}(p) & = & \Phi_D(p, v) \end{array} \right. \quad (5)$$

where $v \in \mathbb{R}^w$ is a vector of design variables, and $\Phi_A(p, v)$, $\Phi_B(p, v)$, $\Phi_C(p, v)$ and $\Phi_D(p, v)$ are matrix polynomials in p and v. The vector of design variables is searched for in the semi-algebraic set

$$\mathcal{V} = \{v \in \mathbb{R}^w : \ b_i(v) \geq 0, \ i = 1 \ldots, n_b\} \quad (6)$$

where $b_i(v)$, $i = 1, \ldots, n_b$, are polynomials. We denote the set of controllers (4) obtainable for $v \in \mathcal{V}$ as $\mathcal{C}$, i.e.,

$$\mathcal{C} = \{\langle \Phi_A(p, v), \Phi_B(p, v), \Phi_C(p, v), \Phi_D(p, v)\rangle : \ v \in \mathcal{V}\}. \quad (7)$$

The problem addressed in this paper is as follows.

Problem 1: Find a fixed-order fixed-degree output feedback controller (4) in the set $\mathcal{C}$ such that the closed-loop system (1)–(4) is well-posed and asymptotically stable for all parameters $p \in \mathcal{P}$.

□

Let us observe that Problem 1 contains several specific problems of interest, in particular the design of:

1) fixed-order (such as static) output feedback controllers for parameter-free systems, i.e., with no dependence on p;
2) common fixed-order (such as static) output feedback controllers for systems affected by parameters;
3) parameter-dependent (such as linearly) fixed-order (such as static) output feedback controllers for systems affected by parameters.

B. SOS Polynomials

Here we briefly review SOS polynomials; see for instance [13] and references therein for details. Let us start by introducing the following definition.

Definition 1: A polynomial $h(v)$ is said to be *SOS* if there exist polynomials $h_i(v)$, $i = 1, \ldots, k$, such that

$$h(v) = \sum_{i=1}^{k} h_i(v)^2. \quad (8)$$

□

A necessary and sufficient condition for establishing whether a polynomial is SOS can be given in terms of feasibility of an LMI. Specifically, let $d \in \mathbb{N}$ be such that

$$\deg(h(v)) \leq 2d. \quad (9)$$

Let $v^{\{d\}} \in \mathbb{R}^{\sigma(w,d)}$ be a vector whose entries are the monomials of degree not greater than d in x, e.g., according to

$$v^{\{d\}} = \left(1, v_1, \ldots, v_w, v_1^2, v_1 v_2, \ldots, v_w^d\right)' \quad (10)$$

where $\sigma(w,d)$ is the total number of such monomials given by

$$\sigma(w,d) = \frac{(w+d)!}{w!d!}. \quad (11)$$

Then, $h(v)$ can be expressed as

$$h(v) = v^{\{d\}'} \left(H + L(\alpha)\right) v^{\{d\}} \quad (12)$$

where $H \in \mathbb{R}^{\sigma(w,d)\times\sigma(w,d)}$ is a symmetric matrix such that

$$h(v) = v^{\{d\}'} H v^{\{d\}}, \quad (13)$$

$L : \mathbb{R}^{\omega(w,d)} \to \mathbb{R}^{\sigma(w,d)\times\sigma(w,d)}$ is a linear parametrization of the linear subspace

$$\mathcal{L}(w,d) = \{L = L' : \ w^{\{d\}'} L w^{\{d\}} = 0\}, \quad (14)$$

and $\alpha \in \mathbb{R}^{\omega(w,d)}$ is a free vector, where $\omega(w,d)$ is the dimension of $\mathcal{L}(w,d)$ given by

$$\omega(w,d) = \frac{1}{2}\sigma(w,d)(\sigma(w,d)+1) - \sigma(w,2d). \quad (15)$$

The representation (12) is known as Gram matrix method and square matricial representation (SMR). This representation allows one to establish whether a polynomial is SOS via an LMI feasibility test, which amounts to solving a convex optimization problem. Indeed, $h(v)$ is SOS if and only if there exists α satisfying the LMI

$$H + L(\alpha) \geq 0. \quad (16)$$

III. Motivation

This section explains the motivation for the proposed study. Specifically, Section III-A presents an example where the set of controllers that solve Problem 1 is nonconvex, and Section III-B presents an example where Problem 1 can be solved with a parameter-dependent controller but cannot be solved with a common controller.

A. Example 1

Hereafter, we present an example that highlights the difficulty of solving Problem 1, in particular showing that the set of controllers that solve this problem can be nonconvex.

Indeed, let us consider the plant (1) with

$$\begin{cases} A_{pla}(p) &= \begin{pmatrix} 0.4 & -0.5p-0.5 & -2 \\ 4 & 0.3 & 0.7p-3.5 \\ 0.8p+2.2 & 3 & -1.3 \end{pmatrix} \\ B_{pla}(p) &= \begin{pmatrix} 0 & -0.6 \\ -0.5 & 0.3 \\ 0 & -0.9 \end{pmatrix} \\ C_{pla}(p) &= \begin{pmatrix} 1 & 1 & -1.6 \end{pmatrix} \\ D_{pla}(p) &= \begin{pmatrix} 0 & 0 \end{pmatrix} \end{cases}$$

where the parameter p is constrained into the set

$$\mathcal{P} = [-1, 1].$$

This plant is controlled in closed-loop by the feedback controller (4) chosen of the form

$$\begin{cases} u(t) &= D_{con}(p)y(t) \\ \deg(D_{con}(p)) &= 0, \end{cases}$$

i.e., a common static output feedback controller. This controller is expressed as in (5) with

$$\Phi_D(p, v) = v$$

where $v \in \mathbb{R}^2$ is the vector of design variables constrained into the set

$$\mathcal{V} = [-3, 3]^2.$$

Figure 1 shows the set of controllers v that solve Problem 1 found by brute force. As it can be seen, this set is nonconvex in this case.

