

MIXED PRODUCT POISSON STRUCTURES ASSOCIATED TO POISSON LIE GROUPS AND LIE BIALGEBRAS

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ABSTRACT. We introduce and study some mixed product Poisson structures on product manifolds associated to Poisson Lie groups and Lie bialgebras. For quasitriangular Lie bialgebras, our construction is equivalent to that of fusion products of quasi-Poisson G -manifolds introduced by Alekseev, Kosmann-Schwarzbach, and Meinrenken. Our primary examples include four series of holomorphic Poisson structures on products of flag varieties and related spaces of complex semi-simple Lie groups.

1. INTRODUCTION AND OUTLINES OF MAIN RESULTS

1.1. Introduction. This paper is the first of a series of papers devoted to a detailed study of some naturally defined holomorphic Poisson structures on products of flag varieties and related spaces of complex semi-simple Lie groups (see §1.3 and §8.6 for the definitions of these Poisson structures). Aspects of the Poisson structures to be investigated include calculations in coordinates, symplectic groupoids over (extended) Bruhat cells, orbits of symplectic leaves under the action of a maximal torus T of G , also called the T -leaves, log-moment maps for the T -action, cluster structures on the T -leaves, and quantization. This paper sets up the general framework, namely that of *mixed product Poisson structures* and Poisson structures *defined by quasitriangular r -matrices*, in which the above mentioned aspects of the Poisson structures will be studied in forthcoming papers. Although we are motivated by particular examples, the theory developed in the paper is applicable to much wider classes of Poisson structures and is of independent interest.

Definition 1.1. Given two manifolds Y_1 and Y_2 , by a *mixed product Poisson structure* on the product manifold $Y_1 \times Y_2$ we mean a Poisson bi-vector field π on $Y_1 \times Y_2$ that projects to well-defined Poisson structures on Y_1 and Y_2 . Given Y_i , $1 \leq i \leq n$, where $n \geq 2$, a Poisson structure π on the product manifold $Y = Y_1 \times \cdots \times Y_n$ is said to be a *mixed product* if the projection of π to $Y_i \times Y_j$ is a well-defined mixed product Poisson structure on $Y_i \times Y_j$ for any $1 \leq i < j \leq n$, and in this case, we also call the pair $(Y_1 \times \cdots \times Y_n, \pi)$ a *mixed product Poisson manifold*.

Mixed product Poisson manifolds are semiclassical analogs of *locally factored algebras* introduced by P. Etingof and D. Kazhdan in [16]. When each Y_j is a finite dimensional vector space, *linear mixed product Poisson structures* on the direct sum vector space $Y = Y_1 \oplus \cdots \oplus Y_n$ are in one-to-one correspondence with Lie algebra structures on its dual space $Y^* \cong Y_1^* \oplus \cdots \oplus Y_n^*$ such that each $Y_i^* \hookrightarrow Y^*$ is a Lie subalgebra and $[Y_i^*, Y_j^*] \subset Y_i^* \oplus Y_j^*$ for each pair $i \neq j$. In this paper, we study a class of mixed product Poisson structures in the category of Poisson manifolds with Poisson actions by Lie bialgebras.

The constructions in this paper are all special cases of *twists* of Lie bialgebras and Poisson actions thereof as introduced by Drinfeld [11] in the context of quasi-Hopf algebras. More specifically, given Lie bialgebras $(\mathfrak{g}_j, \delta_{\mathfrak{g}_j})$, $1 \leq j \leq n$, we introduce (§3.1) the notion of *mixed twists* of their direct product Lie bialgebra, which are twists by *mixed twisting elements* of the direct product Lie bialgebra structure on $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_n$. If (Y_j, π_j, λ_j) is a $(\mathfrak{g}_j, \delta_{\mathfrak{g}_j})$ -Poisson space for each $1 \leq j \leq n$ (see Definition 2.12), a mixed twisting element t is used to obtain a mixed product Poisson structure π_Y on the product manifold $Y = Y_1 \times \cdots \times Y_n$ with the properties 1) the projection of π_Y to each Y_j is π_j , and 2) the direct product

action of \mathfrak{g} on (Y, π_Y) is now a Poisson action of the twist by t of the direct product Lie bialgebra structure on \mathfrak{g} (see Proposition 3.4).

As our first example of mixed twists of direct product Lie bialgebras, we introduce in §5.1 the notion of *polyubles* of an arbitrary Lie bialgebra, generalizing that of the Drinfeld double of a Lie bialgebra. We then construct mixed product Poisson structures (§4 and §5) that have Poisson actions by polyuble Lie bialgebras or their duals (Proposition 5.7 and Remark 5.8). Polyubles for factorizable Lie bialgebras have been previously constructed by V. Fock and A. Rosly [19, 20], from whom we have borrowed the term.

Polyubles of Lie bialgebras are special cases of a more general construction: if $r \in \mathfrak{g} \otimes \mathfrak{g}$ is a quasitriangular r -matrix on a Lie algebra \mathfrak{g} , we construct in §6.1, for each $n \geq 2$, a quasitriangular r -matrix $r^{(n)}$ on the direct product Lie algebra \mathfrak{g}^n , which is a certain alternating sum of r plus a mixed twisting element of the n -fold direct product of the quasitriangular Lie bialgebra (\mathfrak{g}, r) with itself. The dual of the quasitriangular Lie bialgebra $(\mathfrak{g}^n, r^{(n)})$ is precisely the *locally factored Lie bialgebra with equal components* constructed from (\mathfrak{g}, r) by P. Etingof and D. Kazhdan in [16, Proposition 1.9]. If (Y_j, π_j, λ_j) is a (\mathfrak{g}, r) -Poisson space for each $1 \leq j \leq n$ (see Definition 2.12), one then obtains a mixed product Poisson structure on the product manifold $Y = Y_1 \times \cdots \times Y_n$ such that the direct product action of \mathfrak{g}^n on Y is a Poisson action for the quasitriangular Lie bialgebra $(\mathfrak{g}^n, r^{(n)})$ (see Theorem 6.8). In particular, we are naturally lead to the notion of *fusion products* of Poisson spaces of a quasitriangular Lie bialgebra (Definition 6.9), which, under twisting, is equivalent to that of fusion products of quasi-Poisson spaces introduced by Alekseev, Kosmann-Schwarzbach, and Meinrenken [2] (see §6.3).

Another emphasis of our paper is on *Poisson structures defined by quasitriangular r -matrices*, i.e., Poisson structures on a manifold Y of the form $\lambda(r)$, where r is a quasitriangular r -matrix on a Lie algebra \mathfrak{g} and λ is a Lie algebra action of \mathfrak{g} on Y (see §1.2 and §2.6). §7 and §8 are devoted to our primary examples, which are mixed product Poisson structures arising from quotients of product Poisson Lie groups, which, at the same time, are defined by quasitriangular r -matrices of the form $r^{(n)}$ (Theorem 7.8 and Theorem 8.2). This class of examples includes the four series of holomorphic Poisson structures on products of flag varieties and related spaces of complex semi-simple Lie groups (see §1.3).

We also give in §2 a somewhat detailed review on Lie bialgebras. A partial purpose of the review is to fix our conventions on signs and constants that are to be used in this and subsequent papers.

1.2. Poisson structures defined by quasitriangular r -matrices. It is a simple observation, see Proposition 2.18 (see also [27, Section 2.1]), that given a quasitriangular r -matrix $r = \sum_i x_i \otimes y_i \in \mathfrak{g} \otimes \mathfrak{g}$ on a Lie algebra \mathfrak{g} and a left Lie algebra action λ of \mathfrak{g} on a manifold Y , the 2-tensor field

$$\lambda(r) := \sum_i \lambda(x_i) \otimes \lambda(y_i)$$

on Y is Poisson as long as it is skew-symmetric. In this case, $(Y, -\lambda(r), \lambda)$ is a left (\mathfrak{g}, r) -Poisson space, and we say that the Poisson structure $-\lambda(r)$ is *defined by the action λ and the quasitriangular r -matrix r* . An equivalent condition for $\lambda(r)$ to be skew-symmetric is that the stabilizer subalgebra of \mathfrak{g} at each point of Y be coisotropic with respect to the symmetric part of r (see §2.6).

1.3. Primary Examples. For a Lie group G and an integer $n \geq 1$, let G^n act on itself from the right by

$$(1.1) \quad (g_1, g_2, \dots, g_n) \cdot (h_1, h_2, \dots, h_n) = (g_1 h_1, h_1^{-1} g_2 h_2, \dots, h_{n-1}^{-1} g_n h_n), \quad g_i, h_i \in G.$$

If Q_1, \dots, Q_n are closed subgroups of G , let

$$(1.2) \quad Z = G \times_{Q_1} \cdots \times_{Q_{n-1}} G / Q_n$$

be the quotient manifold of G^n by the action of $Q_1 \times \cdots \times Q_n \subset G^n$ given in (1.1). Assume now that (G, π_G) is a Poisson Lie group and that Q_1, \dots, Q_n are closed Poisson Lie subgroups of (G, π_G) . Then (see Lemma-Definition 7.2) the direct product Poisson structure π_G^n on G^n projects to a well-defined Poisson structure $\pi_Z = \varpi_Z(\pi_G^n)$ on Z , where $\varpi_Z : G^n \rightarrow Z$ is the quotient map, and we refer to (Z, π_Z) as a *quotient Poisson manifold* of the product Poisson Lie group (G^n, π_G^n) .

When (G, π_{st}) is a *standard complex semi-simple Poisson Lie group*, defined by the choice of the triple $(B, B_-, \langle \cdot, \cdot \rangle_{\mathfrak{g}})$, where (B, B_-) is a pair of opposite Borel subgroups of the connected complex semi-simple Lie group G and $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$ a fixed symmetric non-degenerate invariant bilinear form on the Lie algebra of \mathfrak{g} (see §8.6), both B and B_- are Poisson Lie subgroups of (G, π_{st}) . For each integer $n \geq 1$, one thus has the quotient Poisson manifolds (F_n, π_n) and $(\tilde{F}_n, \tilde{\pi}_n)$, where

$$F_n = G \times_B \cdots \times_B G/B \quad \text{and} \quad \tilde{F}_n = G \times_B \cdots \times_B G$$

are both quotients of G^n , and $\pi_n = \pi_{F_n}$ and $\tilde{\pi}_n = \pi_{\tilde{F}_n}$ are the respective projections of π_{st}^n to F_n and \tilde{F}_n . Associated to (G, π_{st}) is its Drinfeld double Poisson Lie group $(G \times G, \Pi_{\text{st}})$, where $G \times G$ has the product Lie group structure, and $B \times B_-$ is a Poisson Lie subgroup of $(G \times G, \Pi_{\text{st}})$. For each integer $n \geq 1$, one also has the quotient Poisson manifolds (\mathbb{F}_n, Π_n) and $(\tilde{\mathbb{F}}_n, \tilde{\Pi}_n)$, where

$$\begin{aligned} \mathbb{F}_n &= (G \times G) \times_{(B \times B_-)} \cdots \times_{(B \times B_-)} (G \times G) / (B \times B_-), \\ \tilde{\mathbb{F}}_n &= (G \times G) \times_{(B \times B_-)} \cdots \times_{(B \times B_-)} (G \times G) \end{aligned}$$

are both quotients of $(G \times G)^n$, and $\Pi_n = \pi_{\mathbb{F}_n}$ and $\tilde{\Pi}_n = \pi_{\tilde{\mathbb{F}}_n}$ are the respective projections of Π_{st}^n to \mathbb{F}_n and $\tilde{\mathbb{F}}_n$. These four series of Poisson manifolds are the focus of study in this and forthcoming papers.

Returning to the quotient Poisson manifold (Z, π_Z) in (1.2) for an arbitrary Poisson Lie group (G, π_G) and closed Poisson Lie subgroups Q_1, \dots, Q_n , note that the manifold Z is diffeomorphic to the product manifold $(G/Q_1) \times \cdots \times (G/Q_n)$ via the diffeomorphism $J_Z : Z \rightarrow (G/Q_1) \times \cdots \times (G/Q_n)$ given by

$$J_Z(\varpi_Z(g_1, g_2, \dots, g_n)) = (g_1 Q_1, g_1 g_2 Q_2, \dots, g_1 g_2 \cdots g_n Q_n), \quad g_1, \dots, g_n \in G.$$

We study π_Z via the Poisson structure $J_Z(\pi_Z)$ on the product manifold $(G/Q_1) \times \cdots \times (G/Q_n)$. The main results (Theorem 7.8 and Theorem 8.2) on the Poisson structure $J_Z(\pi_Z)$ are summarized as follows.

Theorem 1.2. 1) *As Poisson structures on the product manifold $(G/Q_1) \times \cdots \times (G/Q_n)$,*

$$J_Z(\pi_Z) = -\sigma \left(r_{\mathfrak{d}}^{(n)} \right),$$

where $r_{\mathfrak{d}}$ is the quasitriangular r -matrix on the double Lie algebra of the Lie bialgebra $(\mathfrak{g}, \delta_{\mathfrak{g}})$ of (G, π_G) , and σ is the direct product action of \mathfrak{d}^n on $(G/Q_1) \times \cdots \times (G/Q_n)$ of naturally defined (see (7.3)) actions of \mathfrak{d} on each factor G/Q_j ;

2) *If $(\mathfrak{g}, \delta_{\mathfrak{g}})$ has a quasitriangular structure $r \in \mathfrak{g} \otimes \mathfrak{g}$ (see Definition 2.3) and if $Q_1 = \cdots = Q_n = Q$, where the Lie algebra \mathfrak{q} of Q satisfies $\text{Im}(r_+) \subset \mathfrak{q}$ (see (2.14)), then*

$$J_Z(\pi_Z) = -\lambda(r^{(n)}),$$

where λ is the direct product action of \mathfrak{g}^n on $(G/Q)^n$ induced by left translation.

Theorem 1.2 applies directly to the Poisson manifolds (F_n, π_n) and (\mathbb{F}_n, Π_n) . We can thus identify both π_n and Π_n as mixed product Poisson structures, respectively on $(G/B)^n$ and $((G \times G)/(B \times B_-))^n$, that are also defined by quasitriangular r -matrices. Similar results holds for the Poisson structures $\tilde{\pi}_n$ and $\tilde{\Pi}_n$ by adjusting the quasitriangular r -matrices (see §8.2).

Let $T = B \cap B_-$, a maximal torus of G . Then T acts on G by left translation and on $G \times G$ by left translation via $T \cong \{(t, t) : t \in T\}$. The actions of T on $Z \in \{F_n, \tilde{F}_n, \mathbb{F}_n, \tilde{\mathbb{F}}_n\}$, induced by the

T -actions on the first factors of G^n and $(G \times G)^n$, preserve the Poisson structures π_Z . In [30], a sequel to the present paper, we describe the T -leaves and compute the ranks of all the symplectic leaves in (Z, π_Z) for $Z \in \{F_n, \widehat{F}_n, \mathbb{F}_n, \widehat{\mathbb{F}}_n\}$ by first developing a general theory on torus orbits of symplectic leaves for Poisson structures defined by quasitriangular r -matrices. Our descriptions of the T -leaves of these Poisson manifolds are in terms of *extended Bruhat cells*, *extended Richardson varieties*, and *extended Double Bruhat cells associated to conjugacy classes* (see [30] for detail). In [14], the induced Poisson structures on extended Bruhat cells are computed explicitly in the so-called Bott-Samelson coordinates in terms of root strings and the structure constants of the Lie algebra \mathfrak{g} , and they are shown to give rise to polynomial Poisson algebras that are *symmetric nilpotent semi-quadratic Poisson-Ore extensions* of \mathbb{C} in the sense of [21, Definition 4]. Symplectic groupoids of extended Bruhat cells in terms of extended double Bruhat cells will be studied in another paper.

We would like to point out that some examples of Poisson structures defined by quasitriangular r -matrices and Lie algebra actions, together with their quantizations, have been studied by several authors (see, for example, [8, 15, 17, 42]). Our paper gives new classes of such Poisson structures, and it would be very interesting to study their quantizations.

