Cross-Hedging Ambiguous Exchange Rate Risk *

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This paper examines the behavior of an exporting firm that sells its output to two foreign countries, only one of which has futures and options available for its currency. The firm possesses smooth ambiguity preferences and faces multiple sources of ambiguous exchange rate risk. We show that the separation theorem fails to hold in that the firm's production and export decisions depend on the firm's attitude towards ambiguity and on the incident to the underlying ambiguity. Given that the random spot exchange rates are first-order independent with respect to each plausible subjective distribution, we derive necessary and sufficient conditions under which the full-hedging theorem applies to the firm's cross-hedging decisions. When these conditions are violated, we show that the firm includes options in its optimal hedge position. This paper as such offers a rationale for the hedging role of options under smooth ambiguity preferences and cross-hedging of ambiguous exchange rate risk.

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1. Introduction

According to the 2013 triennial central bank survey of turnover in foreign exchange markets coordinated by the Bank for International Settlements (BIS), the US dollar remained the dominant vehicle currency that accounted for 87% of the total daily global turnover in April 2013. The euro and the Japanese yen were the second and third most traded currencies, respectively. Trading activities increased strongly in currency forwards and options. Trading volumes of currency forwards reached $680 billion in 2013 from $475 billion in 2010, a 43% increase, while those of currency options increased the most, by more than 60%. The rise in turnover of currency forwards and options together accounted for almost a quarter of global foreign exchange turnover growth between 2010 and 2013.

The vast majority of trading of foreign exchange instruments continues to be conducted over the counter and not traded on organized exchanges, rendering the prevailing use of non-standardized contractual terms. The minor currencies of the bottom 29 out of 53 countries included in the BIS triennial survey had an average of 0.134% of the total daily global foreign exchange turnover in April 2013, which consisted of 0.043% in spot transactions, 0.041% in outright forwards, 0.042% in foreign exchange swaps, 0.001% in currency swaps, and 0.007% in foreign exchange options. These figures suggest that trading of foreign exchange instruments of these minor currencies is likely to be thin and illiquid, thereby resulting in prohibitively high transaction costs. Firms that have positions in these minor currencies are as such induced to look for foreign exchange instruments of a related currency for hedging purposes. Such an exchange rate risk management technique is referred to as "cross-hedging."

The purpose of this paper is to provide theoretical insights into the decision making of an exporting firm that sells its output to two foreign countries, only one of which has futures and options available for its currency. To this end, we depart from the extant literature that is developed within the standard von Neumann-Morgenstern expected utility

paradigm (Adam-Müller, 2000; Broll and Zilcha, 1992; Change and Wong, 2003; Viaene and Zilcha, 1998; Wong, 2013a, 2013b). Taking into account the possibility that the firm cannot unambiguously assign a probability distribution that uniquely describes the exchange rate risk, we define uncertainty in the sense of Knight (1921) to be made up of two components, risk and ambiguity. While risk aversion is the aversion to a set of outcomes with a known probability distribution, ambiguity aversion is the additional aversion to being unsure about the probabilities of outcomes.[1] The distinction between the known-unknown and the unknown-unknown is relevant, which is justifiable by ample experiments (Chow and Sarin, 2001; Einhorn and Hogarth, 1986; Sarin and Weber, 1993) and surveys (Chesson and Viscusi, 2003; Viscusi and Chesson, 1999) that document convincing evidence that individuals appear to prefer gambles with known rather than unknown probabilities.

In this paper, we adopt the approach of Klibanoff et al. (2005) (hereafter referred to as the KMM model) to characterize the firm's preferences by "smooth ambiguity aversion." The KMM model has the following recursive structure. First, ambiguity is represented by a second-order probability distribution that captures the firm's uncertainty about which of the subjective beliefs govern the exchange rate risk. Second, the firm's expected utility under ambiguity is measured by taking the second-order expectation of a concave transformation of the first-order expected utility of profit conditional on each plausible subjective distribution of the exchange rate risk. This recursive structure creates a crisp separation between ambiguity and ambiguity aversion, i.e., between beliefs and tastes, which allows the conventional techniques in the decision making under uncertainty to be applicable in the context of ambiguity (Alary et al., 2013; Broll and Wong, 2015; Cherbonnier and Gollier, 2015; Gollier, 2011, 2014; Iwaki and Osaki, 2014; Snow, 2010, 2011; Taboga, 2005; Treich, 2010; Wong, 2015a, 2016a).

As a benchmark, we examine first the case of perfect hedging wherein futures and options are available for both foreign currencies. In this benchmark case, we show that the

[1] Dated back to the Ellsberg's (1961) paradox, ambiguity has been alluded to the violation of the independence axiom, which is responsible for the decision criterion being linear in the outcome probabilities. See also Dillenberger and Segal (2015) and Machina (2014).

celebrated separation and full-hedging theorems of Danthine (1978), Feder et al. (1980), and Holthausen (1979) hold. Specifically, the separation theorem states that the firm's production and export decisions are independent of the firm's preferences and the underlying exchange rate uncertainty. The full-hedging theorem states that the firm optimally opts for a double full-hedge via the fairly priced futures for both foreign currencies so as to completely eliminate its exposure to the exchange rate risk.[2] Since there is no residual exchange rate risk that is hedgeable, the fairly priced options for both foreign currencies play no role as a hedging instrument and are not used by the firm.

When the firm can only cross-hedge its exposure to the exchange rate risk, the separation theorem no longer holds. We show that there is a wedge between the expected marginal revenue from exporting to the foreign country that has the fairly priced futures and options available for its currency and that from exporting to the other foreign country, which is the ambiguity risk premium demanded by the firm. Since financial hedging is imperfect, the firm employs operational hedging by producing less, exporting more to the foreign country with the direct hedging opportunities and less to the other foreign country, as compared to the benchmark case of perfect hedging. Given that the random spot exchange rates are first-order independent with respect to each plausible subjective distribution, we derive necessary and sufficient conditions under which the full-hedging theorem applies to the firm's cross-hedging decisions. When these conditions are violated, we show that the firm optimally uses the fairly priced options for hedging purposes, which is driven by the second-order dependence structure between the ambiguous spot exchange rates. We as such offer a rationale for the hedging role of options under smooth ambiguity preferences and cross-hedging of ambiguous exchange rate risk.

The rest of this paper is organized as follows. Section 2 delineates the KMM model of an exporting firm under exchange rate uncertainty. Section 3 examines a benchmark case wherein the firm can perfectly hedge its exchange rate risk exposure. Section 4 contrasts the

[2]The full-hedging theorem is analogous to a well-known result in the insurance literature that a risk-averse individual fully insures at an actuarially fair price (Mossin, 1968).

firm's production and export decisions to those in the benchmark case of perfect hedging. Section 5 examines the firm's cross-hedging decisions and shows the hedging role of options. Section 6 concludes.

2. The model

Consider an exporting firm that operates for one period with two dates, 0 and 1. To begin, the firm produces a single commodity in the home country according to a deterministic cost function, $C(Q)$, where $Q \geq 0$ is the output level, and $C(Q)$ is compounded to date 1. The firm's production technology exhibits decreasing returns to scale so that the cost function, $C(Q)$, satisfies that $C(0) = C'(0) = 0$, and $C'(Q) > 0$ and $C''(Q) > 0$ for all $Q > 0$.

At date 1, the firm exports its entire output, Q, to two foreign countries, indexed by $i = 1$ and 2. The firm receives revenue from its export, Q_i, to country i according to a deterministic revenue function, $R_i(Q_i)$, where $Q_i \geq 0$, $Q_1 + Q_2 = Q$, and $R_i(Q_i)$ is denominated in country i's currency for $i = 1$ and 2. The revenue function, $R_i(Q_i)$, satisfies that $R_i(0) = 0$, and $R_i'(Q_i) > 0$ and $R_i''(Q_i) < 0$ for all $Q_i \geq 0$ and $i = 1$ and 2.