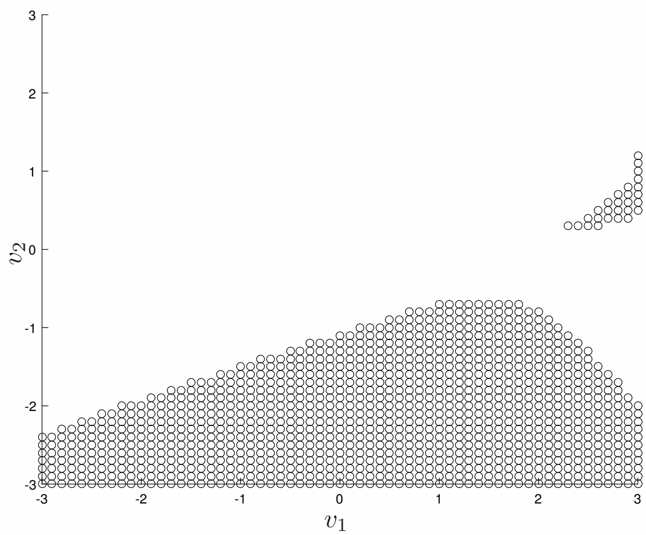

Fig. 1. Example 1. Set of controllers v that solve Problem 1 found by brute force. As it can be seen, the set is nonconvex in this case.

B. Example 2

Hereafter, we present an example that motivates the search for controllers that depend on the parameters for solving Problem 1, in particular showing that there exists such a controller but there does not exist any common controller that solves Problem 1.

Indeed, let us consider the plant (1) with

$$\begin{cases} A_{pla}(p) &= \begin{pmatrix} -1 & 0 & -p+1 \\ 0 & -1 & 1 \\ p+1 & 0 & 0 \end{pmatrix} \\ B_{pla}(p) &= \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \\ C_{pla}(p) &= \begin{pmatrix} 1 & p & 0 \end{pmatrix} \\ D_{pla}(p) &= 0 \end{cases}$$

where the parameter p is constrained into the set

$$\mathcal{P} = [-2, 2].$$

This plant is controlled in closed-loop by the feedback controller (4) chosen of the form

$$\begin{cases} u(t) & = & D_{con}(p)y(t) \\ \deg(D_{con}(p)) & \leq & 1, \end{cases}$$

i.e., a parameter-dependent static output feedback controller of degree not greater than 1 in the parameter. This controller is expressed as in (5) with

$$\Phi_D(p, v) = v_1 + v_2 p$$

where $v \in \mathbb{R}^2$ is the vector of design variables constrained into the set

$$\mathcal{V} = [-2, 2]^2.$$

Figure 2 shows the set of controllers v that solve Problem 1 found by brute force. As it can be seen, there exist parameter-dependent controllers but there do not exist common controllers that solve Problem 1 in this case.

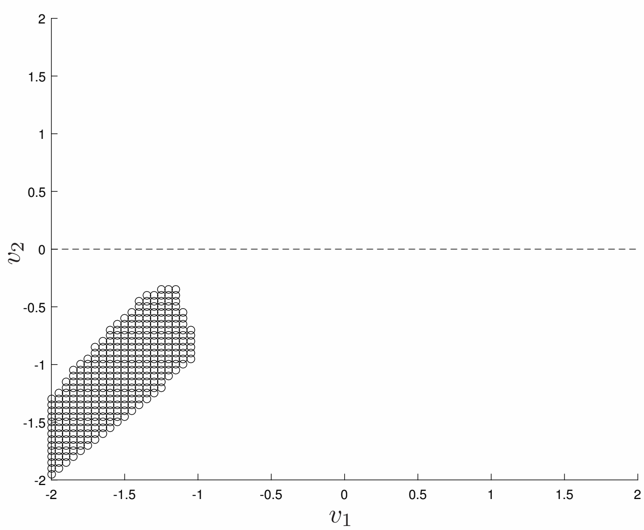

Fig. 2. Example 2. Set of controllers v that solve Problem 1 found by brute force. The dashed line denotes the region of common controllers. As it can be seen, there exist parameter-dependent controllers but there do not exist common controllers that solve Problem 1 in this case.

IV. PROPOSED APPROACH

This section provides the proposed approach. Specifically, Section IV-A derives the equation of the closed-loop system, Section IV-B investigates the well-posedness, Section IV-C tackles the asymptotical stability, Section IV-D derives the optimization problem used to determine a sought solution, Section IV-E explains how to determine such a solution, Section IV-F investigates the non-conservatism of the proposed approach, and Section IV-G reports some remarks.