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1.5. Notation. Throughout the paper, vector spaces are understood to be over $\mathbf{k} = \mathbb{R}$ or \mathbb{C} . The non-degenerate bilinear pairing between a finite dimensional vector space V and its dual space V^* will always be denoted by \langle, \rangle . The annihilator of a vector subspace U of V is, by definition, $U^0 = \{\xi \in V^* : \langle \xi, U \rangle = 0\} \subset V^*$. For each integer $k \geq 1$, we identify the tensor product $V^{\otimes k}$ with the space of multi-linear maps from $V^* \times \cdots \times V^*$ (k -times) to \mathbf{k} , so that

$$(v_1 \otimes v_2 \otimes \cdots \otimes v_k)(\xi_1, \xi_2, \dots, \xi_k) = \langle v_1, \xi_1 \rangle \langle v_2, \xi_2 \rangle \cdots \langle v_k, \xi_k \rangle, \quad v_j \in V, \xi_j \in V^*.$$

Let $\wedge^k V$ be the subspace of skew-symmetric elements in $V^{\otimes k}$, and define, for $v_1, \dots, v_k \in V$,

$$(1.3) \quad v_1 \wedge v_2 \wedge \cdots \wedge v_k = \sum_{\sigma \in S_k} \text{sign}(\sigma) v_{\sigma(1)} \otimes v_{\sigma(2)} \otimes \cdots \otimes v_{\sigma(k)} \in \wedge^k V \subset V^{\otimes k}.$$

If $\langle, \rangle_{(V, W)}$ is a bilinear pairing between V and another vector space W , for each integer $k \geq 1$, we extend $\langle, \rangle_{(V, W)}$ to a bilinear pairing between $\wedge^k V$ and $\wedge^k W$, also denoted as $\langle, \rangle_{(V, W)}$, via

$$\langle x_1 \wedge \cdots \wedge x_k, y_1 \wedge \cdots \wedge y_k \rangle_{(V, W)} = \det((\langle x_i, y_j \rangle_{(V, W)})_{i,j=1,\dots,k}), \quad x_j \in V, y_j \in W.$$

In particular, we identify $\wedge^k(V^*)$ with $(\wedge^k V)^*$ via the bilinear pairing between $\wedge^k V$ and $\wedge^k(V^*)$ which is the extension of the pairing \langle, \rangle between V and V^* . For any element $r = \sum_i u_i \otimes v_i \in V \otimes V$, define

$$(1.4) \quad r^\# : V^* \longrightarrow V, \quad r^\#(\xi) = \sum_i \langle \xi, u_i \rangle v_i, \quad \xi \in V^*.$$

If $\Lambda = \sum_i u_i \wedge v_i \in \wedge^2 V$, then $\Lambda^\# : V^* \rightarrow V$ is given by $\Lambda^\#(\xi) = \sum_i \langle \xi, u_i \rangle v_i - \langle \xi, v_i \rangle u_i$, or

$$(1.5) \quad \langle \Lambda^\#(\xi), \eta \rangle = \Lambda(\xi, \eta) = \langle \Lambda, \xi \wedge \eta \rangle, \quad \xi, \eta \in V^*.$$

If $\sigma : V \rightarrow W$ is a linear map, denote also by σ the linear map $V^{\otimes k} \rightarrow W^{\otimes k}$ given by $\sigma(v_1 \otimes \cdots \otimes v_k) = \sigma(v_1) \otimes \cdots \otimes \sigma(v_k)$. If $V = V_1 \oplus \cdots \oplus V_n$ is a direct sum of vector spaces, and if $A_j \in V_j^{\otimes k}$, $1 \leq j \leq n$,

denote by $(0, \dots, A_j, \dots, 0) \in V^{\otimes k}$ the image of A_j under the embedding of V_j into V , and let

$$(A_1, \dots, A_n) = \sum_{j=1}^n (0, \dots, A_j, \dots, 0) \in V^{\otimes k}.$$

We will work in the category of finite dimensional smooth real or complex manifolds. If Y is a real or complex manifold, for an integer $k \geq 1$, $\mathcal{V}^k(Y)$ denotes the space of k -vector fields on Y , i.e., smooth or holomorphic sections of $\wedge^k TY$, where TY is the smooth or holomorphic tangent bundle of Y . One has the Schouten bracket $[\cdot, \cdot] : \mathcal{V}^k(Y) \times \mathcal{V}^l(Y) \rightarrow \mathcal{V}^{k+l-1}(Y)$ which restricts to the Lie brackets on $\mathcal{V}^1(Y)$ and satisfies $[A, f] = A(f)$ for $A \in \mathcal{V}^1(Y)$, $f \in \mathcal{V}^0(Y)$, and

$$\begin{aligned} [A, B \wedge C] &= [A, B] \wedge C + (-1)^{(k-1)l} B \wedge [A, C], \quad A \in \mathcal{V}^k(Y), B \in \mathcal{V}^l(Y), C \in \mathcal{V}^m(Y), \\ [A, B] &= -(-1)^{(k-1)(l-1)} [B, A], \quad A \in \mathcal{V}^k(Y), B \in \mathcal{V}^l(Y). \end{aligned}$$

A real (resp. complex) *Poisson manifold* is a pair (Y, π_Y) , where Y is a real (resp. complex) manifold and π_Y is a smooth (resp. holomorphic) bi-vector field on Y such that $[\pi_Y, \pi_Y] = 0$. Given Poisson manifolds (Y_j, π_{Y_j}) , $1 \leq j \leq n$, the direct product Poisson structure on the product manifold $Y_1 \times \dots \times Y_n$ is $(\pi_{Y_1}, \dots, \pi_{Y_n})$, also denoted by $\pi_{Y_1} \times \dots \times \pi_{Y_n}$. The n -fold product of a Poisson manifold (Y, π_Y) with itself is denoted by (Y^n, π_Y^n) .

Let G be a Lie group with Lie algebra \mathfrak{g} . Identifying $\wedge \mathfrak{g}$ with the space of left invariant multivector fields on G , one has the Schouten bracket on $\wedge \mathfrak{g}$ which is also denoted by $[\cdot, \cdot]$. In particular, for $\Lambda \in \wedge^2 \mathfrak{g}$ and using our convention in (1.3), $[\Lambda, \Lambda] \in \wedge^3 \mathfrak{g}$ is given by

$$(1.6) \quad \frac{1}{2} [\Lambda, \Lambda](\xi, \eta, \zeta) = \langle \xi, [\Lambda^\#(\eta), \Lambda^\#(\zeta)] \rangle + \langle \eta, [\Lambda^\#(\zeta), \Lambda^\#(\xi)] \rangle + \langle \zeta, [\Lambda^\#(\xi), \Lambda^\#(\eta)] \rangle,$$

for $\xi, \eta, \zeta \in \mathfrak{g}^*$. A *left action* of \mathfrak{g} on a manifold Y is a Lie algebra anti-homomorphism $\lambda : \mathfrak{g} \rightarrow \mathcal{V}^1(Y)$, while a *right action* of \mathfrak{g} on Y is a Lie algebra homomorphism $\rho : \mathfrak{g} \rightarrow \mathcal{V}^1(Y)$. If $\lambda : G \times Y \rightarrow Y, (g, y) \mapsto gy$ is a left action of G on Y , one has the induced left action of \mathfrak{g} on Y , also denoted by λ , given by

$$\lambda : \mathfrak{g} \longrightarrow \mathcal{V}^1(Y), \quad \lambda(x)(y) = \frac{d}{dt}|_{t=0} \exp(tx)y, \quad x \in \mathfrak{g}, y \in Y.$$

Similarly, a right Lie group action $\rho : Y \times G \rightarrow Y, (y, g) \mapsto yg$ induces a right Lie algebra action

$$\rho : \mathfrak{g} \longrightarrow \mathcal{V}^1(Y), \quad \rho(x)(y) = \frac{d}{dt}|_{t=0} y \exp(tx), \quad x \in \mathfrak{g}, y \in Y.$$

2. REVIEW ON POISSON LIE GROUPS, LIE BIALGEBRAS, AND POISSON ACTIONS

In this section, we set up our conventions (on signs and constants) and review some facts on Lie bialgebras that will be used in this paper and in subsequent papers. Except for that in §2.6, all materials are standard and well-known, and we refer to [5, 9, 10, 18, 24, 25, 31, 36, 37, 38, 39] for more details. All Lie bialgebras in this paper are assumed to be finite dimensional.

2.1. Lie bialgebras. A *Lie bialgebra* is a pair $(\mathfrak{g}, \delta_{\mathfrak{g}})$ where \mathfrak{g} is a Lie algebra and $\delta_{\mathfrak{g}} : \mathfrak{g} \rightarrow \wedge^2 \mathfrak{g}$ a linear map, called the *co-bracket*, such that 1) the dual map $\delta_{\mathfrak{g}}^* : \wedge^2 \mathfrak{g}^* \rightarrow \mathfrak{g}^*$ of $\delta_{\mathfrak{g}}$ is a Lie bracket on the dual space \mathfrak{g}^* of \mathfrak{g} , and 2) $\delta_{\mathfrak{g}}$ satisfies the 1-cocycle condition

$$(2.1) \quad \delta_{\mathfrak{g}}[x, y] = [x, \delta_{\mathfrak{g}}(y)] + [\delta_{\mathfrak{g}}(x), y], \quad x, y \in \mathfrak{g}.$$

A *Lie bialgebra homomorphism* from one Lie bialgebra $(\mathfrak{g}_1, \delta_{\mathfrak{g}_1})$ to another $(\mathfrak{g}_2, \delta_{\mathfrak{g}_2})$ is a Lie algebra homomorphism $\phi : \mathfrak{g}_1 \rightarrow \mathfrak{g}_2$ such that $\phi(\delta_{\mathfrak{g}_1}(x)) = \delta_{\mathfrak{g}_2}(\phi(x))$ for all $x \in \mathfrak{g}_1$.

If $(\mathfrak{g}, \delta_{\mathfrak{g}})$ is a Lie bialgebra, so is $(\mathfrak{g}^*, \delta_{\mathfrak{g}^*})$, where the Lie bracket on \mathfrak{g}^* is given by $\delta_{\mathfrak{g}}^* : \wedge^2 \mathfrak{g}^* \rightarrow \mathfrak{g}^*$, the dual map of $\delta_{\mathfrak{g}}$, and $\delta_{\mathfrak{g}^*} : \mathfrak{g}^* \rightarrow \wedge^2 \mathfrak{g}^*$ is the dual map of the Lie bracket on \mathfrak{g} . The Lie bialgebra $(\mathfrak{g}^*, \delta_{\mathfrak{g}^*})$

is called the *dual Lie bialgebra* of $(\mathfrak{g}, \delta_{\mathfrak{g}})$. If $(\mathfrak{g}', \delta_{\mathfrak{g}'})$ is a Lie bialgebra isomorphic to the dual Lie bialgebra $(\mathfrak{g}^*, \delta_{\mathfrak{g}^*})$, we will call $(\mathfrak{g}, \delta_{\mathfrak{g}})$ and $(\mathfrak{g}', \delta_{\mathfrak{g}'})$ a *dual pair of Lie bialgebras*.

A *sub-Lie bialgebra* of a Lie bialgebra $(\mathfrak{g}, \delta_{\mathfrak{g}})$ is a Lie bialgebra $(\mathfrak{p}, \delta_{\mathfrak{p}})$ with an injective Lie bialgebra homomorphism $(\mathfrak{p}, \delta_{\mathfrak{p}}) \rightarrow (\mathfrak{g}, \delta_{\mathfrak{g}})$. A *quotient Lie bialgebra* of $(\mathfrak{g}, \delta_{\mathfrak{g}})$ is a Lie bialgebra $(\mathfrak{q}, \delta_{\mathfrak{q}})$ with a surjective Lie bialgebra homomorphism $(\mathfrak{g}, \delta_{\mathfrak{g}}) \rightarrow (\mathfrak{q}, \delta_{\mathfrak{q}})$.

2.2. The Double Lie (bi)algebra of a Lie bialgebra. Recall that a quadratic Lie algebra is a pair $(\mathfrak{d}, \langle \cdot, \cdot \rangle_{\mathfrak{d}})$, where \mathfrak{d} is a Lie algebra and $\langle \cdot, \cdot \rangle_{\mathfrak{d}}$ is a non-degenerate symmetric ad-invariant bilinear form on \mathfrak{d} . Let $(\mathfrak{g}, \delta_{\mathfrak{g}})$ be a Lie bialgebra and let $(\mathfrak{g}^*, \delta_{\mathfrak{g}^*})$ be its dual Lie bialgebra. Consider the direct sum vector space $\mathfrak{d} = \mathfrak{g} \oplus \mathfrak{g}^*$ and denote its elements by $x + \xi$, where $x \in \mathfrak{g}$ and $\xi \in \mathfrak{g}^*$. Then \mathfrak{d} carries the symmetric bilinear form

$$\langle x + \xi, y + \eta \rangle_{\mathfrak{d}} = \langle x, \eta \rangle + \langle \xi, y \rangle, \quad x, y \in \mathfrak{g}, \xi, \eta \in \mathfrak{g}^*,$$

and a unique Lie bracket $[\cdot, \cdot]$ extending those on \mathfrak{g} and \mathfrak{g}^* such that $\langle \cdot, \cdot \rangle_{\mathfrak{d}}$ is ad-invariant. Namely

$$(2.2) \quad [x + \xi, y + \eta] = [x, y] + \text{ad}_{\xi}^* y - \text{ad}_{\eta}^* x + [\xi, \eta] + \text{ad}_x^* \eta - \text{ad}_y^* \xi, \quad x, y \in \mathfrak{g}, \xi, \eta \in \mathfrak{g}^*,$$

where ad^* denotes both the coadjoint action of \mathfrak{g} on \mathfrak{g}^* and of \mathfrak{g}^* on \mathfrak{g} , i.e.,

$$\langle \text{ad}_x^* \xi, y \rangle = \langle \xi, [y, x] \rangle, \quad \langle \text{ad}_{\xi}^* x, \eta \rangle = \langle x, [\eta, \xi] \rangle, \quad x, y \in \mathfrak{g}, \xi \in \mathfrak{g}^*.$$

The quadratic Lie algebra $(\mathfrak{d}, \langle \cdot, \cdot \rangle_{\mathfrak{d}})$ is called the *(Drinfeld) double Lie algebra* of the Lie bialgebra $(\mathfrak{g}, \delta_{\mathfrak{g}})$.

Definition 2.1. Assume that $(\mathfrak{d}, \langle \cdot, \cdot \rangle_{\mathfrak{d}})$ is an even dimensional quadratic Lie algebra. By a *Lagrangian subalgebra* of $(\mathfrak{d}, \langle \cdot, \cdot \rangle_{\mathfrak{d}})$ we mean a Lie subalgebra \mathfrak{g} of \mathfrak{d} which is also a Lagrangian, i.e., maximal isotropic, with respect to $\langle \cdot, \cdot \rangle_{\mathfrak{d}}$. A decomposition $\mathfrak{d} = \mathfrak{g} + \mathfrak{g}'$ of vector spaces, where both \mathfrak{g} and \mathfrak{g}' are Lagrangian subalgebras of $(\mathfrak{d}, \langle \cdot, \cdot \rangle_{\mathfrak{d}})$, is called a *Lagrangian splitting* of $(\mathfrak{d}, \langle \cdot, \cdot \rangle_{\mathfrak{d}})$, and in this case $((\mathfrak{d}, \langle \cdot, \cdot \rangle_{\mathfrak{d}}), \mathfrak{g}, \mathfrak{g}')$ is also called a *Manin triple*. Given a Lagrangian splitting $\mathfrak{d} = \mathfrak{g} + \mathfrak{g}'$, by identifying \mathfrak{g} and \mathfrak{g}' as the dual spaces of each other using $\langle \cdot, \cdot \rangle_{\mathfrak{d}}$, one obtains $\delta_{\mathfrak{g}} : \mathfrak{g} \rightarrow \wedge^2 \mathfrak{g}$ and $\delta_{\mathfrak{g}'} : \mathfrak{g}' \rightarrow \wedge^2 \mathfrak{g}'$, the dual maps of the Lie brackets on \mathfrak{g}' and on \mathfrak{g} respectively, and one has the dual pair of Lie bialgebras $(\mathfrak{g}, \delta_{\mathfrak{g}})$ and $(\mathfrak{g}', \delta_{\mathfrak{g}'})$. Their double Lie algebra is clearly isomorphic to $(\mathfrak{d}, \langle \cdot, \cdot \rangle_{\mathfrak{d}})$. The notion of Lie bialgebras is therefore equivalent to that of even-dimensional quadratic Lie algebras with Lagrangian splittings.