The firm faces multiple sources of exchange rate uncertainty in that the spot exchange rates at date 1, $\tilde{S}_1$ and $\tilde{S}_2$, are not known ex ante, where $\tilde{S}_i$ is expressed in units of the home currency per unit of country i's currency at date 1 for $i = 1$ and 2.[3] Let $F_i^\circ(S_i)$ be the objective marginal cumulative distribution function (CDF) of $\tilde{S}_i$ over support $[\underline{S}_i, \overline{S}_i]$, and $\mathrm{E}_{F_i^\circ}(\cdot)$ be the expectation operator with respect to $F_i^\circ(S_i)$, where $0 < \underline{S}_i < \overline{S}_i$ for $i = 1$ and 2. To allow for possible correlation between $\tilde{S}_1$ and $\tilde{S}_2$, we let $G^\circ(S_1, S_2)$ be their objective joint CDF over support $[\underline{S}_1, \overline{S}_1] \times [\underline{S}_2, \overline{S}_2]$, and $\mathrm{E}_{G^\circ}(\cdot)$ be the expectation operator with respect to $G^\circ(S_1, S_2)$. These two random variables are independent if, and only if, $G^\circ(S_1, S_2) = F_1^\circ(S_1)F_2^\circ(S_2)$ for all $(S_1, S_2) \in [\underline{S}_1, \overline{S}_1] \times [\underline{S}_2, \overline{S}_2]$.

[3] Throughout the paper, random variables have a tilde (~) while their realizations do not.

To cross-hedge against the exchange rate risk, the firm can trade infinitely divisible currency futures and put option contracts at date 0, each of which calls for delivery of the home currency per unit of country 1's currency at date 1.[4] The futures exchange rate is predetermined at S_1^f at date 0, where $\underline{S}_1 < S_1^f < \overline{S}_1$. The currency put option contracts have a single strike price, K_1, and an exogenously given option premium, P_1, per contract, where $\underline{S}_1 < K_1 < \overline{S}_1$ and $P_1 > 0$. The option premium, P_1, is denominated in the home currency and compounded to date 1. To focus on the firm's hedging motive, we assume throughout the paper that the currency futures and put option contracts are fairly priced in that $S_1^f = \mathrm{E}_{F_1^\circ}(\tilde{S}_1)$ and $P_1 = \mathrm{E}_{F_1^\circ}[\max(K_1 - \tilde{S}_1, 0)]$.

The firm's profit at date 1, denominated in the home currency, is given by

$$\Pi(\tilde{S}_1, \tilde{S}_2) = \sum_{i=1}^{2} \tilde{S}_i R_i(Q_i) + (S_1^f - \tilde{S}_1)X_1 + [P_1 - \max(K_1 - \tilde{S}_1, 0)]Y_1 - C(Q), \tag{1}$$

where $Q = Q_1 + Q_2$, and X_1 and Y_1 are the numbers of the currency futures and put option contracts sold (purchased if negative) by the firm at date 0, respectively. We refer to the pair, (X_1, Y_1), as the firm's hedge position. We say that the futures position, X_1, is an under-hedge, a full-hedge, or an over-hedge if X_1 is smaller than, equal to, or greater than the revenue from country 1, $R_1(Q_1)$, respectively. We say that the put option position, Y_1, is a short (long) position if $Y_1 > (<)\ 0$. The firm possesses a von Neumann-Morgenstern utility function, $u(\Pi)$, defined over its home currency profit at date 1, Π, with $u'(\Pi) > 0$ and $u''(\Pi) \leq 0$. The firm is risk neutral or risk averse, depending on whether $u(\Pi) = \Pi$ or $u''(\Pi) < 0$, respectively.

The firm faces ambiguity in that it is uncertain about the objective CDFs, $F_1^\circ(S_1)$, $F_2^\circ(S_2)$, and $G^\circ(S_1, S_2)$. Let $F_1(S_1|\theta)$, $F_2(S_2|\theta)$, and $G(S_1, S_2|\theta)$ be the firm's subjective CDFs of $\tilde{S}_1$ and $\tilde{S}_2$, respectively, where θ is the realization of an unknown parameter, $\tilde{\theta}$. The KMM model represents ambiguity by a second-order subjective CDF of $\tilde{\theta}$, $H(\theta)$, over support $[\underline{\theta}, \overline{\theta}]$ with $\underline{\theta} < \overline{\theta}$, which captures the firm's uncertainty about which of the

[4] Because of the put-call parity, payoffs of any combinations of futures, calls, and puts can be replicated by any two of these three financial instruments, thereby rendering one of them to be redundant. Restricting the firm to use only currency futures and put option contracts is without any loss of generality.

subjective CDFs govern $\tilde{S}_1$ and $\tilde{S}_2$. Following Gollier (2011), Snow (2010, 2011), and Wong (2015b, 2016b), we assume that the firm's ambiguous beliefs are unbiased in the following sense:

$$\int_{\underline{\theta}}^{\overline{\theta}} F_i(S_i|\theta)\mathrm{d}H(\theta) = F_i^{\circ}(S_i), \tag{2}$$

for all $S_i \in [\underline{S}_i, \overline{S}_i]$, and $i = 1$ and 2, and

$$\int_{\underline{\theta}}^{\overline{\theta}} G(S_1, S_2|\theta)\mathrm{d}H(\theta) = G^{\circ}(S_1, S_2), \tag{3}$$

for all $(S_1, S_2) \in [\underline{S}_1, \overline{S}_1] \times [\underline{S}_2, \overline{S}_2]$. We denote $\mathrm{E}_{F_i}(\cdot|\theta)$, $\mathrm{E}_G(\cdot|\theta)$, and $\mathrm{E}_H(\cdot)$ as the expectation operators with respect to the subjective CDFs, $F_i(S_i|\theta)$, $G(S_1, S_2|\theta)$, and $H(\theta)$, respectively, for $i = 1$ and 2.

We say that the two random spot exchange rates, $\tilde{S}_1$ and $\tilde{S}_2$, are first-order independent if, and only if, $G(S_1, S_2|\theta) = F_1(S_1|\theta)F_2(S_2|\theta)$ for all $(S_1, S_2) \in [\underline{S}_1, \overline{S}_1] \times [\underline{S}_2, \overline{S}_2]$ and $\theta \in [\underline{\theta}, \overline{\theta}]$. Even when $\tilde{S}_1$ and $\tilde{S}_2$ are first-order independent, they are deemed to be second-order dependent as θ varies. To see this, we calculate the covariance between $A(\tilde{S}_1)$ and $B(\tilde{S}_2)$:

$$\begin{aligned}
&\int_{\underline{\theta}}^{\overline{\theta}} \int_{\underline{S}_1}^{\overline{S}_1} \int_{\underline{S}_2}^{\overline{S}_2} \{A(S_1) - \mathrm{E}_{F_1^{\circ}}[A(\tilde{S}_1)]\}\{B(S_2) - \mathrm{E}_{F_2^{\circ}}[B(\tilde{S}_2)]\}\mathrm{d}F_1(S_1|\theta)\mathrm{d}F_2(S_2|\theta)\mathrm{d}H(\theta) \\
&= \int_{\underline{\theta}}^{\overline{\theta}} \{\mathrm{E}_{F_1}[A(\tilde{S}_1)|\theta] - \mathrm{E}_{F_1^{\circ}}[A(\tilde{S}_1)]\}\{\mathrm{E}_{F_2}[B(\tilde{S}_2)|\theta] - \mathrm{E}_{F_2^{\circ}}[B(\tilde{S}_2)]\}\mathrm{d}H(\theta) \\
&= \mathrm{Cov}_H\{\mathrm{E}_{F_1}[A(\tilde{S}_1)|\tilde{\theta}], \mathrm{E}_{F_2}[B(\tilde{S}_2)|\tilde{\theta}]\},
\end{aligned} \tag{4}$$

where $A(\cdot)$ and $B(\cdot)$ are two arbitrarily chosen functions, and $\mathrm{Cov}_H(\cdot,\cdot)$ is the covariance operator with respect to the second-order CDF, $H(\theta)$. Equation (4) implies that $A(\tilde{S}_1)$ and $B(\tilde{S}_2)$ are second-order positively (negatively) dependent if $\mathrm{Cov}_H\{\mathrm{E}_{F_1}[A(\tilde{S}_1)|\tilde{\theta}], \mathrm{E}_{F_2}[B(\tilde{S}_2)|\tilde{\theta}]\} >$ $(<)$ 0. For example, this is the case when changes in θ affect $\mathrm{E}_{F_1}[A(\tilde{S}_1)|\theta]$ and $\mathrm{E}_{F_2}[B(\tilde{S}_2)|\theta]$ in the same direction (opposite directions) for all $\theta \in [\underline{\theta}, \overline{\theta}]$. We can justify the second-order

dependence structure between $\tilde{S}_1$ and $\tilde{S}_2$ by a learning framework, whereby the first-order priors are the Bayesian posterior distributions, and the second-order priors are the unconditional likelihoods. In this context, we can interpret θ as a latent business cycle indicator that affects the two random spot exchange rates simultaneously.