A. Closed-Loop System

The first step of the proposed approach is to express the closed-loop system (1)–(4) as

$$\dot{x}(t) = A(p, v)x(t) \tag{17}$$

where $x \in \mathbb{R}^n$ is the state

$$x(t) = \begin{pmatrix} x_{pla}(t) \\ x_{con}(t) \end{pmatrix} \tag{18}$$

of dimension

$$n = n_{pla} + n_{con}, \tag{19}$$

and $A(p, v)$ is a matrix rational function in p and v. In particular, the expression of $A(p, v)$ is given by

$$A(p, v) = \begin{pmatrix} A_1(p, v) & A_2(p, v) \\ A_3(p, v) & A_4(p, v) \end{pmatrix} \tag{20}$$

where

$$\begin{cases} A_1(p, v) & = & B_{pla}(p)E(p, v)^{-1}\Phi_D(p, v)C_{pla}(p) \\ & & +A_{pla}(p) \\ A_2(p, v) & = & B_{pla}(p)E(p, v)^{-1}\Phi_C(p, v) \\ A_3(p, v) & = & \Phi_B(p, v)\left(D_{pla}(p)E(p, v)^{-1}\Phi_D(p, v)\right. \\ & & \left.+I\right)C_{pla}(p) \\ A_4(p, v) & = & \Phi_B(p, v)D_{pla}(p)E(p, v)^{-1}\Phi_C(p, v) \\ & & +\Phi_A(p, v) \end{cases} \tag{21}$$

and

$$E(p, v) = I - \Phi_D(p, v)D_{pla}(p). \tag{22}$$

The matrix rational function $A(p, v)$ can be expressed as

$$A(p, v) = \frac{A_{num}(p, v)}{A_{den}(p, v)} \tag{23}$$

where $A_{num}(p, v)$ is a matrix polynomial and $A_{den}(p, v)$ is a polynomial in p and v. Let us observe that $A_{num}(p, v)$ and $A_{den}(p, v)$ are non-unique since they are defined up to a scaling function. Hereafter, we adopt the following expressions for $A_{num}(p, v)$ and $A_{den}(p, v)$:

$$\begin{cases} A_{num}(p, v) & = & adj(E(p, v)) \\ A_{den}(p, v) & = & \det(E(p, v)). \end{cases} \tag{24}$$

B. Well-Posedness

The second step of the proposed approach addresses the well-posedness of the closed-loop system (1)–(4). Let us start by formally defining this concept as follows.

Definition 2: The closed-loop system (1)–(4) is said to be *well-posed* for all parameters $p \in \mathcal{P}$ for some controller v if the matrix $A(p, v)$ in (17) does exist for all $p \in \mathcal{P}$.

□

Let us observe that the matrix $A(p,v)$ in (17) does exist for all parameters $p \in \mathcal{P}$ for some controller v whenever

$$E(p,v) \ \ \text{is non-singular} \ \ \forall p \in \mathcal{P}. \tag{25}$$

In this paper we impose well-posedness of the closed-loop system (1)–(4) by requiring that

$$|A_{den}(p,v)| > \rho_{wep} \ \ \forall p \in \mathcal{P} \tag{26}$$

where $\rho_{wep} \geq 0$ is an arbitrary chosen threshold.

C. Asymptotical Stability

The third step of the proposed approach consists of ensuring asymptotical stability of the closed-loop system (1)–(4). Let us start by formally defining this concept as follows.

Definition 3: The closed-loop system (1)–(4) is said to be *asymptotically stable* for all parameters $p \in \mathcal{P}$ for some controller v if

$$\Re(\lambda) < 0 \ \ \forall \lambda \in spec\,(A(p,v)) \ \ \forall p \in \mathcal{P}. \tag{27}$$

□

In this paper we impose asymptotical stability of the closed-loop system (1)–(4) by requiring that

$$\Re(\lambda) < -\rho_{sta} \ \ \forall \lambda \in spec\,(A(p,v)) \ \ \forall p \in \mathcal{P} \tag{28}$$

where $\rho_{sta} \geq 0$ is an arbitrary chosen threshold.

In order to impose this constraint, let $\theta \in \Theta$ be an auxiliary variable, where

$$\Theta = \{-1, 1\}, \tag{29}$$

and let us define the characteristic polynomial of $\theta A_{num}(p,v)$ as

$$c(\lambda,p,v) = \det\left(\lambda I - \theta\left(A_{num}(p,v) + \rho_{sta} A_{den}(p,v) I\right)\right) \tag{30}$$

where $\lambda \in \mathbb{R}$ is an auxiliary variable. Let us express this characteristic polynomial as

$$c(\lambda,p,v) = \sum_{i=0}^{n} \tilde{c}_i(p,v)\lambda^i \tag{31}$$

where $\tilde{c}_i(p,v)$, $i = 1,\ldots,n$, are polynomials in p and v. Let us build the table

$$\begin{array}{ccc} r_{1,1}(p,v) & r_{1,2}(p,v) & \ldots \\ r_{2,1}(p,v) & r_{2,2}(p,v) & \ldots \\ \vdots & \vdots & \ddots \end{array} \tag{32}$$

where the generic entry in position (i,j), $i = 3,\ldots,n+1$ and $j = 1,2,\ldots$, is given by

$$\begin{array}{rcl} r_{i,j}(p,v) & = & r_{i-1,j}(p,v) r_{i-2,j+1}(p,v) \\ & & -r_{i-2,j}(p,v) r_{i-1,j+1}(p,v) \end{array} \tag{33}$$

by using the initialization

$$\left\{ \begin{array}{rcl} r_{1,j}(p,v) & = & \tilde{c}_{n+1-2j}(p,v) \\ r_{2,j}(p,v) & = & \tilde{c}_{n-2j}(p,v). \end{array} \right. \tag{34}$$

Let us observe that the entries of the built table are polynomials in p and v.