Let $(\mathfrak{g}, \delta_{\mathfrak{g}})$ be a Lie bialgebra, with dual Lie bialgebra $(\mathfrak{g}^*, \delta_{\mathfrak{g}^*})$ and double Lie algebra $(\mathfrak{d}, \langle \cdot, \cdot \rangle_{\mathfrak{d}})$. Consider the direct product Lie algebra $\mathfrak{d}^2 = \mathfrak{d} \oplus \mathfrak{d}$ and denote its elements by (a, b) , where $a, b \in \mathfrak{d}$. Let $\langle \cdot, \cdot \rangle_{\mathfrak{d}^2}$ be the bilinear form on \mathfrak{d}^2 given by

$$(2.3) \quad \langle (a_1, a_2), (a'_1, a'_2) \rangle_{\mathfrak{d}^2} = \langle a_1, a'_1 \rangle_{\mathfrak{d}} - \langle a_2, a'_2 \rangle_{\mathfrak{d}}, \quad a_1, a_2, a'_1, a'_2 \in \mathfrak{d}.$$

The quadratic Lie algebra $(\mathfrak{d}^2, \langle \cdot, \cdot \rangle_{\mathfrak{d}^2})$ has the Lagrangian splitting $\mathfrak{d}^2 = \mathfrak{d}_{\text{diag}} + \mathfrak{d}'$, where

$$\mathfrak{d}_{\text{diag}} = \{(a, a) : a \in \mathfrak{d}\} \quad \text{and} \quad \mathfrak{d}' = \mathfrak{g}^* \oplus \mathfrak{g} = \{(\xi, x) : \xi \in \mathfrak{g}^*, x \in \mathfrak{g}\}.$$

Identifying $\mathfrak{d} \cong \mathfrak{d}_{\text{diag}}$ via $a \mapsto (a, a)$, one obtains the dual pair of Lie bialgebras $(\mathfrak{d}, \delta_{\mathfrak{d}})$ and $(\mathfrak{d}', \delta_{\mathfrak{d}'})$. The Lie bialgebra $(\mathfrak{d}, \delta_{\mathfrak{d}})$ is called the *(Drinfeld) double Lie bialgebra* of the Lie bialgebra $(\mathfrak{g}, \delta_{\mathfrak{g}})$. Note that the non-degenerate pairing between $\mathfrak{d} \cong \mathfrak{d}_{\text{diag}}$ and \mathfrak{d}' induced by $\langle \cdot, \cdot \rangle_{\mathfrak{d}^2}$ is given by

$$(2.4) \quad \langle a, (\xi, x) \rangle_{(\mathfrak{d}, \mathfrak{d}')} = \langle \xi - x, a \rangle_{\mathfrak{d}}, \quad a \in \mathfrak{d}, (\xi, x) \in \mathfrak{d}'.$$

It is straightforward to check (see also [18, §4.2] and [41, §1.1]) that

$$(2.5) \quad \delta_{\mathfrak{d}}|_{\mathfrak{g}} = \delta_{\mathfrak{g}} : \mathfrak{g} \longrightarrow \wedge^2 \mathfrak{g} \quad \text{and} \quad \delta_{\mathfrak{d}}|_{\mathfrak{g}^*} = -\delta_{\mathfrak{g}^*} : \mathfrak{g}^* \longrightarrow \wedge^2 \mathfrak{g}^*.$$

In particular, both $(\mathfrak{g}, \delta_{\mathfrak{g}})$ and $(\mathfrak{g}^*, -\delta_{\mathfrak{g}^*})$ are sub-Lie bialgebras of $(\mathfrak{d}, \delta_{\mathfrak{d}})$.

Example 2.2. Let $(\mathfrak{g}, \delta_{\mathfrak{g}})$ be a Lie bialgebra and $(\mathfrak{p}, \delta_{\mathfrak{p}})$ a sub-Lie bialgebra of $(\mathfrak{g}, \delta_{\mathfrak{g}})$. The double Lie bialgebra $(\mathfrak{d}_{\mathfrak{p}}, \delta_{\mathfrak{d}_{\mathfrak{p}}})$ of $(\mathfrak{p}, \delta_{\mathfrak{p}})$ can be identified with a “sub-quotient” of the double Lie bialgebra $(\mathfrak{d}, \delta_{\mathfrak{d}})$ of $(\mathfrak{g}, \delta_{\mathfrak{g}})$: indeed, $(\mathfrak{p} + \mathfrak{g}^*, \delta_{\mathfrak{d}|_{(\mathfrak{p} + \mathfrak{g}^*)}})$ is a sub-Lie bialgebra of $(\mathfrak{d}, \delta_{\mathfrak{d}})$, and the map $\mathfrak{p} + \mathfrak{g}^* \rightarrow \mathfrak{d}_{\mathfrak{p}} : x + \xi \mapsto x + \xi|_{\mathfrak{p}}$, $x \in \mathfrak{p}, \xi \in \mathfrak{g}^*$, is a surjective Lie bialgebra homomorphism from $(\mathfrak{p} + \mathfrak{g}^*, \delta_{\mathfrak{d}|_{(\mathfrak{p} + \mathfrak{g}^*)}})$ to $(\mathfrak{d}_{\mathfrak{p}}, \delta_{\mathfrak{d}_{\mathfrak{p}}})$. \diamond

2.3. Quasitriangular r -matrices. Let \mathfrak{g} be any Lie algebra. For $r = \sum_i x_i \otimes y_i \in \mathfrak{g} \otimes \mathfrak{g}$, define

$$(2.6) \quad \delta_r : \mathfrak{g} \longrightarrow \mathfrak{g} \otimes \mathfrak{g}, \quad \delta_r(x) = \text{ad}_x r = \sum_i ([x, x_i] \otimes y_i + x_i \otimes [x, y_i]), \quad x \in \mathfrak{g}.$$

An element $r \in \mathfrak{g} \otimes \mathfrak{g}$ is called a *classical r -matrix* (or simply an *r -matrix*) on \mathfrak{g} if (\mathfrak{g}, δ_r) is a Lie bialgebra. Writing r as $r = \Lambda + s$ with $\Lambda \in \wedge^2 \mathfrak{g}$ and $s \in S^2 \mathfrak{g}$, it follows from the definition that r is a classical r -matrix on \mathfrak{g} if and only if $s \in (S^2 \mathfrak{g})^{\mathfrak{g}}$ and $[\Lambda, \Lambda] \in (\wedge^3 \mathfrak{g})^{\mathfrak{g}}$, where $(S^2 \mathfrak{g})^{\mathfrak{g}}$ and $(\wedge^3 \mathfrak{g})^{\mathfrak{g}}$ denote respectively the ad-invariant subspaces of $S^2 \mathfrak{g}$ and $\wedge^3 \mathfrak{g}$, and $[\cdot, \cdot]$ is the Schouten bracket on $\wedge \mathfrak{g}$ (see §1.5 and (1.6)). In this paper, we will refer to classical r -matrices simply as r -matrices. Given a Lie bialgebra $(\mathfrak{g}, \delta_{\mathfrak{g}})$, an element $r \in \mathfrak{g} \otimes \mathfrak{g}$ such that $\delta_{\mathfrak{g}} = \delta_r$ is called an *r -matrix for $(\mathfrak{g}, \delta_{\mathfrak{g}})$* . A Lie bialgebra $(\mathfrak{g}, \delta_{\mathfrak{g}})$ is said to be *co-boundary* if it has an r -matrix.

Recall [10] the classical Yang-Baxter operator $\text{CYB} : \mathfrak{g} \otimes \mathfrak{g} \rightarrow \mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g}$ given by

$$(2.7) \quad \text{CYB}(r) = [r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] \in \mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g} \subset U(\mathfrak{g})^{\otimes 3}, \quad r \in \mathfrak{g} \otimes \mathfrak{g},$$

where $U(\mathfrak{g})$ is the universal enveloping algebra of \mathfrak{g} , $[\cdot, \cdot]$ on the right hand side denotes the commutator bracket in $U(\mathfrak{g})^{\otimes 3}$, and $r_{12}, r_{13}, r_{23} \in U(\mathfrak{g})^{\otimes 3}$ are respectively given by

$$r_{12} = \sum_i x_i \otimes y_i \otimes 1, \quad r_{23} = \sum_i 1 \otimes x_i \otimes y_i, \quad r_{13} = \sum_i x_i \otimes 1 \otimes y_i$$

if $r = \sum_i x_i \otimes y_i \in \mathfrak{g} \otimes \mathfrak{g}$. One checks directly [9, 24] (note our convention in §1.5 and (1.4)) that

$$(2.8) \quad \text{CYB}(\Lambda) = \frac{1}{2}[\Lambda, \Lambda] \in \wedge^3 \mathfrak{g} \text{ if } \Lambda \in \wedge^2 \mathfrak{g};$$

$$(2.9) \quad \text{CYB}(s) \in (\wedge^3 \mathfrak{g})^{\mathfrak{g}} \text{ and } \text{CYB}(s)(\xi, \eta, \zeta) = \langle \xi, [s^{\#}(\eta), s^{\#}(\zeta)] \rangle, \quad \xi, \eta, \zeta \in \mathfrak{g}^*, \text{ if } s \in (S^2 \mathfrak{g})^{\mathfrak{g}};$$

$$(2.10) \quad \text{CYB}(r) = \text{CYB}(\Lambda) + \text{CYB}(s) \in \wedge^3 \mathfrak{g} \text{ if } r = \Lambda + s \text{ with } \Lambda \in \wedge^2 \mathfrak{g} \text{ and } s \in (S^2 \mathfrak{g})^{\mathfrak{g}}.$$

An element $r \in \mathfrak{g} \otimes \mathfrak{g}$ is called a *quasitriangular r -matrix* on \mathfrak{g} if the symmetric part of r is ad-invariant and if r satisfies the *Classical Yang Baxter Equation* $\text{CYB}(r) = 0$. By (2.8) - (2.10), a quasitriangular r -matrix on \mathfrak{g} is indeed an r -matrix on \mathfrak{g} (see also [10, 24, 33]). Let

$$r^{21} = \sum_i y_i \otimes x_i \quad \text{if} \quad r = \sum_i x_i \otimes y_i,$$

It is clear that $r \in \mathfrak{g} \otimes \mathfrak{g}$ is a quasitriangular r -matrix on \mathfrak{g} if and only if r^{21} is.

Definition 2.3. 1) A quasitriangular r -matrix for a Lie bialgebra $(\mathfrak{g}, \delta_{\mathfrak{g}})$ is also called a *quasitriangular structure* of $(\mathfrak{g}, \delta_{\mathfrak{g}})$. If r is a quasitriangular r -matrix on \mathfrak{g} , we call the pair (\mathfrak{g}, r) a *quasitriangular Lie bialgebra* (with $\delta_r : \mathfrak{g} \rightarrow \wedge^2 \mathfrak{g}$ as its co-bracket). A quasitriangular r -matrix r on \mathfrak{g} is said to be *factorizable* if its symmetric part is non-degenerate, and in this case we call (\mathfrak{g}, r) a factorizable Lie bialgebra;

2) Given two quasitriangular Lie bialgebras (\mathfrak{g}_1, r_1) and (\mathfrak{g}_2, r_2) , we say $\phi : (\mathfrak{g}_1, r_1) \rightarrow (\mathfrak{g}_2, r_2)$ is a *morphism of quasitriangular Lie bialgebras* if $\phi : \mathfrak{g}_1 \rightarrow \mathfrak{g}_2$ is a Lie algebra homomorphism and $\phi(r_1) = r_2$.

Remark 2.4. A quasitriangular Lie bialgebra is thus the Lie bialgebra together with a quasitriangular quasitriangular structure. A morphism $\phi : (\mathfrak{g}_1, r_1) \rightarrow (\mathfrak{g}_2, r_2)$ of quasitriangular Lie bialgebras induces a morphism $\phi : (\mathfrak{g}_1, \delta_{r_1}) \rightarrow (\mathfrak{g}_2, \delta_{r_2})$ of Lie bialgebras, but the converse fails in general (for example, when both \mathfrak{g}_1 and \mathfrak{g}_2 are abelian Lie algebras). \diamond

Let $(\mathfrak{g}, \delta_{\mathfrak{g}})$ be a Lie bialgebra and $(\mathfrak{d}, \langle, \rangle_{\mathfrak{d}})$ its double Lie algebra. Let $\dim \mathfrak{g} = m$ and let $\text{Gr}_0(m, \mathfrak{d})$ be the set of all m -dimensional subspaces of \mathfrak{d} complementary to \mathfrak{g} . Then

$$(2.11) \quad \mathfrak{g} \otimes \mathfrak{g} \longrightarrow \text{Gr}_0(m, \mathfrak{d}), \quad r \longmapsto \mathfrak{k}_r = \{-r^{\#}(\xi) + \xi : \xi \in \mathfrak{g}^*\}$$

is a bijection, where $r^{\#} : \mathfrak{g}^* \rightarrow \mathfrak{g}$ is given in (1.4). Note that $\mathfrak{k}_r^{\perp} = \mathfrak{k}_{-r^{21}}$ for any $r \in \mathfrak{g} \otimes \mathfrak{g}$, where $\mathfrak{q}^{\perp} = \{a \in \mathfrak{d} : \langle a, \mathfrak{q} \rangle_{\mathfrak{d}} = 0\}$ for $\mathfrak{q} \subset \mathfrak{d}$. In particular, \mathfrak{k}_r is a Lagrangian subspace of \mathfrak{d} with respect to $\langle, \rangle_{\mathfrak{d}}$ if and only if $r \in \wedge^2 \mathfrak{g}$. A proof of the following result of Drinfeld can be found in [3].

Lemma 2.5. (Drinfeld [11]) *An element $r \in \mathfrak{g} \otimes \mathfrak{g}$ is an r -matrix for $(\mathfrak{g}, \delta_{\mathfrak{g}})$ if and only if $[\mathfrak{g}, \mathfrak{k}_r] \subset \mathfrak{k}_r$, and r is a quasitriangular r -matrix for $(\mathfrak{g}, \delta_{\mathfrak{g}})$ if and only if \mathfrak{k}_r is a Lie ideal of \mathfrak{d} .*

Applying Lemma 2.5 to the double Lie bialgebra $(\mathfrak{d}, \delta_{\mathfrak{d}})$ of any Lie bialgebra $(\mathfrak{g}, \delta_{\mathfrak{g}})$, one sees that $(\mathfrak{d}, \delta_{\mathfrak{d}})$ is co-boundary: the two Lie ideals $0 \oplus \mathfrak{d}$ and $\mathfrak{d} \oplus 0$ of $\mathfrak{d}^2 = \mathfrak{d} \oplus \mathfrak{d}$ give rise to the two factorizable quasitriangular matrices $r_{\mathfrak{d}}$ and $-(r_{\mathfrak{d}})^{21}$ for $(\mathfrak{d}, \delta_{\mathfrak{d}})$ respectively, with

$$(2.12) \quad r_{\mathfrak{d}} = \sum_{i=1}^m x_i \otimes \xi_i \in \mathfrak{d} \otimes \mathfrak{d},$$

where $\{x_i\}_{i=1}^m$ is any basis of \mathfrak{g} and $\{\xi_i\}_{i=1}^m$ its dual basis of \mathfrak{g}^* . It is straightforward to check [18, §4.2] that $\delta_{\mathfrak{d}} = \delta_{r_{\mathfrak{d}}} : \mathfrak{d} \rightarrow \wedge^2 \mathfrak{d}$.

Definition 2.6. We call $r_{\mathfrak{d}} \in \mathfrak{d} \otimes \mathfrak{d}$ in (2.12) *the quasitriangular r -matrix on \mathfrak{d} associated to the Lagrangian splitting $\mathfrak{d} = \mathfrak{g} + \mathfrak{g}^*$ of $(\mathfrak{d}, \langle, \rangle_{\mathfrak{d}})$* . Its skew-symmetric part

$$(2.13) \quad \Lambda_{\mathfrak{g}, \mathfrak{g}^*} = \frac{1}{2} \sum_{i=1}^m x_i \wedge \xi_i \in \wedge^2 \mathfrak{d}$$

will be called the *skew-symmetric r -matrix* on \mathfrak{d} associated to the Lagrangian splitting $\mathfrak{d} = \mathfrak{g} + \mathfrak{g}^*$.