The recursive structure of the KMM model implies that we can compute the firm's expected utility under ambiguity in three steps. First, we calculate the firm's expected utility for each first-order joint CDF of $\tilde{S}_1$ and $\tilde{S}_2$:

$$U(\theta) = \int_{\underline{S}_1}^{\overline{S}_1} \int_{\underline{S}_2}^{\overline{S}_2} u[\Pi(S_1, S_2)] \mathrm{d}G(S_1, S_2|\theta), \tag{5}$$

where $\Pi(S_1, S_2)$ is given by Equation (1). Second, we transform each first-order expected utility obtained in Equation (5) by an ambiguity function, $\varphi(U)$, where $\varphi'(U) > 0$ and U is the firm's utility level. Finally, we take the expectation of the transformed first-order expected utility obtained in the second step with respect to the second-order CDF of $\tilde{\theta}$. The firm's ex-ante decision problem as such is given by

$$\max_{Q_1, Q_2, X_1, Y_1} \int_{\underline{\theta}}^{\overline{\theta}} \varphi[U(\theta)] \mathrm{d}H(\theta), \tag{6}$$

where $U(\theta)$ is given by Equation (5). Inspection of the objective function of program (6) reveals that the effect of ambiguity, represented by the second-order CDF, $H(\theta)$, and the effect of ambiguity preferences, represented by the shape of the ambiguity function, $\varphi(U)$, can be separated and thus studied independently.

The firm is ambiguity averse if, for any given allocation of exports, (Q_1, Q_2), and hedge position, (X_1, Y_1), the objective function of program (6) decreases when the firm's ambiguous beliefs, specified by $H(\theta)$, change in a way that induces a mean-preserving-spread in the distribution of the firm's first-order expected utility. According to this definition, Klibanoff et al. (2005) show that ambiguity aversion implies that the ambiguity function, $\varphi(U)$, is concave in U.[5] The firm is ambiguity neutral or ambiguity averse, depending on whether

[5] When $\varphi(U) = [1 - \exp(-\eta U)]/\eta$, Klibanoff et al. (2005) show that the maxmin expected utility model of Gilboa and Schmeidler (1989) is the limiting case as the constant absolute ambiguity aversion, η, approaches infinity under some conditions.

$\varphi(U) = U$ or $\varphi''(U) < 0$, respectively.

The first-order conditions for program (6) are given by

$$\int_{\underline{\theta}}^{\overline{\theta}} \varphi'[U^*(\theta)]\mathrm{E}_G\{u'[\Pi^*(\tilde{S}_1, \tilde{S}_2)][\tilde{S}_1 R_1'(Q_1^*) - C'(Q^*)]|\theta\}\mathrm{d}H(\theta) = 0, \tag{7}$$

$$\int_{\underline{\theta}}^{\overline{\theta}} \varphi'[U^*(\theta)]\mathrm{E}_G\{u'[\Pi^*(\tilde{S}_1, \tilde{S}_2)][\tilde{S}_2 R_2'(Q_2^*) - C'(Q^*)]|\theta\}\mathrm{d}H(\theta) = 0, \tag{8}$$

$$\int_{\underline{\theta}}^{\overline{\theta}} \varphi'[U^*(\theta)]\mathrm{E}_G\{u'[\Pi^*(\tilde{S}_1, \tilde{S}_2)](S_1^f - \tilde{S}_1)|\theta\}\mathrm{d}H(\theta) = 0, \tag{9}$$

and

$$\int_{\underline{\theta}}^{\overline{\theta}} \varphi'[U^*(\theta)]\mathrm{E}_G\{u'[\Pi^*(\tilde{S}_1, \tilde{S}_2)][P_1 - \max(K_1 - \tilde{S}_1, 0)]|\theta\}\mathrm{d}H(\theta) = 0, \tag{10}$$

where $Q^* = Q_1^* + Q_2^*$, and an asterisk (*) indicates an optimal level. The second-order conditions for program (6) are satisfied given the assumed properties of $\varphi(U)$, $u(\Pi)$, $R_1(Q_1)$, $R_2(Q_2)$, and $C(Q)$.

3. Benchmark case of perfect hedging

In this section, we consider a benchmark wherein the firm can trade fairly priced futures and put option contracts for the currencies of both foreign countries. In this benchmark case, the firm's home currency profit at date 1 is given by

$$\hat{\Pi}(\tilde{S}_1, \tilde{S}_2) = \sum_{i=1}^{2} \tilde{S}_i R_i(Q_i) + (S_i^f - \tilde{S}_i)X_i + [P_i - \max(K_i - \tilde{S}_i, 0)]Y_i - C(Q). \tag{11}$$

The firm's ex-ante decision problem in this benchmark case is given by

$$\max_{Q_1, Q_2, X_1, X_2, Y_1, Y_2} \int_{\underline{\theta}}^{\overline{\theta}} \varphi[\hat{U}(\theta)]\mathrm{d}H(\theta), \tag{12}$$

where $\hat{U}(\theta)$ is given by Equation (5) with $\Pi(S_1, S_2)$ replaced by $\hat{\Pi}(S_1, S_2)$ as defined in Equation (11).

The first-order conditions for program (12) are given by

$$\int_{\underline{\theta}}^{\overline{\theta}} \varphi'[\hat{U}^\circ(\theta)]\mathrm{E}_G\{u'[\hat{\Pi}^\circ(\tilde{S}_1, \tilde{S}_2)][\tilde{S}_i R_i'(Q_i^\circ) - C'(Q^\circ)]|\theta\}\mathrm{d}H(\theta) = 0, \tag{13}$$

$$\int_{\underline{\theta}}^{\overline{\theta}} \varphi'[\hat{U}^\circ(\theta)]\mathrm{E}_G\{u'[\hat{\Pi}^\circ(\tilde{S}_1, \tilde{S}_2)](S_i^f - \tilde{S}_i)|\theta\}\mathrm{d}H(\theta) = 0, \tag{14}$$

and

$$\int_{\underline{\theta}}^{\overline{\theta}} \varphi'[\hat{U}^\circ(\theta)]\mathrm{E}_G\{u'[\hat{\Pi}^\circ(\tilde{S}_1, \tilde{S}_2)][P_i - \max(K_i - \tilde{S}_i, 0)]|\theta\}\mathrm{d}H(\theta) = 0, \tag{15}$$

for $i = 1$ and 2, where $Q^\circ = Q_1^\circ + Q_2^\circ$, and a nought ($^\circ$) indicates an optimal level. Solving Equations (13), (14), and (15) simultaneously yields our first proposition.

Proposition 1. *Given that the ambiguity-averse exporting firm can trade the fairly priced futures and put option contracts for the currencies of both foreign countries, the firm optimally allocates its exports, Q_1° and Q_2°, that solve the following system of equations:*

$$\mathrm{E}_{F_1^\circ}(\tilde{S}_1)R_1'(Q_1^\circ) = C'(Q^\circ), \tag{16}$$

and

$$\mathrm{E}_{F_2^\circ}(\tilde{S}_2)R_2'(Q_2^\circ) = C'(Q^\circ). \tag{17}$$

The firm's optimal hedge position, $(X_1^\circ, X_2^\circ, Y_1^\circ, Y_2^\circ)$, consists of a double full-hedge, i.e., $X_1^\circ = R_1(Q_1^\circ)$ and $X_2^\circ = R_2(Q_2^\circ)$, and no options, i.e., $Y_1^\circ = Y_2^\circ = 0$.