The closed-loop system (1)–(4) is asymptotically stable for all parameters $p \in \mathcal{P}$ for some controller v if and only if

$$\left\{ \begin{array}{rcl} r_{i,1}(p,v) & > & 0 \ \ \forall i = 2,\ldots,n+1 \ \forall p \in \mathcal{P} \\ \theta A_{den}(p,v) & > & 0. \end{array} \right. \tag{35}$$

D. Optimization Problem

The fourth step of the proposed approach consists of introducing an optimization problem that will be used to determine a controller that solves Problem 1, if any.

Let us start by defining the polynomials $f_i(p,v)$, $i = 1,\ldots,n+1$, in p and v as

$$\left\{ \begin{array}{rcl} f_1(p,v) & = & \theta A_{den}(p,v) - \rho_{wep} \\ f_i(p,v) & = & r_{i,1}(p,v) \ \ \forall i = 2,\ldots,n+1. \end{array} \right. \tag{36}$$

Let $\xi(v)$, $\beta_{i,j}(p,v)$, $\gamma_{i,k}(p,v)$ and $\delta_k(v)$ be auxiliary polynomial variables, $i = 1,\ldots,n+1$, $j = 1,\ldots,n_a$ and $k = 1,\ldots,n_b$, and let us define

$$\left\{ \begin{array}{rcl} g_i(p,v) & = & f_i(p,v) - \xi(v) - \displaystyle\sum_{j=1}^{n_a} a_j(p)\beta_{i,j}(p,v) \\ & & - \displaystyle\sum_{k=1}^{n_b} b_k(v)\gamma_{i,k}(p,v) \\ h(v) & = & \rho_{pos} - \xi(v) - \displaystyle\sum_{k=1}^{n_b} b_k(v)\delta_k(v) \end{array} \right. \tag{37}$$

where $\rho_{pos} > 0$ is a chosen threshold whose role will be clarified in the sequel. Let us define the integral of $\xi(v)$ over $\mathcal{V}$ as

$$\Xi = \int_{\mathcal{V}} \xi(v) dv. \tag{38}$$

Let us observe that Ξ is a linear function of the coefficients of $\xi(v)$.

Let us define the optimization problem

$$\begin{array}{l} \Xi^* = \displaystyle\sup_{\xi,\beta_{i,j},\gamma_{i,k},\delta_k} \Xi \\ \text{s.t.} \left\{ \begin{array}{l} g_i(p,v),\ h(v) \ \ \text{are SOS} \\ \beta_{i,j}(p,v),\ \gamma_{i,k}(p,v),\ \delta_k(v) \ \ \text{are SOS} \\ \forall i = 1,\ldots,n+1 \\ \forall j = 1,\ldots,n_a \\ \forall k = 1,\ldots,n_b. \end{array} \right. \end{array} \tag{39}$$

The optimization problem (39) is convex. Indeed, the cost function is linear in the decision variables, which are the coefficients of the polynomials $\xi(v)$, $\beta_{i,j}(p,v)$, $\gamma_{i,k}(p,v)$ and $\delta_k(v)$. Moreover, the constraints impose that some polynomials, which depend affine linearly on the decision variables, are SOS. From Section II-B it follows that these constraints are equivalent to LMIs in the decision variables and auxiliary variables. Therefore, the optimization problem (39) is convex since the cost function is convex and since the feasible set is convex.

Let us observe that the polynomial $\xi(v)$ quantifies a stability margin of the closed-loop system depending on the controller. This polynomial generalizes the concept of robust stabilizability function introduced in [12] for the design of robust static output feedback controllers.

E. Determining The Controller

The fifth step of the proposed approach consists of determining a controller that solves Problem 1, if any, from the solution of the optimization problem (39).

Specifically, let $h^*(v)$ and $\xi^*(v)$ be the optimal values of the polynomials $h(v)$ and $\xi(v)$ in the optimization problem (39). Let us define the set

$$\mathcal{H} = \{ v \in \mathbb{R}^w : \; h^*(v) = 0, \; \xi^*(v) = \rho_{pos}, \; v \in \mathcal{V} \}. \quad (40)$$

The following theorem explains how Problem 1 can be solved with the proposed approached.

Theorem 1: All vectors v in the set $\mathcal{H}$, if any, define a controller (4) with matrices given by (5) that solves Problem 1.

Proof. See Appendix 1. □

Theorem 1 provides a sufficient condition for the solution of Problem 1. As it will be explained in the next section, this condition is nonconservative provided that some assumptions hold. The condition of Theorem 1 is based on the set $\mathcal{H}$, which is determined once that the optimization problem (39) has been solved.

How to determine the set $\mathcal{H}$? This can be done according to the following two steps:

1) search for the zeros of $h^*(v)$;
2) keep the zeros of $h^*(v)$ that satisfy $\xi^*(v) = \rho_{pos}$ and $v \in \mathcal{V}$.