Let $(\mathfrak{g}, \delta_{\mathfrak{g}})$ be a Lie bialgebra with a quasitriangular structure $r \in \mathfrak{g} \otimes \mathfrak{g}$, and let $(\mathfrak{g}^*, \delta_{\mathfrak{g}^*})$ and $(\mathfrak{d}, \delta_{\mathfrak{d}})$ be respectively the dual and the double Lie bialgebra of $(\mathfrak{g}, \delta_{\mathfrak{g}})$. Define (see (1.4))

$$(2.14) \quad r_+ = r^{\#} : \mathfrak{g}^* \longrightarrow \mathfrak{g}, \quad r_- = -(r^{21})^{\#} = -r_+^* : \mathfrak{g}^* \longrightarrow \mathfrak{g},$$

$$(2.15) \quad p_{\pm} : \mathfrak{d} \longrightarrow \mathfrak{g}, \quad p_+(x + \xi) = x + r_+(\xi), \quad p_-(x + \xi) = x + r_-(\xi), \quad x \in \mathfrak{g}, \xi \in \mathfrak{g}^*.$$

It follows from Lemma 2.5 (see also [12, 18, 23, 24, 36]) that both p_+ and p_- are Lie algebra homomorphisms. With $r_{\mathfrak{d}} \in \mathfrak{d} \otimes \mathfrak{d}$ given in (2.12), one has

$$(2.16) \quad p_+(r_{\mathfrak{d}}) = \sum_{i=1}^m x_i \otimes r_+(\xi_i) = r, \quad p_-(r_{\mathfrak{d}}) = \sum_{i=1}^m x_i \otimes r_-(\xi_i) = -r^{21}.$$

In particular, p_+ and p_- are also Lie bialgebra homomorphisms from $(\mathfrak{d}, \delta_{\mathfrak{d}})$ to $(\mathfrak{g}, \delta_{\mathfrak{g}})$, and it follows that $r_+, r_- : (\mathfrak{g}^*, -\delta_{\mathfrak{g}^*}) \rightarrow (\mathfrak{g}, \delta_{\mathfrak{g}})$ are also Lie bialgebra homomorphisms. Let

$$(2.17) \quad \mathfrak{f}_- = \text{Im}(r_-) \subset \mathfrak{g}, \quad \mathfrak{f}_+ = \text{Im}(r_+) \subset \mathfrak{g}.$$

Then $(\mathfrak{f}_-, \delta_{\mathfrak{g}}|_{\mathfrak{f}_-})$ and $(\mathfrak{f}_+, \delta_{\mathfrak{g}}|_{\mathfrak{f}_+})$ are sub-Lie bialgebras of $(\mathfrak{g}, \delta_{\mathfrak{g}})$. Consider the well-defined non-degenerate bilinear pairing $\langle, \rangle_{(\mathfrak{f}_-, \mathfrak{f}_+)}$ between \mathfrak{f}_- and \mathfrak{f}_+ given by

$$(2.18) \quad \langle r_-(\xi), r_+(\eta) \rangle_{(\mathfrak{f}_-, \mathfrak{f}_+)} = -\langle \xi, r_+(\eta) \rangle = \langle r_-(\xi), \eta \rangle, \quad \xi, \eta \in \mathfrak{g}^*.$$

Lemma 2.7. (See also [18, Proposition 4.1]) *Under the non-degenerate pairing $\langle, \rangle_{(\mathfrak{f}_-, \mathfrak{f}_+)}$ between \mathfrak{f}_- and \mathfrak{f}_+ , $(\mathfrak{f}_-, \delta_{\mathfrak{g}}|_{\mathfrak{f}_-})$ and $(\mathfrak{f}_+, -\delta_{\mathfrak{g}}|_{\mathfrak{f}_+})$ form a dual pair of Lie bialgebras.*

Proof. Let $x_-, y_- \in \mathfrak{f}_-$ and $x_+, y_+ \in \mathfrak{f}_+$. Let $x_- = r_-(\xi)$ and $x_+ = r_+(\eta)$, where $\xi, \eta \in \mathfrak{g}^*$. As $r_\pm : (\mathfrak{g}^*, -\delta_{\mathfrak{g}^*}) \rightarrow (\mathfrak{g}, \delta_{\mathfrak{g}})$ are Lie bialgebra homomorphisms, one has

$$\begin{aligned} \langle x_- \wedge y_-, \delta_{\mathfrak{g}}(x_+) \rangle_{(\mathfrak{f}_-, \mathfrak{f}_+)} &= \langle x_- \wedge y_-, \delta_{\mathfrak{g}}(r_+(\eta)) \rangle_{(\mathfrak{f}_-, \mathfrak{f}_+)} = -\langle x_- \wedge y_-, r_+(\delta_{\mathfrak{g}^*}(\eta)) \rangle_{(\mathfrak{f}_-, \mathfrak{f}_+)} \\ &= -\langle x_- \wedge y_-, \delta_{\mathfrak{g}^*}(\eta) \rangle = -\langle [x_-, y_-], \eta \rangle = -\langle [x_-, y_-], x_+ \rangle_{(\mathfrak{f}_-, \mathfrak{f}_+)}, \\ \langle \delta_{\mathfrak{g}}(x_-), x_+ \wedge y_+ \rangle_{(\mathfrak{f}_-, \mathfrak{f}_+)} &= \langle \delta_{\mathfrak{g}}(r_-(\xi)), x_+ \wedge y_+ \rangle_{(\mathfrak{f}_-, \mathfrak{f}_+)} = -\langle r_-(\delta_{\mathfrak{g}^*}(\xi)), x_+ \wedge y_+ \rangle_{(\mathfrak{f}_-, \mathfrak{f}_+)} \\ &= -\langle \delta_{\mathfrak{g}^*}(\xi), x_+ \wedge y_+ \rangle = -\langle \xi, [x_+, y_+] \rangle = \langle x_-, [x_+, y_+] \rangle_{(\mathfrak{f}_-, \mathfrak{f}_+)}. \end{aligned}$$

Q.E.D.

Remark 2.8. Let $\{x_i\}_{i=1}^m$ be a basis of \mathfrak{g} such that $\{x_i\}_{i=1}^l$ is a basis of \mathfrak{f}_- , and let $\{\xi_i\}_{i=1}^m$ be the dual basis of \mathfrak{g}^* . As $\text{Span}\{\xi_{l+1}, \dots, \xi_m\} = \mathfrak{f}_-^0 = \ker r_+$, $\{r_+(\xi_i)\}_{i=1}^l$ is a basis of \mathfrak{f}_+ , dual to the basis $\{x_i\}_{i=1}^l$ of \mathfrak{f}_- under the pairing $\langle \cdot, \cdot \rangle_{(\mathfrak{f}_-, \mathfrak{f}_+)}$. Consequently,

$$(2.19) \quad r = \sum_{i=1}^m x_i \otimes r_+(\xi_i) = \sum_{i=1}^l x_i \otimes r_+(\xi_i) \in \mathfrak{f}_- \otimes \mathfrak{f}_+ \subset \mathfrak{g} \otimes \mathfrak{g}.$$

◇

Let $(\mathfrak{d}_{\mathfrak{f}_-}, \delta_{\mathfrak{d}_{\mathfrak{f}_-}})$ be the double Lie bialgebra of $(\mathfrak{f}_-, \delta_{\mathfrak{g}}|_{\mathfrak{f}_-})$, and let $r_{\mathfrak{d}_{\mathfrak{f}_-}}$ be the quasitriangular r -matrix on $\mathfrak{d}_{\mathfrak{f}_-}$ associated to the Lagrangian splitting $\mathfrak{d}_{\mathfrak{f}_-} = \mathfrak{f}_- + \mathfrak{f}_+$. Consider the linear map

$$(2.20) \quad q : \mathfrak{d}_{\mathfrak{f}_-} \longrightarrow \mathfrak{g}, \quad q(x_-, x_+) = x_- + x_+, \quad x_- \in \mathfrak{f}_-, \quad x_+ \in \mathfrak{f}_+.$$

Lemma 2.9. ([18, Proposition 4.1]) *The map $q : (\mathfrak{d}_{\mathfrak{f}_-}, \delta_{\mathfrak{d}_{\mathfrak{f}_-}}) \rightarrow (\mathfrak{g}, \delta_{\mathfrak{g}})$ is a Lie bialgebra homomorphism. In fact, $q(r_{\mathfrak{d}_{\mathfrak{f}_-}}) = r$.*

Proof. Identify $(\mathfrak{d}_{\mathfrak{f}_-}, \delta_{\mathfrak{d}_{\mathfrak{f}_-}})$ with the quotient of the sub-Lie bialgebra $(\mathfrak{f}_- + \mathfrak{g}^*, \delta_{\mathfrak{d}}|_{\mathfrak{f}_- + \mathfrak{g}^*})$ of $(\mathfrak{d}, \delta_{\mathfrak{d}})$ as in Example 2.2. Then q , being the map induced by the Lie algebra homomorphism $p_+ : \mathfrak{d} \rightarrow \mathfrak{g}$, is a Lie algebra homomorphism. It follows from the definition of $r_{\mathfrak{d}_{\mathfrak{f}_-}}$ and (2.19) that $q(r_{\mathfrak{d}_{\mathfrak{f}_-}}) = r$.

Q.E.D.

2.4. Twists of Lie bialgebras and of quasitriangular r -matrices. Let $(\mathfrak{g}, \delta_{\mathfrak{g}})$ be a Lie bialgebra and $(\mathfrak{d}, \langle \cdot, \cdot \rangle_{\mathfrak{d}})$ its double Lie algebra. An element $t \in \wedge^2 \mathfrak{g}$ is called a *twisting element* for $(\mathfrak{g}, \delta_{\mathfrak{g}})$ if the Lagrangian subspace $\mathfrak{k}_t = \{-t^\#(\xi) + \xi : \xi \in \mathfrak{g}^*\}$ of \mathfrak{d} (see (2.11)) is a Lie subalgebra of \mathfrak{d} . Using (1.6), one readily checks [11] that $t \in \wedge^2 \mathfrak{g}$ is a twisting element for $(\mathfrak{g}, \delta_{\mathfrak{g}})$ if and only if

$$(2.21) \quad \delta_{\mathfrak{g}}(t) + \frac{1}{2}[t, t] = 0,$$

where the co-bracket $\delta_{\mathfrak{g}} : \mathfrak{g} \rightarrow \wedge^2 \mathfrak{g}$ is linearly extended to $\delta_{\mathfrak{g}} : \wedge^2 \mathfrak{g} \rightarrow \wedge^3 \mathfrak{g}$ via

$$(2.22) \quad \delta_{\mathfrak{g}}(x \wedge y) = \delta_{\mathfrak{g}}(x) \wedge y - x \wedge \delta_{\mathfrak{g}}(y), \quad x, y \in \mathfrak{g}.$$

A twisting element t for $(\mathfrak{g}, \delta_{\mathfrak{g}})$ gives rise to the Lagrangian splitting $\mathfrak{d} = \mathfrak{g} + \mathfrak{k}_t$ and thus another co-bracket $\delta_{\mathfrak{g}, t} : \mathfrak{g} \rightarrow \wedge^2 \mathfrak{g}$ on \mathfrak{g} given by

$$(2.23) \quad \delta_{\mathfrak{g}, t}(x) = \delta_{\mathfrak{g}}(x) - [x, t] = \delta_{\mathfrak{g}}(x) + [t, x], \quad x \in \mathfrak{g}.$$

The Lie bialgebra $(\mathfrak{g}, \delta_{\mathfrak{g}, t})$ is called the *twist* of $(\mathfrak{g}, \delta_{\mathfrak{g}})$ by the twisting element t , and we also say that the Lagrangian splitting $\mathfrak{d} = \mathfrak{g} + \mathfrak{k}_t$ is the *twist* of the Lagrangian splitting $\mathfrak{d} = \mathfrak{g} + \mathfrak{g}^*$ by t . Note that the two quasitriangular r -matrices $r_{\mathfrak{d}}$ and $r_{\mathfrak{d}, t}$ on \mathfrak{d} associated to the two Lagrangian splittings $\mathfrak{d} = \mathfrak{g} + \mathfrak{g}^*$ and $\mathfrak{d} = \mathfrak{g} + \mathfrak{k}_t$ are now related by $r_{\mathfrak{d}, t} = r_{\mathfrak{d}} - t$, where $t \in \wedge^2 \mathfrak{g}$ is regarded as an element in $\wedge^2 \mathfrak{d}$ via the embedding $\mathfrak{g} \hookrightarrow \mathfrak{d}$.

Lemma 2.10. *Assume that the Lie bialgebra $(\mathfrak{g}, \delta_{\mathfrak{g}})$ has a quasitriangular structure $r \in \mathfrak{g} \otimes \mathfrak{g}$. Then*

$$\wedge^2 \mathfrak{g} \longrightarrow \mathfrak{g} \otimes \mathfrak{g}, \quad t \longmapsto r - t,$$

is a one-to-one correspondence between twisting elements for the Lie bialgebra $(\mathfrak{g}, \delta_{\mathfrak{g}})$ and quasitriangular r -matrices on \mathfrak{g} that have the same symmetric part as r .

Proof. Let $r = \Lambda + s$, where $\Lambda \in \wedge^2 \mathfrak{g}$. Let $t \in \wedge^2 \mathfrak{g}$. By (2.8) - (2.10),

$$\text{CYB}(r - t) = \frac{1}{2}[\Lambda - t, \Lambda - t] + \text{CYB}(s) = -[t, \Lambda] + \frac{1}{2}[t, t] = \delta_{\mathfrak{g}}(t) + \frac{1}{2}[t, t].$$

Thus $r - t$ is a quasitriangular matrix on \mathfrak{g} if and only if t is a twisting element for $(\mathfrak{g}, \delta_{\mathfrak{g}})$.

Q.E.D.

Definition 2.11. Given a quasitriangular r -matrix r on a Lie algebra \mathfrak{g} , any quasitriangular r -matrix r' on \mathfrak{g} with the same symmetric part as r will be called a *twist* of r .

2.5. Poisson Lie groups and Poisson actions. A Poisson bi-vector field π_G on a Lie group G is said to be *multiplicative* if the group multiplication map $(G \times G, \pi_G \times \pi_G) \rightarrow (G, \pi_G)$, $(g_1, g_2) \mapsto g_1 g_2$, is Poisson. A *Poisson Lie group* is a pair (G, π_G) , where G is a Lie group and π_G is a multiplicative Poisson bi-vector field on G . A Lie subgroup of G that is also a Poisson submanifold with respect to π_G is called a *Poisson Lie subgroup* of (G, π_G) .

Let (G, π_G) be a Poisson Lie group and \mathfrak{g} the Lie algebra of G . Then $\pi_G(e) = 0$, where $e \in G$ is the identity element. Let $d_e \pi_G : \mathfrak{g} \rightarrow \wedge^2 \mathfrak{g}$ be the linearization of π_G at e , given by $(d_e \pi_G)(x) = [\tilde{x}, \pi_G](e)$, where for $x \in \mathfrak{g}$, \tilde{x} is any local vector field on G with $\tilde{x}(e) = x$. Then the pair $(\mathfrak{g}, \delta_{\mathfrak{g}} = d_e \pi_G)$ is a Lie bialgebra, called the *Lie bialgebra of the Poisson Lie group* (G, π_G) .

Let (G, π_G) be a Poisson Lie group with Lie bialgebra $(\mathfrak{g}, \delta_{\mathfrak{g}})$. Any Poisson Lie group (G^*, π_{G^*}) whose Lie bialgebra is isomorphic to the dual Lie bialgebra of $(\mathfrak{g}, \delta_{\mathfrak{g}})$ is called a *dual Poisson Lie group* of (G, π_G) , and we also call (G, π_G) and (G^*, π_{G^*}) a *dual pair of Poisson Lie groups*. If **CHECK** (D, π_D) is a connected Poisson Lie group whose Lie bialgebra $(\mathfrak{d}, \delta_{\mathfrak{d}})$ is isomorphic to the double Lie bialgebra of $(\mathfrak{g}, \delta_{\mathfrak{g}})$, and if the Lie algebra embedding $\mathfrak{g} \rightarrow \mathfrak{d}$ integrates to a Lie group homomorphism $G \rightarrow D$, we call (D, π_D) a *Drinfeld double* of (G, π_G) .