Proof. Multiplying $R_i'(Q_i^\circ)$ to Equation (14) and adding the resulting equation to Equation (13) yields

$$[S_i^f R_i'(Q_i^\circ) - C'(Q^\circ)] \int_{\underline{\theta}}^{\overline{\theta}} \varphi'[\hat{U}^\circ(\theta)]\mathrm{E}_G\{u'[\hat{\Pi}^\circ(\tilde{S}_1, \tilde{S}_2)]|\theta\}\mathrm{d}H(\theta) = 0, \tag{18}$$

for $i = 1$ and 2. Since $S_i^f = \mathrm{E}_{F_i^\circ}(\tilde{S}_i)$, Equation (18) reduces to Equations (16) and (17).

Suppose that the firm adopts a double full-hedge, i.e., $X_1^\circ = R_1(Q_1^\circ)$ and $X_2^\circ = R_2(Q_2^\circ)$, and uses no options, i.e., $Y_1^\circ = Y_2^\circ = 0$. In this case, the firm's home currency profit at date 1 becomes $\hat{\Pi}^\circ = \sum_{i=1}^2 \mathrm{E}_{F_i^\circ}(\tilde{S}_i)R_i(Q_i^\circ) - C(Q^\circ)$, which is non-stochastic. It then follows from Equations (2) and (3), $S_i^f = \mathrm{E}_{F_i^\circ}(\tilde{S}_i)$, and $P_i = \mathrm{E}_{F_i^\circ}[\max(K_i - \tilde{S}_i, 0)]$ that Equations (14) and (15) hold for $i = 1$ and 2, thereby implying that the double full-hedge and zero option position are indeed optimal. □

The intuition for Proposition 1 is as follows. The firm can always lock in the marginal revenue from exporting to country i by trading the fairly priced futures contracts for country i's currency, where $i = 1$ and 2. As such, the usual optimality condition applies in that the marginal cost of production, $C'(Q^\circ)$, must be equated to the known marginal revenue from exporting to either foreign country, $\mathrm{E}_{F_i^\circ}(\tilde{S}_i)R_i'(Q_i^\circ)$ for $i = 1$ and 2, which determines the optimal allocation of exports, Q_1° and Q_2°. Since the futures exchange rates are unbiased, the firm, being ambiguity averse, finds it optimal to adopt a double full-hedge, i.e., $X_1^\circ = R_1(Q_1^\circ)$ and $X_2^\circ = R_2(Q_2^\circ)$, that completely eliminates the ambiguous exchange rate risk. Since there is no residual risk that is hedgeable, the fairly priced put option contracts play no role as a hedging instrument and are not used by the firm, i.e., $Y_1^\circ = Y_2^\circ = 0$. Proposition 1 as such extends the celebrated separation and full-hedging theorems to the case of smooth ambiguity preferences and multiple ambiguous spot exchange rates.

4. Optimal production and export decisions

In this section, we examine the firm's optimal production and export decisions when the firm can only cross-hedge the ambiguous exchange rate risk by trading the fairly priced futures and put option contracts for country 1's currency. To this end, we contrast the firm's decisions to those in the benchmark case of perfect hedging. We state and prove the

following proposition.

Proposition 2. *Given that the ambiguity-averse exporting firm can only trade the fairly priced futures and put option contracts for country 1's currency, the firm optimally allocates its exports, Q_1^* and Q_2^*, such that*

$$\mathrm{E}_{F_1^\circ}(\tilde{S}_1)R_1'(Q_1^*) = C'(Q^*), \tag{19}$$

and

$$\mathrm{E}_{F_2^\circ}(\tilde{S}_2)R_2'(Q_2^*) > C'(Q^*), \tag{20}$$

so that the firm optimally exports more to country 1 and less to country 2, i.e., $Q_1^* > Q_1^\circ$ and $Q_2^* < Q_2^\circ$, and produces less in the home country, i.e., $Q^* < Q^\circ$, as compared to the benchmark case of perfect hedging if the following condition holds:

$$\mathrm{Cov}_H\left\{\varphi'[U^*(\tilde{\theta})], \mathrm{E}_G\left\{u'[\Pi^*(\tilde{S}_1,\tilde{S}_2)]\{\Pi^*(\tilde{S}_1,\tilde{S}_2) - \mathrm{E}_{G^\circ}[\Pi^*(\tilde{S}_1,\tilde{S}_2)]\}\middle|\tilde{\theta}\right\}\right\} \leq 0. \tag{21}$$

Proof. Multiplying $R_1'(Q_1^*)$ to Equation (9) and adding the resulting equation to Equation (7) yields

$$[S_1^f R_1'(Q_1^*) - C'(Q^*)]\int_{\underline{\theta}}^{\overline{\theta}} \varphi'[U^*(\theta)]\mathrm{E}_G\{u'[\Pi^*(\tilde{S}_1,\tilde{S}_2)]|\theta\}\mathrm{d}H(\theta) = 0. \tag{22}$$

Since $S_1^f = \mathrm{E}_{F_1^\circ}(\tilde{S}_1)$, Equation (22) reduces to Equation (19). Rewrite Equation (8) as

$$[\mathrm{E}_{F_2^\circ}(\tilde{S}_2)R_2'(Q_2^*) - C'(Q^*)]\int_{\underline{\theta}}^{\overline{\theta}} \varphi'[U^*(\theta)]\mathrm{E}_G\{u'[\Pi^*(\tilde{S}_1,\tilde{S}_2)]|\theta\}\mathrm{d}H(\theta)$$

$$= \int_{\underline{\theta}}^{\overline{\theta}} \varphi'[U^*(\theta)]\mathrm{E}_G\{u'[\Pi^*(\tilde{S}_1,\tilde{S}_2)][\mathrm{E}_{F_2^\circ}(\tilde{S}_2) - \tilde{S}_2]|\theta\}R_2'(Q_2^*)\mathrm{d}H(\theta). \tag{23}$$

It follows from Equations (9) and (10) that

$$\int_{\underline{\theta}}^{\overline{\theta}} \varphi'[U^*(\theta)]\mathrm{E}_G\left\{u'[\Pi^*(\tilde{S}_1,\tilde{S}_2)]\{\Pi^*(\tilde{S}_1,\tilde{S}_2) - \mathrm{E}_{G^\circ}[\Pi^*(\tilde{S}_1,\tilde{S}_2)]\}\middle|\theta\right\}\mathrm{d}H(\theta)$$

$$= \int_{\underline{\theta}}^{\overline{\theta}} \varphi'[U^*(\theta)]\mathrm{E}_G\{u'[\Pi^*(\tilde{S}_1,\tilde{S}_2)][\tilde{S}_2 - \mathrm{E}_{F_2^\circ}(\tilde{S}_2)]|\theta]\}R_2(Q_2^*)\mathrm{d}H(\theta). \tag{24}$$

Note that

$$\int_{\underline{\theta}}^{\overline{\theta}} \varphi'[U^*(\theta)]\mathrm{E}_G\Big\{u'[\Pi^*(\tilde{S}_1,\tilde{S}_2)]\{\Pi^*(\tilde{S}_1,\tilde{S}_2) - \mathrm{E}_{G^\circ}[\Pi^*(\tilde{S}_1,\tilde{S}_2)]\}\Big|\theta\Big\}\mathrm{d}H(\theta)$$

$$= \mathrm{E}_H\{\varphi'[U^*(\tilde{\theta})]\}\mathrm{Cov}_{G^\circ}\{u'[\Pi^*(\tilde{S}_1,\tilde{S}_2)],\Pi^*(\tilde{S}_1,\tilde{S}_2)\}$$

$$+\mathrm{Cov}_H\Big\{\varphi'[U^*(\tilde{\theta})],\mathrm{E}_G\Big\{u'[\Pi^*(\tilde{S}_1,\tilde{S}_2)]\{\Pi^*(\tilde{S}_1,\tilde{S}_2) - \mathrm{E}_{G^\circ}[\Pi^*(\tilde{S}_1,\tilde{S}_2)]\}\Big|\tilde{\theta}\Big\}\Big\}, \tag{25}$$

where the equality follows from Equation (3), and $\mathrm{Cov}_{G^\circ}(\cdot,\cdot)$ is the covariance operator with respect to the objective CDF, $G^\circ(S_1,S_2)$. Risk aversion implies that the covariance in the first term on the right-hand side of Equation (25) is negative. Condition (21) implies that the second term on right-hand side of Equation (25) is non-positive. Hence, Equation (20) follows immediately from Equations (23) and (24).