The first step can be addressed via linear algebra operations once that the optimization problem (39) has been solved. A possibility consists of using the method proposed in [14] for solving systems of polynomial equations as explained hereafter:

1) once that the optimization problem (39) has been solved, one obtains from the LMI solver a positive semidefinite Gram matrix of $h^*(v)$, i.e., a matrix $H^* \geq 0$ such that
$$h^*(v) = v^{\{d\}'} H^* v^{\{d\}} \quad (41)$$
where $v^{\{d\}}$ is a vector of monomials in v;
2) since $H^* \geq 0$, one has that $h^*(v) = 0$ if and only if
$$v^{\{d\}} \in \ker(H^*). \quad (42)$$
Hence, the problem of finding the zeros of $h^*(v)$ is equivalent to the problem of finding vectors of monomials in $\ker(H^*)$;
3) the problem of finding vectors of monomials in $\ker(H^*)$ can be addressed by pivoting operations that reduce the problem to finding the roots of a polynomial in a single variable whenever the dimension of $\ker(H^*)$ is smaller than a certain value as shown in [14].

The second step is trivial since the number of zeros of $h^*(v)$ is finite for non-degenerate cases (one just keep the zeros that satisfy $\xi^*(v) = \rho_{pos}$ and $v \in \mathcal{V}$ through individual tests).

F. Non-Conservatism

The previous section has provided a sufficient condition for the solution of Problem 1 through Theorem 1. As it will be explained in the next section, this condition may be nonconservative. Let us start by introducing the following assumption.

Assumption 1: The sets $\mathcal{P}$ and $\mathcal{V}$ are compact. Moreover, the polynomials $a_i(p)$, $i = 1, \ldots, n_a$, in (3) and $b_i(v)$, $i = 1, \ldots, n_b$, in (6) have even degree, and their highest degree forms have no common root except zero.

□

The following theorem explains that the sufficient condition provided by Theorem 1 is nonconservative whenever Assumption 1 holds.

Theorem 2: Suppose that there exists a controller (4) with matrices given by (5) for some $v \in \mathcal{V}$ that solves Problem 1. Also, suppose that Assumption 1 holds. Then, the set $\mathcal{H}$ is nonempty for some $\theta \in \Theta$ for any sufficiently small threshold $\rho_{pos} > 0$.

Proof. See Appendix 2. □

It is worth observing that Assumption 1 introduces mild assumptions on Problem 1. Indeed, it is reasonable to assume that the sets $\mathcal{P}$ and $\mathcal{V}$ are compact, since the allowed values for parameters and controllers are bounded in practice, and since having $\mathcal{P}$ and $\mathcal{V}$ closed rather than open does not make differences in general. Also, one can assume without loss of generality that the polynomials $a_i(p)$, $i = 1, \ldots, n_a$, in (3) and $b_i(v)$, $i = 1, \ldots, n_b$, in (6) have even degree, because, if not, one could multiply the polynomials with odd degree times linear functions that are positive over $\mathcal{P}$ and $\mathcal{V}$ in order to fill this requirement without modifying $\mathcal{P}$ and $\mathcal{V}$. Lastly, the assumption that the highest degree forms of these polynomials have no common root except zero is automatically satisfied in many cases of interests. For instance, ellipsoids can be described by polynomials such as

$$a_1(p) = p' M_2 p + M_1' p + M_0 \tag{43}$$

where $M_0 \in \mathbb{R}$, $M_1 \in \mathbb{R}^q$ and $M_2 \in \mathbb{R}^{q \times q}$, with $M_2 > 0$, and the highest degree form is $p' M_2 p$ whose only possible root is the origin. Moreover, interval sets can be described by polynomials such as

$$a_i(p) = (m_{i,-} - p_i)(p_i - m_{i,+}) \quad \forall i = 1, \ldots, q \tag{44}$$

where $m_{i,-}, m_{i,+} \in \mathbb{R}$, with $m_{i,-} \leq m_{i,+}$, and the highest degree forms are $-p_i^2$ whose only possible common root is the origin.

G. Remarks

The first remarks concerns the well-posedness of the closed-loop system (1)–(4). Let us observe that, whenever the plant (1) is strictly proper (i.e., $D_{pla}(p) = 0$) or the controller (4) is strictly proper (i.e., $D_{con}(p) = 0$), well-posedness is automatically satisfied, and (26) holds for any $\rho_{wep} \in [0, 1)$. This means that the polynomial $f_1(p, v)$ in (36) does not need to be introduced, and one can simply redefine (36) as

$$f_{i-1}(p, v) = r_{i,1}(p, v) \quad \forall i = 2, \ldots, n+1. \tag{45}$$

The second remark concerns the polynomials $f_i(p, v)$ in (36). The polynomials $f_i(p, v)$ that are known to be positive over $\mathcal{P} \times \mathcal{V}$ (such as positive constants) do not need to be introduced, since the proposed approach aims to collect in the set $\mathcal{H}$ vectors v such that the polynomials $f_i(p, v)$ are positive for all parameters $p \in \mathcal{P}$. This also implies that, if there exists a polynomial $f_i(p, v)$ that is known to be non-positive for all parameters $p \in \mathcal{P}$ (such as non-positive constants), then Problem 1 has no solution.