A *left Poisson action* of a Poisson Lie group (G, π_G) on a Poisson manifold (Y, π_Y) is a left group action $\sigma : G \times Y \rightarrow Y$ such that $\sigma : (G \times Y, \pi_G \times \pi_Y) \rightarrow (Y, \pi_Y)$ is a Poisson map. Similarly, a right group action of G on Y given by $\sigma : Y \times G \rightarrow Y$ is a *right Poisson action* of (G, π_G) on (Y, π_Y) if $\sigma : (Y \times G, \pi_Y \times \pi_G) \rightarrow (Y, \pi_Y)$ is a Poisson map.

A *left (resp. right) Poisson action* of a Lie bialgebra $(\mathfrak{g}, \delta_{\mathfrak{g}})$ on a Poisson manifold (Y, π_Y) is a Lie algebra anti-homomorphism (resp. Lie algebra homomorphism) $\sigma : \mathfrak{g} \rightarrow \mathcal{V}^1(Y)$ such that

$$[\sigma(x), \pi_Y] = \sigma(\delta_{\mathfrak{g}}(x)), \quad \forall x \in \mathfrak{g}.$$

Here recall our convention from §1.5 that for any integer $k \geq 2$, σ also denotes the linear map

$$(2.24) \quad \wedge^k \mathfrak{g} \longrightarrow \mathcal{V}^k(Y), \quad \sigma(x_1 \wedge \cdots \wedge x_k) = \sigma(x_1) \wedge \cdots \wedge \sigma(x_k), \quad x_1, \dots, x_k \in \mathfrak{g}.$$

Definition 2.12. 1) For a Poisson Lie group (G, π_G) , a *left (or right) (G, π_G) -Poisson space* is a triple (Y, π_Y, σ) , where (Y, π_Y) is a Poisson manifold and σ a left (or right) Poisson action of (G, π_G) on (Y, π_Y) ;

2) For a Lie bialgebra $(\mathfrak{g}, \delta_{\mathfrak{g}})$, a *left (or right) $(\mathfrak{g}, \delta_{\mathfrak{g}})$ -Poisson space* is a triple (Y, π_Y, σ) , where (Y, π_Y) is a Poisson manifold and $\sigma : \mathfrak{g} \rightarrow \mathcal{V}^1(Y)$ a left (or right) Poisson action of $(\mathfrak{g}, \delta_{\mathfrak{g}})$ on (Y, π_Y) .

3) For a quasitriangular Lie algebra (\mathfrak{g}, r) (see Definition 2.3), a *left (or right) (\mathfrak{g}, r) -Poisson space* is a left (or right)-Poisson space of the Lie bialgebra (\mathfrak{g}, δ_r) , where $\delta_r : \mathfrak{g} \rightarrow \wedge^2 \mathfrak{g}$ is given in (2.6).

Let (G, π) be a connected Poisson Lie group with Lie bialgebra $(\mathfrak{g}, \delta_{\mathfrak{g}})$, let (Y, π_Y) be a Poisson manifold, and let $\sigma : G \times Y \rightarrow Y$ (resp. $\sigma : Y \times G \rightarrow Y$) be a left (resp. right) action of G on Y . Recall from §1.5 our convention on Lie algebra actions induced from Lie group actions.

Lemma 2.13. [40] *(Y, π_Y, σ) is a (G, π_G) -Poisson space if and only if it is a $(\mathfrak{g}, \delta_{\mathfrak{g}})$ -Poisson space.*

Remark 2.14. By Lemma 2.13, if (Y, π_Y, λ) is a left $(\mathfrak{g}, \delta_{\mathfrak{g}})$ -Poisson space, then $(Y, \pi_Y, -\lambda)$ is a right $(\mathfrak{g}, -\delta_{\mathfrak{g}})$ -Poisson space, and $(Y, -\pi_Y, -\lambda)$ is a right $(\mathfrak{g}, \delta_{\mathfrak{g}})$ -Poisson space. \diamond

Example 2.15. (Dressing actions) Let (G, π_G) be a Poisson Lie group with Lie bialgebra $(\mathfrak{g}, \delta_{\mathfrak{g}})$, and let $(\mathfrak{g}^*, \delta_{\mathfrak{g}^*})$ and $(\mathfrak{d}, \delta_{\mathfrak{d}})$ be respectively the dual and double Lie bialgebra of $(\mathfrak{g}, \delta_{\mathfrak{g}})$. A Poisson map $\Phi : (Y, \pi_Y) \rightarrow (G, \pi_G)$ generates a right Poisson action of $(\mathfrak{g}^*, \delta_{\mathfrak{g}^*})$ on (Y, π_Y) by

$$\mathfrak{g}^* \longrightarrow \mathcal{V}^1(Y), \quad \xi \longmapsto \pi_Y^\#(\Phi^*(\xi^R)), \quad \xi \in \mathfrak{g}^*,$$

where for $\xi \in \mathfrak{g}^*$, ξ^R denotes the right invariant 1-form on G with value ξ at e . In particular, the identity map $G \rightarrow G$ generates [38] the *right dressing action* ϱ of $(\mathfrak{g}^*, \delta_{\mathfrak{g}^*})$ on (G, π_G) given by

$$(2.25) \quad \varrho : \mathfrak{g}^* \longrightarrow \mathcal{V}^1(G), \quad \varrho(\xi) = \pi_G^\#(\xi^R), \quad \xi \in \mathfrak{g}^*.$$

The vector fields $\varrho(\xi)$, $\xi \in \mathfrak{g}^*$, are called the *right dressing vector fields* on G . For $g \in G$, denote by Ad_g the Adjoint action of g on \mathfrak{g} and by l_g and r_g the left and right translations on G by g . Then [13] the adjoint representation of \mathfrak{g} on \mathfrak{d} integrates to an action of G on \mathfrak{d} , again denoted by Ad_g for $g \in G$, via

$$(2.26) \quad \text{Ad}_{g^{-1}}(x + \xi) = \text{Ad}_{g^{-1}}x - l_{g^{-1}}\pi_G^\#(\xi^R)(g) + \text{Ad}_g^*\xi, \quad x \in \mathfrak{g}, \xi \in \mathfrak{g}^*, g \in G.$$

Let $p_{\mathfrak{g}} : \mathfrak{d} \rightarrow \mathfrak{g}$ be the projection with respect to the decomposition $\mathfrak{d} = \mathfrak{g} + \mathfrak{g}^*$. By (2.26),

$$(2.27) \quad \varrho(\xi)(g) = -l_g p_{\mathfrak{g}} \text{Ad}_{g^{-1}}\xi, \quad \xi \in \mathfrak{g}^*.$$

The following “multiplicativity” of the dressing vector fields $\varrho(\xi)$, $\xi \in \mathfrak{g}^*$, can be proved directly using (2.27) (see also [25, Lemma 3.4.8] and [29, Corollary 3.10]):

$$(2.28) \quad \varrho(\xi)(g_1 g_2) = r_{g_2} \varrho(\xi)(g_1) + l_{g_1} \varrho(\text{Ad}_{g_1}^* \xi)(g_2), \quad g_1, g_2 \in G.$$

Let λ be the left action of \mathfrak{g} on G given by

$$\lambda : \mathfrak{g} \longrightarrow \mathcal{V}^1(G), \quad \lambda(x) = x^R, \quad x \in \mathfrak{g},$$

where for $x \in \mathfrak{g}$, x^R denotes the right invariant vector field on G with value x at e . It follows from (2.28) that one has the left Lie algebra action

$$(2.29) \quad \varsigma : \mathfrak{d} \longrightarrow \mathcal{V}^1(G), \quad \varsigma(x + \xi) = \lambda(x) - \varrho(\xi) = x^R - \pi_G^\#(\xi^R), \quad x \in \mathfrak{g}, \xi \in \mathfrak{g}^*,$$

of \mathfrak{d} on G , which, by (2.5), is a left Poisson action of the Lie bialgebra $(\mathfrak{d}, \delta_{\mathfrak{d}})$ on (G, π_G) . \diamond

Recall from §2.4 the notion of twists of Lie bialgebras.

Lemma 2.16. (Twists of Poisson actions.) *Assume that the Lie bialgebra $(\mathfrak{g}, \delta_{\mathfrak{g},t})$ is the twist of a Lie bialgebra $(\mathfrak{g}, \delta_{\mathfrak{g}})$ by a twisting element $t \in \wedge^2 \mathfrak{g}$ of $(\mathfrak{g}, \delta_{\mathfrak{g}})$.*

- 1) *If (Y, π_Y, λ) is a left $(\mathfrak{g}, \delta_{\mathfrak{g}})$ -Poisson space, then $(Y, \pi_Y + \lambda(t), \lambda)$ is a left $(\mathfrak{g}, \delta_{\mathfrak{g},t})$ -Poisson space;*
- 2) *If (Y, π_Y, ρ) is a right $(\mathfrak{g}, \delta_{\mathfrak{g}})$ -Poisson space, then $(Y, \pi_Y - \rho(t), \rho)$ is a right $(\mathfrak{g}, \delta_{\mathfrak{g},t})$ -Poisson space.*

Proof. Assume that (Y, π_Y, λ) is a left $(\mathfrak{g}, \delta_{\mathfrak{g}})$ -Poisson space. Then

$$[\pi_Y + \lambda(t), \pi_Y + \lambda(t)] = 2[\lambda(t), \pi_Y] + [\lambda(t), \lambda(t)] = -\lambda(2\delta_{\mathfrak{g}}(t) + [t, t]) = 0.$$

Thus $\pi_Y + \lambda(t)$ is a Poisson structure on Y . It follows from (2.23) that λ is also a left Poisson action of the Lie bialgebra $(\mathfrak{g}, \delta_{\mathfrak{g},t})$ on the Poisson manifold $(Y, \pi_Y + \lambda(t))$. 2) is proved similarly.

Q.E.D.

2.6. Poisson structures defined by quasitriangular r -matrices. Let \mathfrak{g} be any Lie algebra and let $\sigma : \mathfrak{g} \rightarrow \mathcal{V}^1(Y)$ be a left or right action of \mathfrak{g} on a manifold Y . For $r = \sum_i x_i \otimes y_i \in \mathfrak{g} \otimes \mathfrak{g}$, one has the 2-tensor field $\sigma(r)$ on Y given by

$$\sigma(r) = \sum_i \sigma(x_i) \otimes \sigma(y_i).$$

Let s be the symmetric part of r . Clearly, $\sigma(r)$ is skew-symmetric, i.e., $\sigma(r)$ is a bi-vector field on Y , if and only if $\sigma(s) = 0$. A subspace \mathfrak{c} of \mathfrak{g} is said to be *coisotropic with respect to s* if $s^\#(\mathfrak{c}^0) \subset \mathfrak{c}$, where recall that $\mathfrak{c}^0 = \{\xi \in \mathfrak{g}^* : \langle \xi, \mathfrak{c} \rangle = 0\} \subset \mathfrak{g}^*$.

Lemma 2.17. *For $r \in \mathfrak{g} \otimes \mathfrak{g}$, the 2-tensor field $\sigma(r)$ on Y is skew-symmetric if and only if the stabilizer subalgebra of \mathfrak{g} at every $y \in Y$ is coisotropic with respect to the symmetric part s of r .*

Proof. Let $y \in Y$ and let $\sigma_y : \mathfrak{g} \rightarrow T_y$, $\sigma_y(x) = \sigma(x)(y)$ for $x \in \mathfrak{g}$. Then $\ker \sigma_y \subset \mathfrak{g}$ is the stabilizer subalgebra of \mathfrak{g} at y , and $(\ker \sigma_y)^0 = \text{Im}(\sigma_y^* : T_y^*Y \rightarrow \mathfrak{g}^*)$. It follows from the definitions that $(\sigma(s)(y))^\# = \sigma_y \circ s^\# \circ \sigma_y^* : T_y^*Y \rightarrow T_yY$. Thus $\sigma(s)(y) = 0$ if and only if $\ker \sigma_y$ is coisotropic with respect to s .

Q.E.D.

Let $r \in \mathfrak{g} \otimes \mathfrak{g}$ be a quasitriangular r -matrix on a Lie algebra \mathfrak{g} so that (\mathfrak{g}, r) is a quasitriangular Lie bialgebra. Let $\lambda : \mathfrak{g} \rightarrow \mathcal{V}^1(Y)$ be a left Lie algebra action of \mathfrak{g} on a manifold Y .

Proposition 2.18. *If the two tensor field $\lambda(r)$ on Y is skew-symmetric, it is Poisson, and $(Y, -\lambda(r), \lambda)$ is a left (\mathfrak{g}, r) -Poisson space (see Definition 2.12).*

Proof. Let $r = \Lambda + s$, where $\Lambda \in \wedge^2 \mathfrak{g}$ and $s \in (S^2(\mathfrak{g}))^\mathfrak{g}$, and assume that $\lambda(s) = 0$, so that $\lambda(r)$ is a bi-vector field on Y . By (2.8) - (2.10), the Schouten bracket of $\lambda(r)$ with itself is given by

$$[\lambda(r), \lambda(r)] = [\lambda(\Lambda), \lambda(\Lambda)] = -\lambda([\Lambda, \Lambda]) = 2\lambda(\text{CYB}(s)).$$

Let $y \in Y$, and let the notation be as in the proof of Lemma 2.17. Let $\alpha, \beta, \gamma \in T_y^*Y$. Then by (2.9) and by the assumption that $\ker \lambda_y$ is coisotropic with respect to s , one has

$$\lambda(\text{CYB}(s))(y)(\alpha, \beta, \gamma) = \langle \lambda_y^*(\alpha), [s^\#(\lambda_y^*(\beta), s^\#(\lambda_y^*(\gamma)))] \rangle = 0.$$

Since $[\lambda(x), -\lambda(\Lambda)] = -[\lambda(x), \lambda(\Lambda)] = \lambda([x, \Lambda]) = \lambda(\delta_r(x))$ for all $x \in \mathfrak{g}$, where $\delta_r : \mathfrak{g} \rightarrow \wedge^2 \mathfrak{g}$ is defined in (2.6), $(Y, -\lambda(r), \lambda)$ is a left (\mathfrak{g}, r) -Poisson space.

Q.E.D.

Definition 2.19. For a quasitriangular r -matrix r on a Lie algebra \mathfrak{g} and a left Lie algebra action $\lambda : \mathfrak{g} \rightarrow \mathcal{V}^1(Y)$, if $\lambda(r)$ is skew-symmetric (so it is a Poisson structure on Y by Proposition 2.18), we call $-\lambda(r)$ the *Poisson structure on Y defined by λ and r* .

Remark 2.20. 1) When $(\mathfrak{g}, \langle, \rangle_{\mathfrak{g}})$ is a quadratic Lie algebra and the quasitriangular r -matrix on \mathfrak{g} is defined by a Lagrangian splitting of $(\mathfrak{g}, \langle, \rangle_{\mathfrak{g}})$, the construction in Proposition 2.18 has been given in [32, §2], and an interpretation of the construction in the framework of Courant algebroids is given in [26].

2) An observation analogous to Proposition 2.18 for \mathfrak{g} -quasi-Poisson spaces is made in [27, §2.1]. The (\mathfrak{g}, r) -Poisson space $(Y, -\lambda(r), \lambda)$ in Proposition 2.18 corresponds to the \mathfrak{g} -quasi-Poisson space $(Y, 0, \lambda)$ under an equivalence via twisting between the category of (\mathfrak{g}, r) -Poisson spaces and that of \mathfrak{g} -quasi-Poisson spaces, which will be explained in §6.3 and Example 6.14. \diamond

3. MIXED TWISTS OF DIRECT PRODUCT LIE BIALGEBRAS AND MIXED PRODUCT POISSON STRUCTURES

3.1. Mixed twists of direct product Lie bialgebras. Let $(\mathfrak{g}_1, \delta_1), \dots, (\mathfrak{g}_n, \delta_n)$, $n \geq 1$, be Lie bialgebras, and let $(\mathfrak{g}, \delta_{\mathfrak{g}})$ be the direct product Lie bialgebra, i.e., $\mathfrak{g} = \mathfrak{g}_1 \oplus \dots \oplus \mathfrak{g}_n$ with the direct product Lie bracket, and

$$\delta_{\mathfrak{g}}(x_1, \dots, x_n) = (\delta_{\mathfrak{g}_1}(x_1), \dots, \delta_{\mathfrak{g}_n}(x_n)), \quad (x_1, \dots, x_n) \in \mathfrak{g},$$

(see notation in §1.5). For $1 \leq j \leq n$, let $p_j : \mathfrak{g} \rightarrow \mathfrak{g}_j$ be the j 'th projection. Recall from §2.4 the notion of twists of Lie bialgebras. In particular, if $t \in \wedge^2 \mathfrak{g}$ is a twisting element for $(\mathfrak{g}, \delta_{\mathfrak{g}})$, one has the twisted Lie bialgebra $(\mathfrak{g}, \delta_{\mathfrak{g}, t})$, where $\delta_{\mathfrak{g}, t}(x) = \delta_{\mathfrak{g}}(x) - [x, t]$ for $x \in \mathfrak{g}$.