Suppose that $Q_2^* \geq Q_2^\circ$. It follows from $R_2''(Q_2) < 0$ that $R_2'(Q_2^*) \leq R_2'(Q_2^\circ)$. Equations (19) and (20) imply that $\mathrm{E}_{F_1^\circ}(\tilde{S}_1)R_1'(Q_1^*) < \mathrm{E}_{F_2^\circ}(\tilde{S}_2)R_2'(Q_2^*)$. On the other hand, Equations (16) and (17) imply that $\mathrm{E}_{F_1^\circ}(\tilde{S}_1)R_1'(Q_1^\circ) = \mathrm{E}_{F_2^\circ}(\tilde{S}_2)R_2'(Q_2^\circ)$. It then follows that $\mathrm{E}_{F_1^\circ}(\tilde{S}_1)R_1'(Q_1^*) < \mathrm{E}_{F_1^\circ}(\tilde{S}_1)R_1'(Q_1^\circ)$ so that $Q_1^* > Q_1^\circ$ and thereby $Q^* > Q^\circ$. Since $C''(Q) > 0$, Equations (17) and (20) imply that $\mathrm{E}_{F_2^\circ}(\tilde{S}_2)R_2'(Q_2^*) > C'(Q^*) > C'(Q^\circ) = \mathrm{E}_{F_2^\circ}(\tilde{S}_2)R_2'(Q_2^\circ)$, a contradiction to $Q_2^* \geq Q_2^\circ$. Hence, it must be true that $Q_2^* < Q_2^\circ$.

Suppose that $Q_1^* \leq Q_1^\circ$. It follows from $R_1''(Q_1) < 0$ that $R_1'(Q_1^*) \geq R_1'(Q_1^\circ)$. Equations (16) and (19) imply that $C'(Q^*) \geq C'(Q^\circ)$. It follows from $C''(Q) > 0$ that $Q^* \geq Q^\circ$. Since $Q_2^* < Q_2^\circ$, this implies that $Q_1^* > Q_1^\circ$, a contradiction to $Q_1^* \leq Q_1^\circ$. Hence, it must be true that $Q_1^* > Q_1^\circ$.

Since $Q_1^* > Q_1^\circ$, it follows from $R_1''(Q_1) < 0$ that $R_1'(Q_1^*) < R_1'(Q_1^\circ)$. Equations (16) and (19) imply that $C'(Q^*) < C'(Q^\circ)$. It follows from $C''(Q) > 0$ that $Q^* < Q^\circ$. $\square$

To see the intuition for Proposition 2, we define the following two functions:

$$F_2^d(S_2|\theta) = \int_{\underline{S}_1}^{\overline{S}_1} \int_{\underline{S}_2}^{S_2} \frac{u'[\Pi^*(S_1, x)]}{\mathrm{E}_G\{u'[\Pi^*(\tilde{S}_1, \tilde{S}_2)]|\theta\}} \mathrm{d}G(S_1, x|\theta), \tag{26}$$

for all $S_2 \in [\underline{S}_2, \overline{S}_2]$ and $\theta \in [\underline{\theta}, \overline{\theta}]$, and

$$H^d(\theta) = \int_{\underline{\theta}}^{\theta} \frac{\varphi'[U^*(x)]\mathrm{E}_G\{u'[\Pi^*(\tilde{S}_1, \tilde{S}_2)]|x\}}{\mathrm{E}_H\left\{\varphi'[U^*(\tilde{\theta})]\mathrm{E}_G\{u'[\Pi^*(\tilde{S}_1, \tilde{S}_2)]|\tilde{\theta}\}\right\}} \mathrm{d}H(x), \tag{27}$$

for all $\theta \in [\underline{\theta}, \overline{\theta}]$. It is evident from Equations (26) and (27) that we can interpret $F_2^d(S_2|\theta)$ as a distorted CDF of $\tilde{S}_2$, and $H^d(\theta)$ as a distorted CDF of $\tilde{\theta}$. Substituting Equations (26) and (27) into Equation (8) yields

$$\int_{\underline{\theta}}^{\overline{\theta}} \mathrm{E}_{F_2^d}(\tilde{S}_2|\theta)\mathrm{d}H^d(\theta)R_2'(Q^*) = C'(Q^*), \tag{28}$$

where $\mathrm{E}_{F_2^d}(\cdot|\theta)$ is the expectation operator with respect to the distorted CDF, $F_2^d(S_2|\theta)$. The left-hand side of Equation (28) can be interpreted as the certainty equivalent marginal revenue from exporting to country 2, taking the firm's ambiguity preferences and the underlying ambiguity into account. Equation (28) is the usual optimality condition such that the firm's optimal output level, Q^*, equates the marginal cost of production, $C'(Q^*)$, to the certainty equivalent marginal revenue. Using Equations (24) and (25), we have

$$\begin{aligned}
&\mathrm{E}_{F_2^\circ}(\tilde{S}_2) - \int_{\underline{\theta}}^{\overline{\theta}} \mathrm{E}_{F_2^d}(\tilde{S}_2|\theta)\mathrm{d}H^d(\theta) \\
&= -\frac{\mathrm{E}_H\{\varphi'[U^*(\tilde{\theta})]\}\mathrm{Cov}_{G^\circ}\{u'[\Pi^*(\tilde{S}_1, \tilde{S}_2)], \Pi^*(\tilde{S}_1, \tilde{S}_2)\}}{\mathrm{E}_H\left\{\varphi'[U^*(\tilde{\theta})]\mathrm{E}_G\{u'[\Pi^*(\tilde{S}_1, \tilde{S}_2)]|\tilde{\theta}\}\right\}R_2(Q_2^*)} \\
&\quad -\frac{\mathrm{Cov}_H\left\{\varphi'[U^*(\tilde{\theta})], \mathrm{E}_G\left\{u'[\Pi^*(\tilde{S}_1, \tilde{S}_2)]\{\Pi^*(\tilde{S}_1, \tilde{S}_2) - \mathrm{E}_{G^\circ}[\Pi^*(\tilde{S}_1, \tilde{S}_2)]\}\middle|\tilde{\theta}\right\}\right\}}{\mathrm{E}_H\left\{\varphi'[U^*(\tilde{\theta})]\mathrm{E}_G\{u'[\Pi^*(\tilde{S}_1, \tilde{S}_2)]|\tilde{\theta}\}\right\}R_2(Q_2^*)}.
\end{aligned} \tag{29}$$

The right-hand side of Equation (29) is the risk premium demanded by the firm to compensate for its exposure to the ambiguous exchange rate risk. The first term is the component of

the risk premium that is driven mainly by risk aversion, which is always positive. This term goes away if the firm is risk neutral, i.e., $u(\Pi) = \Pi$. The second term is the component of the risk premium that is driven mainly by ambiguity aversion, which is non-negative given condition (21). This term goes away if the firm is ambiguity neutral, i.e., $\varphi(U) = U$.

The intuition for Proposition 2 is as follows. Equation (22) states that the firm optimally equates the marginal cost to the expected marginal revenue from exporting to country 1. Equation (23), however, states that the expected marginal revenue from exporting to country 2 exceeds the marginal cost. Since all sales in country 2 are exposed to the ambiguous exchange rate risk due to $\tilde{S}_2$, which is not directly hedgeable, the firm demands a positive risk premium to compensate for the risk arising from its export to country 2. The wedge between the expected marginal revenue from exporting to country 1 and that from exporting to country 2 is *de facto* the ambiguity risk premium required by the firm. Given that financial hedging via trading the fairly priced futures and put option contracts for country 1's currency is imperfect, the firm employs operational hedging by exporting more to country 1 and less to country 2, i.e., $Q_1^* > Q_i^\circ$ and $Q_2^* < Q_2^\circ$, so as to better cope with the ambiguous exchange rate risk. The firm, being ambiguity averse, produces less in the home country, i.e., $Q^* < Q^\circ$, in response to the residual ambiguous exchange rate risk when hedging is far from perfect.