The third remark concerns the threshold ρ_{pos} introduced in the polynomial $h(v)$ in (37). This threshold has to be chosen as a positive number, and it is introduced in order to ensure that the vectors v in the set $\mathcal{H}$ satisfy $\xi^*(v) > 0$ (since $\xi^*(v) = \rho_{pos}$). As said in the statement of Theorem 2, non-conservatism is ensured whenever ρ_{pos} is a sufficiently small positive number. Hence, one can simply choose ρ_{pos} as the smallest positive number allowed by the used computer.

V. Examples

In this section we present some illustrative examples of the proposed results. The computations are done in Matlab using the toolbox SeDuMi [15]. The threshold ρ_{wep} is not introduced since the considered plants are strict, and the other thresholds are chosen as $\rho_{sta} = 0$ and $\rho_{pos} = 0.1$. The degrees of the polynomials $\beta_{i,j}(p, v)$, $\gamma_{i,k}(p, v)$ and $\delta_k(v)$ are chosen as the largest degrees ensuring that the polynomials $g_{i,j}(p, v)$ and $h(v)$ have their minimum degrees.

A. Example 1 (continued)

Let us continue Example 1. Let us start by observing that the plant (1) is unstable for some values of the parameter, for instance

$$p = 0 \quad \Rightarrow \quad spec(A_{pla}(p)) = \{-1.049, 0.224 \pm j4.066\}.$$

This fact is also shown by Figure 3, which shows the eigenvalues of the plant for some values of the parameters in $\mathcal{P}$.

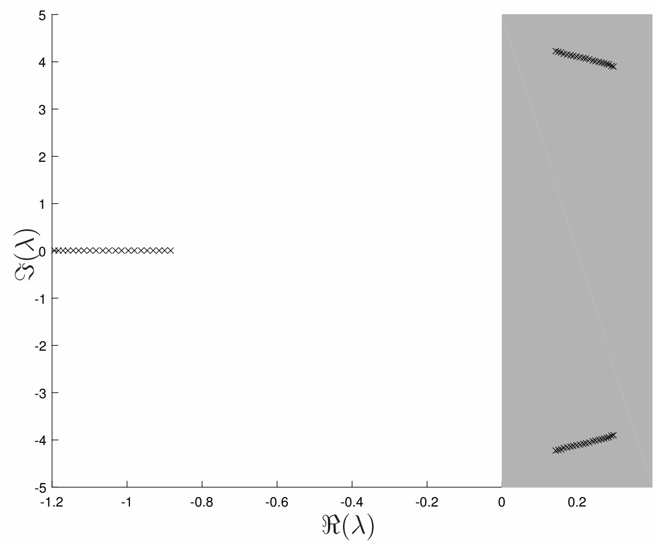

Fig. 3. Example 1. Eigenvalues λ of the plant (1) for some values of the parameters in $\mathcal{P}$. The dashed area denotes the region of unstable eigenvalues.

Let us describe the sets $\mathcal{P}$ and $\mathcal{V}$ as in (3) and (6) with

$$\left\{ \begin{array}{rcl} a_1(p) & = & 1 - p^2 \\ b_i(v) & = & 9 - v_i^2 \quad \forall i = 1, 2. \end{array} \right.$$

Let us solve the optimization problem (39) by using a polynomial variable $\xi(v)$ of degree not greater than 2. We find

$$\left\{ \begin{array}{rcl} \Xi^* & = & -703.772 \\ \xi^*(v) & = & -1.389 v_1^2 + 0.977 v_1 v_2 + 5.686 v_1 - 1.482 v_2^2 \\ & & -7.230 v_2 - 10.936. \end{array} \right.$$

Next, we determine the set $\mathcal{H}$ as explained in Section IV-E, finding

$$\mathcal{H} = \left\{ \left(\begin{array}{c} 1.345 \\ -1.995 \end{array} \right) \right\}.$$

Therefore, from Theorem 1 we conclude that the controller obtained from the vector v in $\mathcal{H}$, i.e.,

$$D_{con}(p) = \begin{pmatrix} 1.345 \\ -1.995 \end{pmatrix}$$

solves Problem 1. This fact is also shown by Figure 4, which shows the eigenvalues of the closed-loop system (1)–(4) obtained with the found controller for some values of the parameters in $\mathcal{P}$.

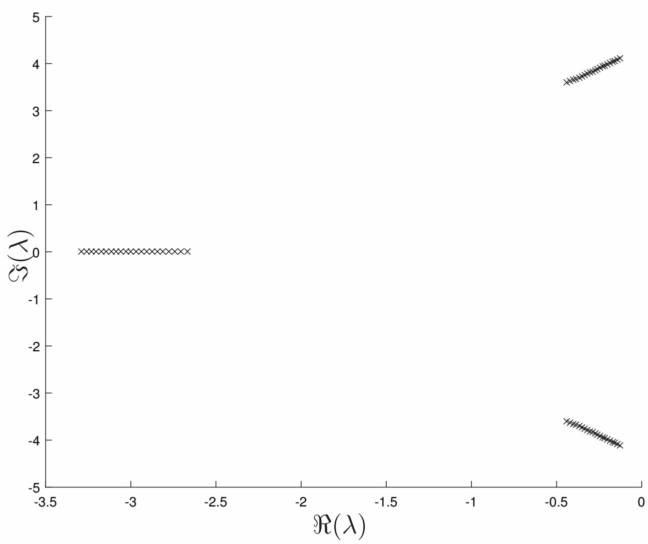

Fig. 4. Example 1. Eigenvalues λ of the closed-loop system (1)–(4) for some values of the parameters in $\mathcal{P}$.