Definition 3.1. A twisting element $t \in \wedge^2 \mathfrak{g}$ of the direct product Lie bialgebra $(\mathfrak{g}, \delta_{\mathfrak{g}})$ is said to be *mixed* if $p_j(t) = 0$ for each $1 \leq j \leq n$, and in this case the twisted Lie bialgebra $(\mathfrak{g}, \delta_{\mathfrak{g}, t})$ (see (2.23)) is called a *mixed twist* of the direct product Lie bialgebra $(\mathfrak{g}, \delta_{\mathfrak{g}})$. Note that when $n = 1$, only the zero element is a mixed twisting element of $(\mathfrak{g}, \delta_{\mathfrak{g}})$.

For $J = \{j_1, \dots, j_k\} \subset \{1, \dots, n\}$ with $1 \leq j_1 < \dots < j_k \leq n$, let $\mathfrak{g}_J = \mathfrak{g}_{j_1} \oplus \dots \oplus \mathfrak{g}_{j_k}$, and let

$$(3.1) \quad p_J : \mathfrak{g} \longrightarrow \mathfrak{g}_J, \quad (x_1, \dots, x_n) \longmapsto (x_{j_1}, \dots, x_{j_k}).$$

Denote by $(\mathfrak{g}_J, \delta_{\mathfrak{g}_J})$ the direct product Lie bialgebra of $(\mathfrak{g}_{j_1}, \delta_{\mathfrak{g}_{j_1}}), \dots, (\mathfrak{g}_{j_k}, \delta_{\mathfrak{g}_{j_k}})$.

Lemma 3.2. If $t \in \wedge^2 \mathfrak{g}$ be a mixed twisting element of the direct product Lie bialgebra $(\mathfrak{g}, \delta_{\mathfrak{g}})$, then $p_J(t) \in \wedge^2 \mathfrak{g}_J$ is a mixed twisting element of the direct product Lie bialgebra $(\mathfrak{g}_J, \delta_{\mathfrak{g}_J})$ for any $J \subset \{1, \dots, n\}$, and

$$(3.2) \quad p_J : (\mathfrak{g}, \delta_{\mathfrak{g}, t}) \longrightarrow (\mathfrak{g}_J, \delta_{\mathfrak{g}_J, p_J(t)})$$

is a Lie bialgebra homomorphism. In particular, $p_j : (\mathfrak{g}, \delta_{\mathfrak{g}, t}) \rightarrow (\mathfrak{g}_j, \delta_{\mathfrak{g}_j})$ is a Lie bialgebra homomorphism for each $1 \leq j \leq n$.

Proof. Let $t_J = p_J(t)$. As $p_J : (\mathfrak{g}, \delta_{\mathfrak{g}}) \rightarrow (\mathfrak{g}_J, \delta_{\mathfrak{g}_J})$ is a Lie bialgebra homomorphism, it follows from $2\delta_{\mathfrak{g}}(t) + [t, t] = 0$ that $2\delta_{\mathfrak{g}_J}(t_J) + [t_J, t_J] = 0$, so t_J is a twisting element, and in fact a mixed twisting element, of $(\mathfrak{g}_J, \delta_{\mathfrak{g}_J})$. It is also clear from the definitions of $\delta_{\mathfrak{g}, t}$ and $\delta_{\mathfrak{g}_J, t_J}$ that p_J in (3.2) is a Lie bialgebra homomorphism.

Q.E.D.

If $(\mathfrak{g}_1, r_1), \dots, (\mathfrak{g}_n, r_n)$ are quasitriangular Lie bialgebras, one has the direct product quasitriangular Lie bialgebra (\mathfrak{g}, r) , where again $\mathfrak{g} = \mathfrak{g}_1 \oplus \dots \oplus \mathfrak{g}_n$ is the direct product Lie algebra and $r = (r_1, \dots, r_n) \in \mathfrak{g} \otimes \mathfrak{g}$. Let $\delta_{\mathfrak{g}_j} = \delta_{r_j}$ (see (2.6)) for $1 \leq j \leq n$. By Lemma 2.10, $t \in \wedge^2 \mathfrak{g}$ is a twisting element for the direct product Lie algebra $(\mathfrak{g}, \delta_{\mathfrak{g}})$ of $(\mathfrak{g}_1, \delta_{\mathfrak{g}_1}), \dots, (\mathfrak{g}_n, \delta_{\mathfrak{g}_n})$ if and only if $r - t$ is quasitriangular r -matrix on \mathfrak{g} .

Definition 3.3. If $t \in \wedge^2 \mathfrak{g}$ is such that $r - t$, where $r = (r_1, \dots, r_n)$, is quasitriangular r -matrix on \mathfrak{g} and $p_j(t) = 0$ for each $1 \leq j \leq n$, we call $r - t$ a *mixed twist* of r , and we also call the quasitriangular Lie bialgebra $(\mathfrak{g}, r - t)$ a *mixed twist of the direct product quasitriangular Lie bialgebra* (\mathfrak{g}, r) .

3.2. Mixed twists and mixed product Poisson structures. Let $(\mathfrak{g}_1, \delta_{\mathfrak{g}_1}), \dots, (\mathfrak{g}_n, \delta_{\mathfrak{g}_n})$, $n \geq 2$, be Lie bialgebras and let $(\mathfrak{g}, \delta_{\mathfrak{g}})$ be the direct product Lie bialgebra as in §3.1. Assume that $t \in \wedge^2 \mathfrak{g}$ is a mixed twisting element of $(\mathfrak{g}, \delta_{\mathfrak{g}})$, and let $(\mathfrak{g}, \delta_{\mathfrak{g}, t})$ be the corresponding twisted Lie bialgebra. As a direct application of Lemma 2.16 and an immediate consequence of t being a mixed twisting element, we have

Proposition 3.4. 1) Let (Y_j, π_j, λ_j) , $1 \leq j \leq n$, be left $(\mathfrak{g}_j, \delta_{\mathfrak{g}_j})$ -Poisson spaces, and let $Y = Y_1 \times \dots \times Y_n$ be the product manifold, equipped with the product Lie algebra action $\lambda = (\lambda_1, \dots, \lambda_n)$ of \mathfrak{g} , i.e.,

$$\lambda : \mathfrak{g} \longrightarrow \mathcal{V}^1(Y), \quad \lambda(x_1, \dots, x_n) = (\lambda_1(x_1), \dots, \lambda_n(x_n)), \quad x_j \in \mathfrak{g}_j.$$

Then $\pi_Y = (\pi_1, \dots, \pi_n) + \lambda(t)$ is a mixed product Poisson structure on Y , and (Y, π_Y, λ) is a left $(\mathfrak{g}, \delta_{\mathfrak{g}, t})$ -Poisson spaces;

2) Let (Y_j, π_j, ρ_j) , $1 \leq j \leq n$, be right $(\mathfrak{g}_j, \delta_{\mathfrak{g}_j})$ -Poisson space, and let

$$\rho = (\rho_1, \dots, \rho_n) : \mathfrak{g} \longrightarrow \mathcal{V}^1(Y), \quad \rho(x_1, \dots, x_n) = (\rho_1(x_1), \dots, \rho_n(x_n)), \quad x_j \in \mathfrak{g}_j,$$

where $Y = Y_1 \times \dots \times Y_n$, Then $\pi_Y = (\pi_1, \dots, \pi_n) - \rho(t)$ is a mixed product Poisson structure on Y , and (Y, π_Y, ρ) is a right $(\mathfrak{g}, \delta_{\mathfrak{g}, t})$ -Poisson spaces;

3) If $r_j \in \mathfrak{g}_j \otimes \mathfrak{g}_j$ is a quasitriangular structure for $(\mathfrak{g}_j, \delta_{\mathfrak{g}_j})$ and $\pi_j = -\lambda_j(r_j)$ in 1) (resp. $\pi_j = \rho_j(r_j)$ in 2)) for $1 \leq j \leq n$, then $\pi_Y = -\lambda(r - t)$ in 1) (resp. $\pi_Y = \rho(r - t)$ in 2)), where $r = (r_1, \dots, r_n)$.

Proposition 3.4 is the key to all the constructions in this paper. Indeed, the mixed product Poisson structures constructed in the following §4, §5, and §6 are all incidences of mixed twists of direct product Lie bialgebras and of their direct product Poisson actions in the spirit of Proposition 3.4.

4. TWO-FOLD MIXED PRODUCT POISSON STRUCTURES ASSOCIATED TO LIE BIALGEBRAS

Although a special case of what to be covered in §5, the *two-fold* mixed product Poisson structure construction in this section is more basic, so we discuss it first. The construction also appeared in [35].

4.1. The construction. Let $(\mathfrak{g}, \delta_{\mathfrak{g}})$ be a Lie bialgebra, with dual Lie bialgebra $(\mathfrak{g}^*, \delta_{\mathfrak{g}^*})$ and double Lie bialgebra $(\mathfrak{d}, \delta_{\mathfrak{d}})$, and let $(\mathfrak{d}', \delta_{\mathfrak{d}'})$ be the dual Lie bialgebra of $(\mathfrak{d}, \delta_{\mathfrak{d}})$. Recall from §2.2 that as a Lie algebra $\mathfrak{d}' = \mathfrak{g}^* \oplus \mathfrak{g}$ with the direct product Lie bracket. Define the direct product Lie co-bracket $\delta'_{\mathfrak{d}'}$ on \mathfrak{d}' by

$$(4.1) \quad \delta'_{\mathfrak{d}'}(\xi, x) = (\delta_{\mathfrak{g}^*}(\xi), -\delta_{\mathfrak{g}}(x)), \quad \xi \in \mathfrak{g}^*, x \in \mathfrak{g}.$$

Let $\{x_i\}_{i=1}^m$ be any basis of \mathfrak{g} and let $\{\xi_i\}_{i=1}^m$ be the dual basis for \mathfrak{g}^* . Let

$$(4.2) \quad t = \sum_{i=1}^m (\xi_i, 0) \wedge (0, x_i) \in \wedge^2 \mathfrak{d}'.$$

Lemma 4.1. *The element $-t \in \wedge^2 \mathfrak{d}'$ is a mixed twisting element for the direct product Lie bialgebra $(\mathfrak{d}', \delta'_{\mathfrak{d}'})$, and the twist of $(\mathfrak{d}', \delta'_{\mathfrak{d}'})$ by $-t$ is the Lie bialgebra $(\mathfrak{d}', \delta_{\mathfrak{d}'})$.*

Proof. It is straightforward to check that $\delta'_{\mathfrak{d}'}$ is defined (see Definition 2.1) by the Lagrangian splitting $\mathfrak{d}^2 = \mathfrak{d}' + (\mathfrak{g} \oplus \mathfrak{g}^*)$ of the quadratic Lie algebra $(\mathfrak{d}^2, \langle \cdot, \cdot \rangle_{\mathfrak{d}^2})$, and that the twist by $-t$ of $\mathfrak{d}^2 = \mathfrak{d}' + (\mathfrak{g} \oplus \mathfrak{g}^*)$ is the Lagrangian splitting $\mathfrak{d}^2 = \mathfrak{d}' + \mathfrak{d}_{\text{diag}}$ which defines the Lie bialgebra $(\mathfrak{d}', \delta_{\mathfrak{d}'})$.

Q.E.D.

Let (X, π_X, ρ) be a right $(\mathfrak{g}^*, \delta_{\mathfrak{g}^*})$ -Poisson space and (Y, π_Y, λ) a left $(\mathfrak{g}, \delta_{\mathfrak{g}})$ -Poisson space. Let again $\{x_i\}_{i=1}^m$ be any basis of \mathfrak{g} and $\{\xi_i\}_{i=1}^m$ the dual basis for \mathfrak{g}^* . Define the bi-vector field $\pi_X \times_{(\rho, \lambda)} \pi_Y$ on the product manifold $X \times Y$ by

$$(4.3) \quad \pi_X \times_{(\rho, \lambda)} \pi_Y = (\pi_X, \pi_Y) - \sum_{i=1}^m (\rho(\xi_i), 0) \wedge (0, \lambda(x_i)).$$

The following Proposition 4.2 is now a direct consequence of Remark 2.14 and Proposition 3.4.

Proposition 4.2. *The bi-vector field $\pi_X \times_{(\rho, \lambda)} \pi_Y$ on $X \times Y$ is Poisson, and the map*

$$\rho_0 : \mathfrak{d}' \longrightarrow \mathcal{V}^1(X \times Y), \quad \rho_0(\xi, x) = (\rho(\xi), 0) + (0, -\lambda(x)), \quad \xi \in \mathfrak{g}^*, x \in \mathfrak{g}$$

is a right Poisson action of the Lie bialgebra $(\mathfrak{d}', \delta_{\mathfrak{d}'})$ on $(X \times Y, \pi_X \times_{(\rho, \lambda)} \pi_Y)$.

Definition 4.3. We call $\pi_X \times_{(\rho, \lambda)} \pi_Y$ the *mixed product Poisson structure* on $X \times Y$ associated to the pair (ρ, λ) of Poisson actions.

Remark 4.4. (Reduction to sub-Lie bialgebras, I.) In the setting of Proposition 4.2, assume that $(\mathfrak{p}, \delta_{\mathfrak{p}})$ is a sub-Lie bialgebra of $(\mathfrak{g}, \delta_{\mathfrak{g}})$ such that $\rho(\xi) = 0$ for all $\xi \in \mathfrak{p}^0 \subset \mathfrak{g}^*$. Let $\lambda_{\mathfrak{p}} : \mathfrak{p} \rightarrow \mathcal{V}^1(Y)$ be the restriction of λ to \mathfrak{p} and $\rho_{\mathfrak{p}^*} : \mathfrak{p}^* \cong \mathfrak{g}^*/\mathfrak{p}^0 \rightarrow \mathcal{V}^1(X)$ the Lie algebra action induced by ρ . Then $(Y, \pi_Y, \lambda_{\mathfrak{p}})$ is a left $(\mathfrak{p}, \delta_{\mathfrak{p}})$ -Poisson space and $(X, \pi_X, \rho_{\mathfrak{p}^*})$ a right $(\mathfrak{p}^*, \delta_{\mathfrak{p}^*})$ -Poisson space. Choosing a basis of \mathfrak{p} and extending it to a basis of \mathfrak{g} , one sees that $\pi_X \times_{(\rho, \lambda)} \pi_Y = \pi_X \times_{(\rho_{\mathfrak{p}^*}, \lambda_{\mathfrak{p}})} \pi_Y$. \diamond

Remark 4.5. Given a right $(\mathfrak{g}^*, \delta_{\mathfrak{g}^*})$ -Poisson space (X, π_X, ρ) and a left $(\mathfrak{g}, \delta_{\mathfrak{g}})$ -Poisson space (Y, π_Y, λ) , let $\hat{\rho} : T^*X \rightarrow \mathfrak{g}$ and $\hat{\lambda} : T^*Y \rightarrow \mathfrak{g}^*$ be respectively given by

$$(4.4) \quad \langle \hat{\rho}(\alpha_p), \xi \rangle = \langle \alpha_p, \rho(\xi)(p) \rangle, \quad p \in X, \alpha_p \in T_p^*X, \xi \in \mathfrak{g}^*,$$

$$(4.5) \quad \langle \hat{\lambda}(\beta_q), x \rangle = \langle \beta_q, \lambda(x)(q) \rangle, \quad q \in Y, \beta_q \in T_q^*Y, x \in \mathfrak{g},$$

and let $p_X : X \times Y \rightarrow X$ and $p_Y : X \times Y \rightarrow Y$ be the two projections. Then a Poisson structure π on $X \times Y$ is equal to $\pi_X \times_{(\rho, \lambda)} \pi_Y$ if and only if the two projections $p_X : (X \times Y, \pi) \rightarrow (X, \pi_X)$ and $p_Y : (X \times Y, \pi) \rightarrow (Y, \pi_Y)$ are Poisson and

$$(4.6) \quad (\pi_X \times_{(\rho, \lambda)} \pi_Y)(p_X^* \alpha, p_Y^* \beta) = -\langle \hat{\rho}(\alpha), \hat{\lambda}(\beta) \rangle.$$

for any 1-form α on X and 1-form β on Y . \diamond

The next Lemma 4.6 follows directly from the definition of the Poisson structure $\pi_X \times_{(\rho, \lambda)} \pi_Y$.