Before leaving this section, it is of interest to contrast the firm's production and export decisions with those under ambiguity neutrality, i.e., $\varphi(U) = U$. Suppose that the firm is risk neutral, i.e., $u(\Pi) = \Pi$. In this case, Equation (29) reduces to

$$\begin{aligned} &\mathrm{E}_{F_2^\circ}(\tilde{S}_2) - \int_{\underline{\theta}}^{\overline{\theta}} \mathrm{E}_{F_2^d}(\tilde{S}_2|\theta)\mathrm{d}H^d(\theta) \\ &= -\frac{\mathrm{Cov}_H\Big\{\varphi'\{\mathrm{E}_G[\Pi^*(\tilde{S}_1,\tilde{S}_2)|\tilde{\theta}]\}, \mathrm{E}_G[\Pi^*(\tilde{S}_1,\tilde{S}_2)|\tilde{\theta}]\Big\}}{\mathrm{E}_H\Big\{\varphi'\{\mathrm{E}_G[\Pi^*(\tilde{S}_1,\tilde{S}_2)|\tilde{\theta}]\Big\}R_2(Q_2^*)}. \end{aligned} \tag{30}$$

The right-hand side of Equation (30) vanishes if $\varphi(U) = U$, and is positive if $\varphi''(U) < 0$.

Hence, making the firm ambiguity averse induces the firm to export more to country 1 and less to country 2, and produce less in the home country, as is shown in Proposition 2.

5. Optimal cross-hedging decisions

In this section, we examine the firm's optimal cross-hedging decisions. To facilitate the exposition, we fix the firm's allocation of exports at $(Q_1, Q_2) = (Q_1^*, Q_2^*)$ and reformulate program (6) as a two-stage optimization problem. In the first stage, we derive the firm's optimal futures position, $X_1(Y_1)$, that maximizes the objective function of program (6) for a given put option position, Y_1. In the second stage, we derive the firm's optimal put option position, Y_1^*, that maximizes the objective function of program (6) with X_1 replaced by $X_1(Y_1)$. The complete solution to program (6) is, therefore, given by Y_1^*, and $X_1^* = X_1(Y_1^*)$. We are particularly interested in studying the robustness of the full-hedging theorem and the hedging role of options in the context of multiple sources of ambiguous exchange rate risk.

Differentiating the objective function of program (6) with $X_1 = X_1(Y_1)$ with respect to Y_1, and evaluating the resulting derivative at $Y_1 = 0$ yields

$$\frac{\partial}{\partial Y_1} \int_{\underline{\theta}}^{\overline{\theta}} \varphi[U(\theta)] \mathrm{d}H(\theta) \Big|_{X_1 = X_1(0), Y_1 = 0}$$

$$= \int_{\underline{\theta}}^{\overline{\theta}} \varphi'[U^\circ(\theta)] \mathrm{E}_G\{u'[\Pi^\circ(\tilde{S}_1, \tilde{S}_2)][P_1 - \max(K_1 - \tilde{S}_1, 0)] | \theta\} \mathrm{d}H(\theta), \tag{31}$$

where $\Pi^\circ(\tilde{S}_1, \tilde{S}_2) = \sum_{i=1}^2 \tilde{S}_i R_i(Q_i^*) + (S_1^f - \tilde{S}_1) X_1(0) - C(Q^*)$, $U^\circ(\theta)$ is given by Equation (5) with $\Pi(S_1, S_2)$ replaced by $\Pi^\circ(S_1, S_2)$, and $X_1(0)$ solves the following first-order condition:

$$\int_{\underline{\theta}}^{\overline{\theta}} \varphi'[U^\circ(\theta)] \mathrm{E}_G\{u'[\Pi^\circ(\tilde{S}_1, \tilde{S}_2)](S_1^f - \tilde{S}_1) | \theta\} \mathrm{d}H(\theta) = 0. \tag{32}$$

If the right-hand side of Equation (31) is negative (positive), it follows from Equation (10) and the second-order conditions for program (6) that $Y_1^* < (>)\ 0$.

The full-hedging theorem applies to the firm's cross-hedging decisions when the firm optimally adopts a full-hedge, i.e., $X_1(0) = R_1(Q_1^*)$, and uses no options, i.e., $Y_1^* = 0$. Using Equations (32) and (31), this is case if, and only if, the following two equations hold simultaneously:

$$\int_{\underline{\theta}}^{\overline{\theta}} \varphi' \Big\{ \mathrm{E}_{F_2}\{u[\Pi^*(\tilde{S}_2)]|\theta\} \Big\} \mathrm{E}_G\{u'[\Pi^*(\tilde{S}_2)](S_1^f - \tilde{S}_1)|\theta\} \mathrm{d}H(\theta) = 0, \tag{33}$$

and

$$\int_{\underline{\theta}}^{\overline{\theta}} \varphi' \Big\{ \mathrm{E}_{F_2}\{u[\Pi^*(\tilde{S}_2)]|\theta\} \Big\} \mathrm{E}_G\{u'[\Pi^*(\tilde{S}_2)][P_1 - \max(K_1 - \tilde{S}_1, 0)]|\theta\} \mathrm{d}H(\theta) = 0, \tag{34}$$

where $\Pi^*(\tilde{S}_2) = S_1^f R_1(Q_1^*) + \tilde{S}_2 R_2(Q_2^*) - C(Q^*)$. The following proposition follows immediately from Equations (33) and (34).

Proposition 3. *Given that the ambiguity-averse exporting firm can only trade the fairly priced futures and put option contracts for country 1's currency, and that the two random spot exchange rates, $\tilde{S}_1$ and $\tilde{S}_2$, are first-order independent, the firm optimally opts for a full-hedge, i.e., $X_1^* = R_1(Q_1^*)$, and uses no options, i.e., $Y_1^* = 0$, if, and only if, the following two equations hold simultaneously:*

$$\mathrm{Cov}_H \Big\{ \varphi' \Big\{ \mathrm{E}_{F_2}\{u[\Pi^*(\tilde{S}_2)]|\tilde{\theta}\} \Big\} \mathrm{E}_{F_2}\{u'[\Pi^*(\tilde{S}_2)]|\tilde{\theta}\}, \mathrm{E}_{F_1}(\tilde{S}_1|\tilde{\theta}) \Big\} = 0, \tag{35}$$

and

$$\mathrm{Cov}_H \Big\{ \varphi' \Big\{ \mathrm{E}_{F_2}\{u[\Pi^*(\tilde{S}_2)]|\tilde{\theta}\} \Big\} \mathrm{E}_{F_2}\{u'[\Pi^*(\tilde{S}_2)]|\tilde{\theta}\}, \mathrm{E}_{F_1}[\max(K_1 - \tilde{S}_1, 0)|\tilde{\theta}] \Big\} = 0, \tag{36}$$

where $\Pi^*(\tilde{S}_2) = S_1^f R_1(Q_1^*) + \tilde{S}_2 R_2(Q_2^*) - C(Q^*)$.

Proof. Rewrite Equations (33) and (34) as

$$\int_{\underline{\theta}}^{\overline{\theta}} \varphi' \Big\{ \mathrm{E}_{F_2}\{u[\Pi^*(\tilde{S}_2)]|\theta\} \Big\} \mathrm{Cov}_G\{u'[\Pi^*(\tilde{S}_2)], \tilde{S}_1|\theta\} \mathrm{d}H(\theta)$$

$$+\mathrm{Cov}_H\left\{\varphi'\left\{\mathrm{E}_{F_2}\{u[\Pi^*(\tilde{S}_2)]|\tilde{\theta}\}\right\}\mathrm{E}_{F_2}\{u'[\Pi^*(\tilde{S}_2)]|\tilde{\theta}\}, \mathrm{E}_{F_1}(\tilde{S}_1|\tilde{\theta})\right\} = 0, \tag{37}$$

and

$$\int_{\underline{\theta}}^{\overline{\theta}} \varphi'\left\{\mathrm{E}_{F_2}\{u[\Pi^*(\tilde{S}_2)]|\theta\}\right\}\mathrm{Cov}_G\{u'[\Pi^*(\tilde{S}_2)], \max(K_1 - \tilde{S}_1, 0)|\theta\}\mathrm{d}H(\theta)$$

$$+\mathrm{Cov}_H\left\{\varphi'\left\{\mathrm{E}_{F_2}\{u[\Pi^*(\tilde{S}_2)]|\tilde{\theta}\}\right\}\mathrm{E}_{F_2}\{u'[\Pi^*(\tilde{S}_2)]|\tilde{\theta}\}, \mathrm{E}_{F_1}[\max(K_1 - \tilde{S}_1, 0)|\tilde{\theta}]\right\} = 0, \tag{38}$$

respectively. Since $\tilde{S}_1$ and $\tilde{S}_2$ are first-order independent, we have $\mathrm{Cov}_G\{u'[\Pi^*(\tilde{S}_2)], \tilde{S}_1|\theta\} = 0$ and $\mathrm{Cov}_G\{u'[\Pi^*(\tilde{S}_2)], \max(K_1 - \tilde{S}_1, 0)|\theta\} = 0$ for all $\theta \in [\underline{\theta}, \overline{\theta}]$. Equations (37) and (38) as such reduce to Equations (35) and (36), respectively. □