B. Example 2 (continued)

Let us continue Example 2. Let us start by observing that the plant (1) is unstable for some values of the parameter, for instance

$$p = 0 \quad \Rightarrow \quad spec(A_{pla}(p)) = \{-1.618, -1, 0.618\}.$$

This fact is also shown by Figure 5, which shows the eigenvalues of the plant for some values of the parameters in $\mathcal{P}$.

Let us describe the sets $\mathcal{P}$ and $\mathcal{V}$ as in (3) and (6) with

$$\begin{cases} a_1(p) & = & 4 - p^2 \\ b_i(v) & = & 4 - v_i^2 \quad \forall i = 1, 2. \end{cases}$$

Let us solve the optimization problem (39) by using a polynomial variable $\xi(v)$ of degree not greater than 3. We find

$$\begin{cases} \Xi^* & = & -160.105 \\ \xi^*(v) & = & 0.149v_1^3 - 0.912v_1^2v_2 - 1.288v_1^2 \\ & & -0.598v_1v_2^2 - 0.676v_1v_2 - 1.501v_1 \\ & & +0.377v_2^3 - 2.822v_2^2 - 6.242v_2 - 4.527. \end{cases}$$

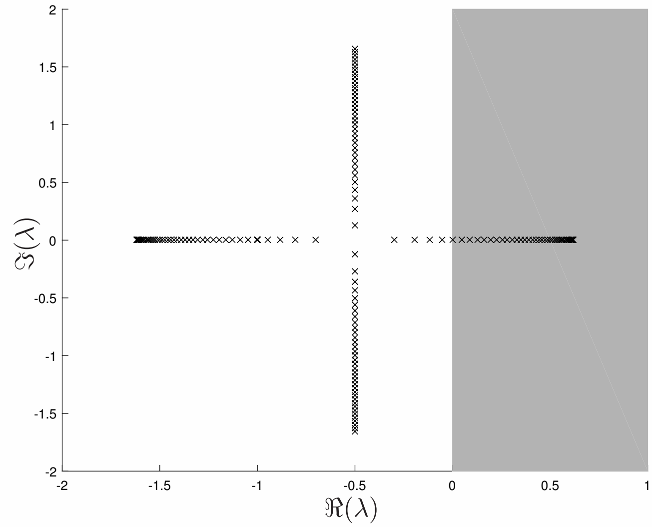

Fig. 5. Example 2. Eigenvalues λ of the plant (1) for some values of the parameters in $\mathcal{P}$. The dashed area denotes the region of unstable eigenvalues.

Next, we determine the set $\mathcal{H}$ as explained in Section IV-E, finding

$$\mathcal{H} = \left\{ \begin{pmatrix} -2.000 \\ -1.663 \end{pmatrix} \right\}.$$

Therefore, from Theorem 1 we conclude that the controller obtained from the vector v in $\mathcal{H}$, i.e.,

$$D_{con}(p) = -2 - 1.663p$$

solves Problem 1. This fact is also shown by Figure 6, which shows the eigenvalues of the closed-loop system (1)–(4) obtained with the found controller for some values of the parameters in $\mathcal{P}$.

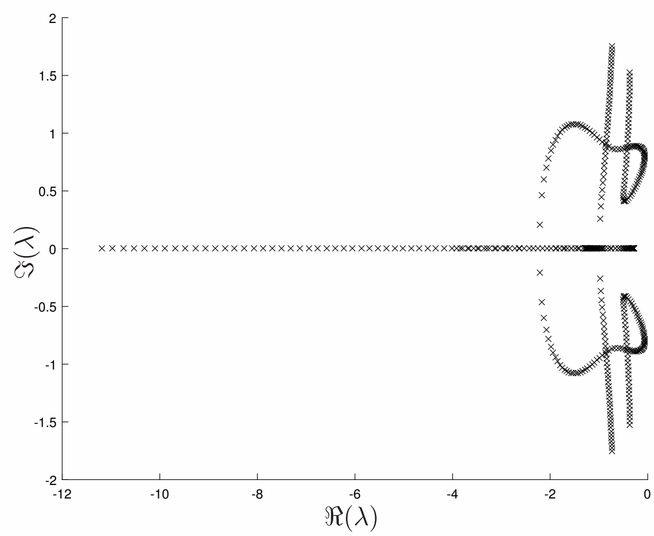

Fig. 6. Example 2. Eigenvalues λ of the closed-loop system (1)–(4) for some values of the parameters in $\mathcal{P}$.

VI. Conclusions

This paper has addressed the design of adaptive output feedback controllers for stabilizing plants affected by time-invariant parameters in the time-invariant case, which is a fundamental problem in engineering. A novel approach has been proposed for designing fixed-order fixed-degree adaptive parameter-dependent output feedback controllers. The proposed approach requires the solution of convex optimization problems with LMIs, and provides a sufficient condition based on the construction of a function that quantifies a stability margin of the closed-loop system depending on the controller. This condition is nonconservative under some mild assumptions by increasing the size of the LMIs. Future work can consider various directions. For instance, one could extend the proposed approach to the case of discrete-time systems. Also, the proposed approach could be generalized in order to deal with time-varying parameters.