Lemma 4.6. *Let $\pi_X \times_{(\rho, \lambda)} \pi_Y$ be the mixed product Poisson structure on $X \times Y$ defined by the pair (ρ, λ) . If $X_1 \subset X$ is a \mathfrak{g}^* -invariant Poisson submanifold of (X, π_X) and $Y_1 \subset Y$ a \mathfrak{g} -invariant Poisson submanifold of (Y, π_Y) , then $X_1 \times Y_1$ is a Poisson submanifold of $(X \times Y, \pi_X \times_{(\rho, \lambda)} \pi_Y)$.*

Example 4.7. Let (G, π_G) be a Poisson Lie group with Lie bialgebra $(\mathfrak{g}, \delta_{\mathfrak{g}})$ and dual Lie bialgebra $(\mathfrak{g}^*, \delta_{\mathfrak{g}^*})$, and let λ be the left Poisson action of $(\mathfrak{g}, \delta_{\mathfrak{g}})$ on (G, π_G) given by $\lambda(x) = x^R$ for $x \in \mathfrak{g}$ (see Example 2.15). For a right $(\mathfrak{g}^*, \delta_{\mathfrak{g}^*})$ -Poisson space (X, π_X, ρ) , the mixed product Poisson structure $\pi_X \times_{(\rho, \lambda)} \pi_G$ on $X \times G$ is called a *semi-direct product Poisson structure* on $X \times G$ in [29]. In this case, the right Poisson action of $(\mathfrak{g}^*, \delta_{\mathfrak{g}^*})$ on $(X \times G, \pi_X \times_{(\rho, \lambda)} \pi_G)$ generated (see Example 2.15) by the Poisson map $\kappa_G : (X \times G, \pi_X \times_{(\rho, \lambda)} \pi_G) \rightarrow (G, \pi_G), (x, g) \rightarrow g$, is the diagonal action

$$\mathfrak{g}^* \longrightarrow \mathcal{V}^1(X \times G), \quad \xi \longmapsto (\rho(\xi), 0) + (0, \varrho(\xi)), \quad \xi \in \mathfrak{g}^*,$$

where ϱ is the right dressing action of $(\mathfrak{g}^*, \delta_{\mathfrak{g}^*})$ on (G, π_G) . \diamond

Example 4.8. Let (G, π_G) be a Poisson Lie group with Lie algebra $(\mathfrak{g}, \delta_{\mathfrak{g}})$ and (G^*, π_{G^*}) any dual Poisson Lie group of (G, π_G) . Consider the dual pair of Poisson Lie groups

$$(H, \pi_H) = (G^{\text{op}}, \pi_G) \times (G, \pi_G) \quad \text{and} \quad (H^*, \pi_{H^*}) = (G^*, -\pi_{G^*}) \times (G^*, \pi_{G^*}),$$

where G^{op} denotes G with the opposite group structure, and the right and left Poisson actions

$$\begin{aligned} \rho : (G^*, \pi_{G^*}) \times (H^*, \pi_{H^*}) &\longrightarrow (G^*, \pi_{G^*}), \quad (g^*, (h_1, h_2)) \longmapsto h_1^{-1} g^* h_2, \quad g^*, h_1, h_2 \in G^*, \\ \lambda : (H, \pi_H) \times (G, -\pi_G) &\longrightarrow (G, -\pi_G), \quad ((g_1, g_2), g) \longmapsto g_1^{-1} g g_2^{-1}, \quad g, g_1, g_2 \in G. \end{aligned}$$

Then $(D', \pi_{D'}) = (G^* \times G, \pi_{G^*} \times_{(\rho, \lambda)} (-\pi_G))$ is a Poisson Lie group with Lie bialgebra $(\mathfrak{d}', \delta_{\mathfrak{d}'})$, where recall from §2.2 that $(\mathfrak{d}', \delta_{\mathfrak{d}'})$ is the dual Lie bialgebra of the double Lie bialgebra $(\mathfrak{d}, \delta_{\mathfrak{d}})$ of $(\mathfrak{g}, \delta_{\mathfrak{g}})$. Indeed, identify the Lie algebra of D' with $\mathfrak{d}' = \mathfrak{g}^* \oplus \mathfrak{g}$. If $\{x_i\}_{i=1}^m$ is a basis of \mathfrak{g} and $\{\xi_i\}_{i=1}^m$ the dual basis in \mathfrak{g}^* , the mixed term of $\pi_{G^*} \times_{(\rho, \lambda)} (-\pi_G)$ is

$$\mu_{(\rho, \lambda)} = - \sum_{i=1}^m (\xi_i^R, 0) \wedge (0, x_i^R) - (\xi_i^L, 0) \wedge (0, x_i^L) = -t^L + t^R,$$

where $t \in \wedge^2 \mathfrak{d}'$ is defined in (4.2). Thus $\pi_{D'} = (\pi_{G^*}, -\pi_G) - t^L + t^R$ is multiplicative. Moreover, since the Lie bialgebra $(\mathfrak{d}', \delta_{\mathfrak{d}'})$ is the twisting by t of the Lie bialgebra $(\mathfrak{d}', \delta_0)$ as in the proof of Proposition 4.2, the Lie bialgebra of $(D', \pi_{D'})$ is $(\mathfrak{d}', \delta_{\mathfrak{d}'})$. \diamond

4.2. Morphisms between mixed product Poisson structures. For $i = 1, 2$, let $(\mathfrak{g}_i, \delta_{\mathfrak{g}_i})$ be a Lie bialgebra and $(\mathfrak{g}_i^*, \delta_{\mathfrak{g}_i^*})$ its dual Lie bialgebra, let $(Y_i, \pi_{Y_i}, \lambda_i)$ be a left $(\mathfrak{g}_i, \delta_{\mathfrak{g}_i})$ -Poisson space and (X_i, π_{X_i}, ρ_i) a right $(\mathfrak{g}_i^*, \delta_{\mathfrak{g}_i^*})$ -Poisson space. Given a triple (Φ, Ψ, ϕ) , where

$$\Phi : (X_1, \pi_{X_1}) \longrightarrow (X_2, \pi_{X_2}) \quad \text{and} \quad \Psi : (Y_1, \pi_{Y_1}) \longrightarrow (Y_2, \pi_{Y_2})$$

are Poisson maps and $\phi : (\mathfrak{g}_1, \delta_{\mathfrak{g}_1}) \rightarrow (\mathfrak{g}_2, \delta_{\mathfrak{g}_2})$ is a Lie bialgebra homomorphism such that for every $x_1 \in \mathfrak{g}_1$, the vector fields $\lambda_1(x_1) \in \mathcal{V}^1(Y_1)$ and $\lambda_2(\phi(x_1)) \in \mathcal{V}^1(Y_2)$ are Ψ -related and that for every $\xi_2 \in \mathfrak{g}_2^*$, the vector fields $\rho_1(\phi^*(\xi_2)) \in \mathcal{V}^1(X_1)$ and $\rho_2(\xi_2) \in \mathcal{V}^1(X_2)$ are Φ -related, let

$$\Phi \times \Psi : X_1 \times Y_1 \longrightarrow X_2 \times Y_2, \quad (p_1, q_1) \longmapsto (\Phi(p_1), \Psi(q_1)), \quad p_1 \in X_1, q_1 \in Y_1.$$

Proposition 4.9. $\Phi \times \Psi : (X_1 \times Y_1, \pi_{X_1} \times_{(\rho_1, \lambda_1)} \pi_{Y_1}) \rightarrow (X_2 \times Y_2, \pi_{X_2} \times_{(\rho_2, \lambda_2)} \pi_{Y_2})$ is Poisson.

Proof. Let $\mu_i = \pi_{X_i} \times_{(\rho_i, \lambda_i)} \pi_{Y_i} - (\pi_{X_i}, 0) - (0, \pi_{Y_i})$ be the mixed terms of $\pi_{X_i} \times_{(\rho_i, \lambda_i)} \pi_{Y_i}$, $i = 1, 2$. Since Φ and Ψ are Poisson, it suffices to show that μ_1 and μ_2 are $(\Phi \times \Psi)$ -related.

Fix $(p_1, q_1) \in X_1 \times Y_1$ and let $p_2 = \Phi(p_1)$ and $q_2 = \Psi(q_1)$. For $i = 1, 2$, let $\hat{\rho}_i : T_{p_i}^* X_i \rightarrow \mathfrak{g}_i$ and $\hat{\lambda}_i : T_{q_i}^* Y_i \rightarrow \mathfrak{g}_i^*$ be defined as in Remark 4.5. Then the assumptions on Φ and Ψ imply that

$$\phi \hat{\rho}_1 \Phi^* = \hat{\rho}_2 : T_{p_2}^* X_2 \longrightarrow \mathfrak{g}_2 \quad \text{and} \quad \hat{\lambda}_1 \Psi^* = \phi^* \hat{\lambda}_2 : T_{q_2}^* Y_2 \longrightarrow \mathfrak{g}_2^*.$$

It follows that for any $(\alpha_2, \beta_2) \in T_{p_2}^* X_2 \times T_{q_2}^* Y_2$, one has

$$\begin{aligned} \mu_1(p_1, q_1)(\Phi^*(\alpha_2), \Psi^*(\beta_2)) &= -\langle \hat{\rho}_1(\Phi^*(\alpha_2)), \hat{\lambda}_1(\Psi^*(\beta_2)) \rangle = -\langle \phi \hat{\rho}_1 \Phi^*(\alpha_2), \hat{\lambda}_2(\beta_2) \rangle \\ &= -\langle \hat{\rho}_2(\alpha_2), \hat{\lambda}_2(\beta_2) \rangle = \mu_2(p_2, q_2)(\alpha_2, \beta_2). \end{aligned}$$

Thus $(\Phi \times \Psi)(\mu_1(p_1, q_1)) = \mu_2(p_2, q_2)$.

Q.E.D.

We now look at a situation in which the mixed term of a mixed product Poisson structure vanishes under a Poisson morphism. Let $((\mathfrak{g}, \delta_{\mathfrak{g}}), (\mathfrak{g}^*, \delta_{\mathfrak{g}^*}))$ be a dual pair of Lie bialgebras.

Lemma 4.10. Let (X, π_X, ρ) be a right $(\mathfrak{g}^*, \delta_{\mathfrak{g}^*})$ -Poisson space and (Y, π_Y, λ) a left $(\mathfrak{g}, \delta_{\mathfrak{g}})$ -Poisson space, and assume that $\Phi : (X, \pi_X) \rightarrow (X', \pi_{X'})$ and $\Psi : (Y, \pi_Y) \rightarrow (Y', \pi_{Y'})$ are Poisson morphisms. If there is a subspace $\mathfrak{a} \subset \mathfrak{g}$ such that $\Psi(\lambda(x)) = 0$ for all $x \in \mathfrak{a}$ and $\Phi(\rho(\xi)) = 0$ for all $\xi \in \mathfrak{a}^0 \subset \mathfrak{g}^*$, then $(\Phi \times \Psi)(\pi_X \times_{(\rho, \lambda)} \pi_Y) = \pi_{X'} \times \pi_{Y'}$.

Proof. Let $\{x_i\}_{i=1}^m$ be a basis of \mathfrak{g} such that $\{x_i\}_{i=1}^{k=\dim \mathfrak{a}}$ is a basis of \mathfrak{a} , and let $\{\xi_i\}_{i=1}^m$ be the dual basis of \mathfrak{g}^* . As $\Psi(\lambda(x_i)) = 0$ for $1 \leq i \leq k$ and $\Phi(\rho(\xi_i)) = 0$ for $k+1 \leq i \leq m$, the mixed term of $\pi_X \times_{(\rho, \lambda)} \pi_Y$ is mapped to zero by $\Phi \times \Psi$.

Q.E.D.

5. POLYUBLES OF LIE BIALGEBRAS AND MIXED PRODUCT POISSON STRUCTURES

5.1. Polyubles of Lie bialgebras. Let $(\mathfrak{g}, \delta_{\mathfrak{g}})$ be a Lie bialgebra, with dual Lie bialgebra $(\mathfrak{g}^*, \delta_{\mathfrak{g}^*})$, double Lie algebra $(\mathfrak{d}, \langle, \rangle_{\mathfrak{d}})$, and double Lie bialgebra $(\mathfrak{d}, \delta_{\mathfrak{d}})$. For an integer $n \geq 1$, consider the quadratic Lie

algebra $(\mathfrak{d}^n, \langle \cdot, \cdot \rangle_{\mathfrak{d}^n})$, where $\mathfrak{d}^n = \mathfrak{d} \oplus \cdots \oplus \mathfrak{d}$ has the direct product Lie bracket, its elements denoted as (a_1, \dots, a_n) with $a_j \in \mathfrak{d}$, and the bilinear form $\langle \cdot, \cdot \rangle_{\mathfrak{d}^n}$ is given by

$$(5.1) \quad \langle (a_1, \dots, a_n), (a'_1, \dots, a'_n) \rangle_{\mathfrak{d}^n} = \sum_{j=1}^n (-1)^{j+1} \langle a_j, a'_j \rangle_{\mathfrak{d}}, \quad a_j, a'_j \in \mathfrak{d}.$$

Let again $\mathfrak{d}_{\text{diag}} = \{(a, a) : a \in \mathfrak{d}\} \subset \mathfrak{d} \oplus \mathfrak{d}$. For an integer $n \geq 1$, define

$$(5.2) \quad \mathfrak{g}_{(2n-1)} = \overbrace{\mathfrak{d}_{\text{diag}} \oplus \cdots \oplus \mathfrak{d}_{\text{diag}}}^{n-1} \oplus \mathfrak{g} \subset \mathfrak{d}^{2n-1}, \quad \mathfrak{g}_{(2n-1)}^* = \mathfrak{g}^* \oplus \overbrace{\mathfrak{d}_{\text{diag}} \oplus \cdots \oplus \mathfrak{d}_{\text{diag}}}^{n-1} \subset \mathfrak{d}^{2n-1},$$

$$(5.3) \quad \mathfrak{g}_{(2n)} = \overbrace{\mathfrak{d}_{\text{diag}} \oplus \cdots \oplus \mathfrak{d}_{\text{diag}}}^n \subset \mathfrak{d}^{2n}, \quad \mathfrak{g}_{(2n)}^* = \mathfrak{g}^* \oplus \overbrace{\mathfrak{d}_{\text{diag}} \oplus \cdots \oplus \mathfrak{d}_{\text{diag}}}^{n-1} \oplus \mathfrak{g} \subset \mathfrak{d}^{2n}.$$

It is straightforward to check that for each $n \geq 1$,

$$(5.4) \quad \mathfrak{d}^n = \mathfrak{g}_{(n)} + \mathfrak{g}_{(n)}^*$$

is a Lagrangian splitting of $(\mathfrak{d}^n, \langle \cdot, \cdot \rangle_{\mathfrak{d}^n})$. Denote by $(\mathfrak{g}_{(n)}, \delta_{\mathfrak{g}_{(n)}})$ and $(\mathfrak{g}_{(n)}^*, \delta_{\mathfrak{g}_{(n)}^*})$ the dual pair of Lie bialgebras defined by the Lagrangian splitting in (5.4), and denote by $(\mathfrak{d}^n, \delta_{\mathfrak{d}^n})$ the corresponding double Lie bialgebra. Note that when $n = 1, 2$, the splitting in (5.4) are respectively $\mathfrak{d} = \mathfrak{g} + \mathfrak{g}^*$ and $\mathfrak{d}^2 = \mathfrak{d}_{\text{diag}} + \mathfrak{d}'$, where $\mathfrak{d}' = \mathfrak{g}^* \oplus \mathfrak{g}$, and we thus recover the two dual pairs of Lie bialgebras $((\mathfrak{g}, \delta_{\mathfrak{g}}), (\mathfrak{g}^*, \delta_{\mathfrak{g}^*}))$ and $((\mathfrak{d}, \delta_{\mathfrak{d}}), (\mathfrak{d}', \delta_{\mathfrak{d}'}))$ (see §2.2).