Given that the two random spot exchange rates, $\tilde{S}_1$ and $\tilde{S}_2$, are first-order independent, Proposition 3 provides the necessary and sufficient conditions under which the full-hedging theorem applies to the firm's cross-hedging decisions. In the absence of ambiguity, i.e., when the firm knows the objective CDFs, $F_1^\circ(S_1)$ and $F_2^\circ(S_2)$, Equations (35) and (36) hold trivially so that a full-hedge, i.e., $X_1^* = R_1(Q_1^*)$, and zero option position, $Y_1^* = 0$, are indeed optimal. To show that these necessary and sufficient conditions may still hold when ambiguity prevails, we consider the special case wherein the firm is risk neutral, i.e., $u(\Pi) = \Pi$. In this case, Equations (35) and (36) reduce to

$$\mathrm{Cov}_H\left\{\varphi'\{\Pi^*[\mathrm{E}_{F_2}(\tilde{S}_2|\tilde{\theta})]\}, \mathrm{E}_{F_1}(\tilde{S}_1|\tilde{\theta})\right\} = 0, \tag{39}$$

and

$$\mathrm{Cov}_H\left\{\varphi'\{\Pi^*[\mathrm{E}_{F_2}(\tilde{S}_2|\tilde{\theta})]\}, \mathrm{E}_{F_1}[\max(K_1 - \tilde{S}_1, 0)|\tilde{\theta}]\right\} = 0, \tag{40}$$

respectively, where $\Pi^*[\mathrm{E}_{F_2}(\tilde{S}_2|\theta)] = S_1^f R_1(Q_1^*) + \mathrm{E}_{F_2}(\tilde{S}_2|\theta)R_2(Q_2^*) - C(Q^*)$. If the first-order expected spot exchange rate, $\mathrm{E}_{F_2}(\tilde{S}_2|\theta)$, is preserved as θ varies, i.e., $\mathrm{E}_{F_2}(\tilde{S}_2|\theta)$ is a constant for all $\theta \in [\underline{\theta}, \overline{\theta}]$, it is evident that Equations (39) and (40) hold simultaneously, thereby rendering the validity of the full-hedging theorem.

In the general case that the firm is risk averse, Equations (35) and (36) need not hold simultaneously even when $\mathrm{E}_{F_2}(\tilde{S}_2|\theta)$ is preserved as θ varies. To see this, suppose that $\tilde{S}_1$ and $\tilde{S}_2$ are first-order independent, and an increase in θ induces mean-preserving-spread increases in risk to the first-order CDFs, $F_1(S_1|\theta)$ and $F_2(S_2|\theta)$, in the sense of Rothschild and Stiglitz (1970).[6] In this case, both $\mathrm{E}_{F_1}(\tilde{S}_1|\theta)$ and $\mathrm{E}_{F_2}(\tilde{S}_2|\theta)$ are preserved as θ varies. It follows that Equation (35) holds so that $X_1(0) = R_1(Q_1^*)$. Equation (31) becomes

$$\begin{aligned}
&\frac{\partial}{\partial Y_1}\int_{\underline{\theta}}^{\overline{\theta}} \varphi[U(\theta)]\mathrm{d}H(\theta)\Big|_{X_1=X_1(0),Y_1=0} \\
&= \int_{\underline{\theta}}^{\overline{\theta}} \varphi'\Big\{\mathrm{E}_{F_2}\{u[\Pi^*(\tilde{S}_2)]|\theta\}\Big\}\mathrm{E}_G\{u'[\Pi^*(\tilde{S}_2)][P_1-\max(K_1-\tilde{S}_1,0)]|\theta\}\mathrm{d}H(\theta) \\
&= -\int_{\underline{\theta}}^{\overline{\theta}} \varphi'\Big\{\mathrm{E}_{F_2}\{u[\Pi^*(\tilde{S}_2)]|\theta\}\Big\}\mathrm{Cov}_G\{u'[\Pi^*(\tilde{S}_2)],\max(K_1-\tilde{S}_1,0)|\theta\}\mathrm{d}H(\theta) \\
&\quad -\mathrm{Cov}_H\Big\{\varphi'\Big\{\mathrm{E}_{F_2}\{u[\Pi^*(\tilde{S}_2)]|\tilde{\theta}\}\Big\}\mathrm{E}_{F_2}\{u'[\Pi^*(\tilde{S}_2)]|\tilde{\theta}\},\mathrm{E}_{F_1}[\max(K_1-\tilde{S}_1,0)|\tilde{\theta}]\Big\}. \qquad (41)
\end{aligned}$$

The first-order independence of $\tilde{S}_1$ and $\tilde{S}_2$ renders the first term on the right-hand side of Equation (41) to vanish. Since $\max(K_1 - S_1, 0)$ is convex in S_1, it follows from Rothschild and Stiglitz (1971) that $\mathrm{E}_{F_1}[\max(K_1 - \tilde{S}_1, 0)|\theta]$ increases with an increase in θ. Risk aversion implies that $\mathrm{E}_{F_2}\{u[\Pi^*(\tilde{S}_2)]|\theta\}$ decreases with an increase in θ. It then follows from ambiguity aversion that $\varphi'\{\mathrm{E}_{F_2}\{u[\Pi^*(\tilde{S}_2)]|\theta\}\}$ increases with an increase in θ. If the firm's utility function exhibits prudence, i.e., $u'''(\Pi) > 0$, in the sense of Kimball (1993), $\mathrm{E}_{F_2}\{u'[\Pi^*(\tilde{S}_2)]|\theta\}$ increases with an increase in θ. Hence, we conclude that $\varphi'\{\mathrm{E}_{F_2}\{u[\Pi^*(\tilde{S}_2)]|\theta\}\}\mathrm{E}_{F_2}\{u'[\Pi^*(\tilde{S}_2)]|\theta\}$ increases with an increase in θ given that $u(\Pi)$ exhibits both risk aversion and prudence. In this case, the second term on the right-hand side of Equation (41) is negative so that $Y_1^* < 0$. It is worth pointing out that this result remains intact even when the firm is ambiguity neutral, i.e., $\varphi(U) = U$. Hence, the full-hedging theorem is not robust to the introduction of ambiguity even in the extreme case

[6] If an increase in θ induces a mean-preserving-spread increase in risk to one first-order CDF, $F_i(S_i|\theta)$, and a mean-preserving-spread decrease in risk to the other first-order CDF, $F_j(S_j|\theta)$, in the sense of Rothschild and Stiglitz (1970) for $i \neq j$, it can be shown analogously that $Y_1^* > 0$ in this case.

of ambiguity neutrality. This is in stark contrast to the extant literature that focuses on a single source of ambiguity, which shows that the behavior of an ambiguity-neutral decision maker is unaffected by the introduction of, or changes in, ambiguity.

We can easily generalize the above result in the following proposition.

Proposition 4. *Given that the ambiguity-averse exporting firm can only trade the fairly priced futures and put option contracts for country 1's currency, the two random spot exchange rates, $\tilde{S}_1$ and $\tilde{S}_2$, are first-order independent, and that the first-order expected spot exchange rate, $\mathrm{E}_{F_1}(\tilde{S}_1|\theta)$, is preserved as the parameter, θ, varies, the firm optimally opts for a long (short) put option position, i.e., $Y_1^* < (>)\ 0$, if, and only if, the following condition holds:*

$$\mathrm{Cov}_H\left\{\varphi'\left\{\mathrm{E}_{F_2}\{u[\Pi^*(\tilde{S}_2)]|\tilde{\theta}\}\right\}\mathrm{E}_{F_2}\{u'[\Pi^*(\tilde{S}_2)]|\tilde{\theta}\}, \mathrm{E}_{F_1}[\max(K_1-\tilde{S}_1,0)|\tilde{\theta}]\right\} > (<)\ 0, \quad (42)$$

where $\Pi^*(\tilde{S}_2) = S_1^f R_1(Q_1^*) + \tilde{S}_2 R_2(Q_2^*) - C(Q^*)$.