Appendix 1

Proof of Theorem 1. Let $g_i^*(p,v)$ be the optimal value of $g_i(p,v)$ in the optimization problem (39). One has that $g_i^*(p,v)$ is a SOS polynomial. It follows that, for all $i = 1,\ldots,n+1$,

$$\begin{array}{rcl} 0 & \leq & g_i^*(p,v) \\ & = & f_i(p,v) - \xi^*(v) - \displaystyle\sum_{j=1}^{n_a} a_j(p)\beta_{i,j}^*(p,v) \\ & & -\displaystyle\sum_{k=1}^{n_b} b_k(v)\gamma_{i,k}^*(p,v) \end{array}$$

where $\xi^*(v)$, $\beta_{i,j}^*(p,v)$ and $\gamma_{i,k}^*(p,v)$ are the optimal values of $\xi(v)$, $\beta_{i,j}(p,v)$ and $\gamma_{i,k}(p,v)$ in the optimization problem (39). One has that $\beta_{i,j}^*(p,v)$ and $\gamma_{i,k}^*(p,v)$ are SOS polynomials. Suppose $v^* \in \mathcal{H}$. Then, $h^*(v^*) = 0$, $\rho_{pos} = \xi^*(v^*)$ and $v^* \in \mathcal{V}$. Let $p^* \in \mathcal{P}$. One has $a_j(p^*) \geq 0$ and $b_k(v^*) \geq 0$. It follows that, for all $i = 1,\ldots,n+1$,

$$\begin{array}{rcl} 0 & \leq & f_i(p^*,v^*) - \xi^*(v^*) - \displaystyle\sum_{j=1}^{n_a} a_j(p^*)\beta_{i,j}^*(p^*,v^*) \\ & & -\displaystyle\sum_{k=1}^{n_b} b_k(v^*)\gamma_{i,k}^*(p^*,v^*) \\ & \leq & f_i(p^*,v^*) - \rho_{pos}. \end{array}$$

Since $\rho_{pos} > 0$ one has

$$f_i(p^*,v^*) > 0 \quad \forall i = 1,\ldots,n+1.$$

From Sections IV-B–IV-C this implies that the controller v^* ensures that the closed-loop system (1)–(4) is well-posed and asymptotically stable for all parameters $p \in \mathcal{P}$. Therefore, v^* solves Problem 1.

Appendix 2

Proof of Theorem 2. Suppose that there exists a controller $\bar{v} \in \mathcal{V}$ that solves Problem 1. From Sections IV-B–IV-C this implies that

$$f_i(p,\bar{v}) > 0 \quad \forall i = 1,\ldots,n+1 \ \forall p \in \mathcal{P}$$

for some $\theta \in \Theta$. Let us define the function

$$\bar{f}(v) = \inf_{\substack{p\in\mathcal{P} \\ i=1,\ldots,n+1}} f_i(p,v).$$

It follows that

$$\bar{f}(\bar{v}) > 0.$$

Suppose that Assumption 1 holds. It follows that $\mathcal{V}$ is compact. Hence, there exists a polynomial $\xi(v)$ that approximates arbitrary well $\bar{f}(v)$ over $\mathcal{V}$, in particular such that

$$\left\{ \begin{array}{rcl} \bar{f}(v) & \geq & \xi(v) \quad \forall v \in \mathcal{V} \\ \rho_{pos} & \geq & \xi(v) \quad \forall v \in \mathcal{V} \\ \xi(\bar{v}) & > & 0. \end{array} \right.$$

Since $\mathcal{P}$ is compact, and since the polynomials $a_i(p)$, $i = 1,\ldots,n_a$, in (3) and $b_i(v)$, $i = 1,\ldots,n_b$, in (6) have even degree, and their highest degree forms have no common root except zero, it follows from [16] that there exist polynomials $\beta_{i,j}(p,v)$, $\gamma_{i,k}(p,v)$ and $\delta_k(v)$ such that the constraints of the optimization problem (39) hold. Since the objective of this optimization problem is to maximize the integral of $\xi(v)$ over $\mathcal{V}$, it follows that there exists $\rho_{pos} > 0$ such that

$$\xi^*(v^*) = \rho_{pos}$$

for some $v^* \in \mathcal{V}$, where $\xi^*(v)$ is the optimal value of $\xi(v)$ in this optimization problem. Since $h^*(v)$ is a SOS polynomial, one has

$$\begin{array}{rcl} 0 & \leq & h^*(v^*) \\ & = & \rho_{pos} - \xi^*(v^*) - \displaystyle\sum_{k=1}^{n_b} b_k(v^*)\delta_k^*(v^*) \\ & = & -\displaystyle\sum_{k=1}^{n_b} b_k(v^*)\delta_k^*(v^*) \end{array}$$

where $\delta_k^*(v)$ is the optimal value of $\delta_k(v)$ in the optimization problem (39). Since $\delta_k^*(v)$ is a SOS polynomial, one concludes that

$$0 \leq h^*(v^*) \leq 0,$$

which implies that $h^*(v^*) = 0$. Therefore, $v^* \in \mathcal{H}$, and the set $\mathcal{H}$ is nonempty for the considered value of $\theta \in \Theta$.

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