Proof. Since $\mathrm{E}_{F_1}(\tilde{S}_1|\theta)$ is a constant for all $\theta \in [\underline{\theta}, \overline{\theta}]$, Equation (35) holds, thereby rendering that $X_1(0) = R_1(Q_1^*)$. The first-order independence of $\tilde{S}_1$ and $\tilde{S}_2$ implies that the first term on the right-hand side of Equation (41) vanishes. The second term on the right-hand side of Equation (41) is negative (positive) if, and only if, condition (42) holds. It then follows from Equation (10) and the second-order conditions for program (6) that $Y_1^* < (>)\ 0$. □

The intuition for Proposition 4 is as follows. Given that covariances can be interpreted as marginal variances, Equation (35) implies that a full-hedge, i.e., $X_1 = R_1(Q_1^*)$, minimizes the variance of the firm's marginal ambiguity given that the firm uses no options, i.e., $Y_1 = 0$. Hence, we have $X_1(0) = R_1(Q_1^*)$ as long as the first moment of $\tilde{S}_1$ is kept fixed for all $\theta \in [\underline{\theta}, \overline{\theta}]$. Since we cannot keep all the higher moments of $\tilde{S}_1$ fixed as θ varies, the volatility effect resulting from the presence of ambiguity matters more for the put option contracts than for the futures contracts. In particular, if $\mathrm{E}_{F_1}[\max(K_1-\tilde{S}_1,0)|\theta]$ changes as θ

changes, it follows from Equation (4) and condition (42) that the firm's marginal ambiguity is positively (negatively) second-order dependent on the payoff of the put option contracts. This implies that the firm can further reduce the variability of its marginal ambiguity by choosing a long (short) put option position, i.e., $Y_1^* < (>)\ 0$, thereby rendering a hedging role of options when multiple sources of ambiguity prevail.

To gain more insight into the firm's optimal hedging decision, we consider the following parametric example. Suppose that the firm is risk neutral, i.e., $u(\Pi) = \Pi$, and possesses a quadratic ambiguity function of the form, $\varphi(\Pi) = \Pi - \gamma\Pi^2/2$, where γ is a positive constant such that $\varphi'(\Pi) = 1 - \gamma\Pi > 0$ for all relevant values of Π. The two random spot exchange rates, $\tilde{S}_1$ and $\tilde{S}_2$, are first-order independent. The firm's optimal hedge position, (X^*, Y^*), is given by

$$X_1^* = R_1(Q_1^*) + \frac{\mathrm{Cov}_H[\mathrm{E}_{F_1}(\tilde{S}_1|\tilde{\theta}), \mathrm{E}_{F_2}(\tilde{S}_2|\tilde{\theta})]R_2(Q_2^*)}{(1-\rho^2)\mathrm{Var}_{F_1^\circ}(\tilde{S}_1)}$$

$$-\frac{\rho^2\mathrm{Cov}_H\{\mathrm{E}_{F_1}[\max(K_1 - \tilde{S}_1, 0)|\tilde{\theta}], \mathrm{E}_{F_2}(\tilde{S}_2|\tilde{\theta})\}R_2(Q_2^*)}{(1-\rho^2)\mathrm{Cov}_{F_1^\circ}[\tilde{S}_1, \max(K_1 - \tilde{S}_1, 0)]}, \tag{43}$$

and

$$Y_1^* = \frac{\mathrm{Cov}_H\{\mathrm{E}_{F_1}[\max(K_1 - \tilde{S}_1, 0)|\tilde{\theta}], \mathrm{E}_{F_2}(\tilde{S}_2|\tilde{\theta})\}R_2(Q_2^*)}{(1-\rho^2)\mathrm{Var}_{F_1^\circ}[\max(K_1 - \tilde{S}_1, 0)]}$$

$$-\frac{\rho^2\mathrm{Cov}_H[\mathrm{E}_{F_1}(\tilde{S}_1|\tilde{\theta}), \mathrm{E}_{F_2}(\tilde{S}_2|\tilde{\theta})]R_2(Q_2^*)}{(1-\rho^2)\mathrm{Cov}_{F_1^\circ}[\tilde{S}_1, \max(K_1 - \tilde{S}_1, 0)]}, \tag{44}$$

where (Q_1^*, Q_2^*) is the firm's optimal allocation of exports, $\mathrm{Var}_{F_1^\circ}(\cdot)$ and $\mathrm{Cov}_{F_1^\circ}(\cdot,\cdot)$ are the variance and covariance operators with respect to the objective CDF, $F_1^\circ(S_1)$, respectively, and $\rho \in (-1, 0)$ is the objective correlation coefficient between $\tilde{S}_1$ and $\max(K_1 - \tilde{S}_1, 0)$.[7] In this example, condition (42) becomes $\mathrm{Cov}_H\{\mathrm{E}_{F_1}[\max(K_1 - \tilde{S}_1, 0)|\tilde{\theta}], \mathrm{E}_{F_2}(\tilde{S}_2|\tilde{\theta})\} < (>)\ 0$. If the first-order expected spot exchange rate, $\mathrm{E}_{F_1}(\tilde{S}_1|\theta)$, is preserved as θ varies, Equations

[7] It is easily shown that the pair, (Q_1^*, Q_2^*), satisfies that $S_1^f R_1'(Q_1^*) = C'(Q^*)$ and $\mathrm{E}_{F_2^\circ}(\tilde{S}_2)R_2'(Q_2^*) > C'(Q^*)$. See also Equation (30).

(43) and (44) then imply that $X^* < (>)\ R_1(Q_1^*)$ and $Y^* < (>)\ 0$, which are consistent with the results of Proposition 4.

It is evident from Equations (4) and (44) that the hedging role of options arises from the second-order dependency between the spot exchange rate, $\tilde{S}_2$, and the payoff of the futures contracts, $S_1^f - \tilde{S}_1$, and that of the put option contracts, $\max(K_1 - \tilde{S}_1, 0)$. When there are multiple sources of ambiguous exchange rate uncertainty, exporting firms that can only cross-hedge their exchange rate risk exposure have to rely on more than one hedging instrument. Options as such are used to achieve better hedging effectiveness.

6. Conclusion

This paper examines the behavior of an exporting firm that sells its output to two foreign countries, only one of which has futures and options available for its currency. The firm's preferences exhibit smooth ambiguity aversion developed by Klibanoff et al. (2005). Within the KMM model, ambiguity is represented by a second-order probability distribution that captures the firm's uncertainty about which of the subjective beliefs govern the exchange rate risk. On the other hand, ambiguity preferences are modeled by the second-order expectation of a concave transformation of the first-order expected utility of profit conditional on each plausible subjective joint distribution of the exchange rate risk.

We show that the separation theorem fails to hold. Since financial hedging is imperfect, the firm employs operational hedging by producing less, exporting more to the foreign country with the direct hedging opportunities and less to the other foreign country in response to the residual exchange rate risk. Given that the random spot exchange rates are first-order independent with respect to each plausible subjective distribution, we derive necessary and sufficient conditions under which the full-hedging theorem applies to the firm's cross-hedging decisions. When these conditions are violated, we show that the firm includes options in its optimal hedge position. This paper as such offers a rationale for the

hedging role of options under smooth ambiguity preferences and cross-hedging of ambiguous exchange rate risk.

While we adopt a static framework of Klibanoff et al. (2005) to study a firm's cross-hedging decision, the KMM model can be readily generalized into a dynamic setting with learning. Indeed, Ju and Miao (2012) develop a generalized recursive smooth ambiguity model that helps resolve the equity premium puzzle. Miao et al. (2014) apply this approach to explain the variance premium puzzle, and Chen et al. (2014) examine the dynamic asset allocation with ambiguous return predictability. It is of great interest to extend our static cross-hedging model to a dynamic one with learning. We leave this challenge for future research.

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