Definition 5.1. The Lie bialgebra $(\mathfrak{g}_{(n)}, \delta_{\mathfrak{g}_{(n)}})$ is called the n -uble of $(\mathfrak{g}, \delta_{\mathfrak{g}})$.

Remark 5.2. 1) The 1-uble and 2-uble of a Lie bialgebra are respectively the Lie bialgebra itself and its Drinfeld double. For $n = 3$, *triples* of quasitriangular Lie bialgebras were constructed by Grabowski in [22]. In the factorizable case, it follows from [22, Theorem 4.5] and our Proposition 6.7 in §6 that the triple of a Lie bialgebra is isomorphic to its 3-uble.

2) *Polyubles* for factorizable Lie bialgebras and factorizable Poisson Lie groups [36] were constructed in [19, 20], and we also refer to [19, 20, 28] for interpretations of such polyubles as moduli spaces of flat connections on (quilted) surfaces. Due to limitation of space, we will only present in this section properties of polyubles (for arbitrary Lie bialgebras) that are needed for the rest of the paper. \diamond

We now show that the Lie bialgebras $(\mathfrak{g}_{(n)}, \delta_{\mathfrak{g}_{(n)}})$ for $n \geq 3$ and $(\mathfrak{g}_{(n)}^*, \delta_{\mathfrak{g}_{(n)}^*})$ for $n \geq 2$ are mixed twists of direct product Lie bialgebras (see §3). Note that the Lie brackets on $\mathfrak{g}_{(n)}$ for $n \geq 3$ and on $\mathfrak{g}_{(n)}^*$ for $n \geq 3$ are indeed direct product Lie brackets: identify $\mathfrak{d}_{\text{diag}} \cong \mathfrak{d}$ via $(a, a) \mapsto a$ for $a \in \mathfrak{d}$ in (5.2) and (5.3), one has the identification of Lie algebras

$$(5.5) \quad \mathfrak{g}_{(2n+2)} \cong \mathfrak{d}^{n+1}, \quad \mathfrak{g}_{(2n)}^* = \mathfrak{g}^* \oplus \mathfrak{d}^{n-1} \oplus \mathfrak{g}, \quad \mathfrak{g}_{(2n+1)} = \mathfrak{d}^n \oplus \mathfrak{g}, \quad \mathfrak{g}_{(2n+1)}^* = \mathfrak{g}^* \oplus \mathfrak{d}^n, \quad n \geq 1.$$

Notation 5.3. Let $n \geq 1$, and for $1 \leq j \leq n+1$ and $a \in \mathfrak{d}$, let $(a)_j = (0, \dots, 0, a, 0, \dots, 0) \in \mathfrak{d}^{n+1}$, where a is at the j 'th entry of $(a)_j$. Let $\{x_i\}_{i=1}^m$ be any basis of \mathfrak{g} and $\{\xi_i\}_{i=1}^m$ the dual basis of \mathfrak{g}^* . Set

$$(5.6) \quad t_{n+1} = \sum_{1 \leq j < k \leq n+1} \sum_{i=1}^m (\xi_i)_j \wedge (x_i)_k \in \wedge^2(\mathfrak{d}^{n+1}).$$

Note that t_{n+1} in fact lies in $\wedge^2(\mathfrak{g}^* \oplus \mathfrak{d}^{n-1} \oplus \mathfrak{g}) \subset (\wedge^2(\mathfrak{d}^n \oplus \mathfrak{g})) \cap (\wedge^2(\mathfrak{g}^* \oplus \mathfrak{d}^n)) \subset \wedge^2(\mathfrak{d}^{n+1})$. Using the identifications in (5.5), we can regard t_{n+1} as an element in $\wedge^2 V$, where V can be either of $\mathfrak{g}_{(2n+2)}$, $\mathfrak{g}_{(2n)}^*$, $\mathfrak{g}_{(2n+1)}$, or $\mathfrak{g}_{(2n+1)}^*$.

Lemma 5.4. *Under the Lie algebra identification $\mathfrak{g}_{(2n+2)} \cong \mathfrak{d}^{n+1}$ in (5.5), where $n \geq 1$, the Lie bialgebra $(\mathfrak{g}_{(2n+2)}, \delta_{\mathfrak{g}_{(2n+2)}})$ is the mixed twist of the direct product Lie bialgebra $(\mathfrak{g}_{(2n+2)}, \delta'_{\mathfrak{g}_{(2n+2)}})$ by the mixed twisting element $t_{n+1} \in \wedge^2 \mathfrak{g}_{(2n+2)}$, where*

$$\delta'_{\mathfrak{g}_{(2n+2)}}(a_1, \dots, a_{n+1}) = (\delta_{\mathfrak{d}}(a_1), \dots, \delta_{\mathfrak{d}}(a_{n+1})), \quad a_1, \dots, a_{n+1} \in \mathfrak{d}.$$

Proof. Consider the two Lagrangian splittings of the quadratic Lie algebra $(\mathfrak{d}^{2n+2}, \langle, \rangle_{\mathfrak{d}^{2n+2}})$:

$$(5.7) \quad \mathfrak{d}^{2n+2} = \mathfrak{g}_{(2n+2)} + \overbrace{\mathfrak{d}' \oplus \dots \oplus \mathfrak{d}'}^{n+1} = \mathfrak{g}_{(2n+2)} + \mathfrak{g}_{(2n+2)}^*,$$

where recall from §2.2 that $\mathfrak{d}' = \mathfrak{g}^* \oplus \mathfrak{g} \subset \mathfrak{d}^2$. Clearly, the co-bracket on $\mathfrak{g}_{(2n+2)}$ defined by the first splitting in (5.7) is $\delta'_{\mathfrak{g}_{(2n+2)}}$. It is straightforward to check that second Lagrangian splitting in (5.7) is the twist by $t_{n+1} \in \wedge^2 \mathfrak{g}_{(2n+2)}$ of the first splitting. It is also evident that the twisting element t_{n+1} is mixed.

Q.E.D.

Let $r_{\mathfrak{d}} \in \mathfrak{d} \otimes \mathfrak{d}$ be again the quasitriangular r -matrix on \mathfrak{d} defined by the Lagrangian splitting $\mathfrak{g} = \mathfrak{g} + \mathfrak{g}^*$ of the quadratic Lie algebra $(\mathfrak{d}, \langle, \rangle_{\mathfrak{d}})$. For $n \geq 1$, let $r_{\mathfrak{d}^{n+1}} \in \mathfrak{d}^{n+1} \otimes \mathfrak{d}^{n+1}$ be the quasitriangular r -matrix on \mathfrak{d}^{n+1} defined by the Lagrangian splitting $\mathfrak{d}^{n+1} = \mathfrak{g}_{(n+1)} + \mathfrak{g}_{(n+1)}^*$ of $(\mathfrak{d}^{n+1}, \langle, \rangle_{\mathfrak{d}^{n+1}})$. Define the $(n+1)$ 'st alternating sum of $r_{\mathfrak{d}}$ by

$$\text{Alt}^{n+1}(r_{\mathfrak{d}}) = \begin{cases} (r_{\mathfrak{d}}, -r_{\mathfrak{d}}^{21}, r_{\mathfrak{d}}, \dots, -r_{\mathfrak{d}}^{21}, r_{\mathfrak{d}}) \in \mathfrak{d}^{n+1} \otimes \mathfrak{d}^{n+1}, & n \text{ even}, \\ (r_{\mathfrak{d}}, -r_{\mathfrak{d}}^{21}, \dots, r_{\mathfrak{d}}, -r_{\mathfrak{d}}^{21}) \in \mathfrak{d}^{n+1} \otimes \mathfrak{d}^{n+1}, & n \text{ odd}. \end{cases}$$

Lemma 5.5. *For $n \geq 1$, the quasitriangular Lie bialgebra $(\mathfrak{d}^{n+1}, r_{\mathfrak{d}^{n+1}})$ is the mixed twist by $t_{n+1} \in \wedge^2 \mathfrak{d}^{n+1}$ of the direct product quasitriangular Lie bialgebra $(\mathfrak{d}^{n+1}, \text{Alt}^{n+1}(r_{\mathfrak{d}}))$, i.e.,*

$$r_{\mathfrak{d}^{n+1}} = \text{Alt}^{n+1}(r_{\mathfrak{d}}) - t_{n+1} \in \mathfrak{d}^{n+1} \otimes \mathfrak{d}^{n+1}.$$

Consequently, $J_{2n} : \mathfrak{g}_{(2n)} \rightarrow \mathfrak{d}^n : (a_1, a_1, a_2, a_2, \dots, a_n, a_n) \mapsto (a_1, a_2, \dots, a_n)$, where $a_1, \dots, a_n \in \mathfrak{d}$, is a Lie bialgebra isomorphism from $(\mathfrak{g}_{(2n)}, \delta_{\mathfrak{g}_{(2n)}})$ to $(\mathfrak{d}^n, \delta_{\mathfrak{d}^n})$.

Proof. For $a \in \mathfrak{d}$, and $1 \leq j \leq k \leq n+1$, let

$$(5.8) \quad (a)_{(j,k)} = (0, \dots, 0, \underset{j^{\text{th}}}{a}, a, \dots, \underset{k^{\text{th}}}{a}, 0, \dots, 0) \in \mathfrak{d}^{n+1}.$$

Assume first that $n = 2q - 1$ is odd. If $\{x_i\}_{i=1}^m$ is a basis of \mathfrak{g} and $\{\xi_i\}_{i=1}^m$ is the dual basis of \mathfrak{g}^* , then $\{(x_i)_{(2r-1, 2r)}, (\xi_i)_{(2r-1, 2r)} : 1 \leq r \leq q\}_{i=1}^m$ is a basis for $\mathfrak{g}_{(n+1)}$, with its dual basis of $\mathfrak{g}_{(n+1)}^*$ given by $\{(\xi_i)_{(1, 2r-1)}, -(x_i)_{(2r, n+1)} : 1 \leq r \leq q\}_{i=1}^m$, and one proves that $r_{\mathfrak{d}^{n+1}} = \text{Alt}^{n+1}(r_{\mathfrak{d}}) - t_{n+1}$ by using the definition of $r_{\mathfrak{d}^{n+1}}$ given in Definition 2.6. The case when n is even is similarly proved. Combining with Lemma 5.4, one sees that the Lie algebra identification J_{2n} is also a Lie bialgebra isomorphism from $(\mathfrak{g}_{(2n)}, \delta_{\mathfrak{g}_{(2n)}})$ to $(\mathfrak{d}^n, \delta_{\mathfrak{d}^n})$.

Q.E.D.

Analogous to Lemma 5.4, we have

Lemma 5.6. *Let $n \geq 1$ and let the Lie algebra identifications be as in (5.5).*

1) *The Lie bialgebra $(\mathfrak{g}_{(2n+1)}, \delta_{\mathfrak{g}_{(2n+1)}})$ is the mixed twist by the element $t_{n+1} \in \wedge^2 \mathfrak{g}_{(2n+1)}$ of the direct product Lie bialgebra $(\mathfrak{g}_{(2n+1)}, \delta'_{\mathfrak{g}_{(2n+1)}})$, where*

$$\delta'_{\mathfrak{g}_{(2n+1)}}(a_1, \dots, a_n, x) = (\delta_{\mathfrak{d}}(a_1), \dots, \delta_{\mathfrak{d}}(a_n), \delta_{\mathfrak{g}}(x)), \quad a_1, \dots, a_n \in \mathfrak{d}, x \in \mathfrak{g};$$

2) The Lie bialgebra $(\mathfrak{g}_{(2n)}^*, \delta_{\mathfrak{g}_{(2n)}^*})$ is the mixed twist by the element $-t_{n+1} \in \wedge^2 \mathfrak{g}_{(2n)}^*$ of the direct product Lie bialgebra $(\mathfrak{g}_{(2n)}^*, \delta'_{\mathfrak{g}_{(2n)}^*})$, where

$$\delta'_{\mathfrak{g}_{(2n)}^*}(\xi, a_1, \dots, a_{n-1}, x) = (\delta_{\mathfrak{g}^*}(\xi), -\delta_{\mathfrak{d}}(a_1), \dots, -\delta_{\mathfrak{d}}(a_{n-1}), -\delta_{\mathfrak{g}}(x)), \quad \xi \in \mathfrak{g}^*, a_1, \dots, a_{n-1} \in \mathfrak{d}, x \in \mathfrak{g};$$

3) The Lie bialgebra $(\mathfrak{g}_{(2n+1)}^*, \delta_{\mathfrak{g}_{(2n+1)}^*})$ is the mixed twist by the element $-t_{n+1} \in \wedge^2 \mathfrak{g}_{(2n+1)}^*$ of the direct product Lie bialgebra $(\mathfrak{g}_{(2n+1)}^*, \delta'_{\mathfrak{g}_{(2n+1)}^*})$, where

$$\delta'_{\mathfrak{g}_{(2n+1)}^*}(\xi, a_1, \dots, a_n) = (\delta_{\mathfrak{g}^*}(\xi), -\delta_{\mathfrak{d}}(a_1), \dots, -\delta_{\mathfrak{d}}(a_n)), \quad \xi \in \mathfrak{g}^*, a_1, \dots, a_n \in \mathfrak{d}.$$

Proof. One checks directly that the embeddings of direct product Lie algebras

$$\begin{aligned} \mathfrak{g}_{(2n+1)} &\longrightarrow \mathfrak{g}_{(2n+2)}, & (a_1, a_1, \dots, a_n, a_n, x) &\longmapsto (a_1, a_1, \dots, a_n, a_n, x, x), \\ \mathfrak{g}_{(2n)}^* &\longrightarrow \mathfrak{g}_{(2n+2)}, & (\xi, a_1, a_1, \dots, a_{n-1}, a_{n-1}, x) &\longmapsto (\xi, \xi, a_1, a_1, \dots, a_{n-1}, a_{n-1}, x, x), \\ \mathfrak{g}_{(2n+1)}^* &\longrightarrow \mathfrak{g}_{(2n+2)}, & (\xi, a_1, a_1, \dots, a_n, a_n) &\longmapsto (\xi, \xi, a_1, a_1, \dots, a_n, a_n), \end{aligned}$$

where $\xi \in \mathfrak{g}^*, x \in \mathfrak{g}$, and $a_j \in \mathfrak{d}$ for $1 \leq j \leq n$, are also respective Lie bialgebra embeddings of $(\mathfrak{g}_{(2n+1)}, \delta_{\mathfrak{g}_{(2n+1)}})$, $(\mathfrak{g}_{(2n)}^*, -\delta_{\mathfrak{g}_{(2n)}^*})$, and $(\mathfrak{g}_{(2n+1)}^*, -\delta_{\mathfrak{g}_{(2n+1)}^*})$ into $(\mathfrak{g}_{(2n+2)}, \delta_{\mathfrak{g}_{(2n+2)}})$. As $t_{n+1} \in \wedge^2 \mathfrak{g}_{(2n+2)}$ actually lies in the images of the above embeddings, Lemma 5.6 follows directly from Lemma 5.4.

Q.E.D.

5.2. Mixed product Poisson structures associated to polyubles. Let again $(\mathfrak{g}, \delta_{\mathfrak{g}})$ be a Lie bialgebra, with dual Lie algebra $(\mathfrak{g}^*, \delta_{\mathfrak{g}^*})$ and double Lie bialgebra $(\mathfrak{d}, \delta_{\mathfrak{d}})$. Assume that (Y_j, π_j, σ_j) , for $1 \leq j \leq n$ and $n \geq 2$, are left $(\mathfrak{d}, \delta_{\mathfrak{d}})$ -Poisson spaces. Set

$$\rho_j = -\sigma_j|_{\mathfrak{g}^*} : \mathfrak{g}^* \longrightarrow \mathcal{V}^1(Y_j) \quad \text{and} \quad \lambda_j = \sigma_j|_{\mathfrak{g}} : \mathfrak{g} \longrightarrow \mathcal{V}^1(Y_j), \quad 1 \leq j \leq n,$$

so that ρ_j and λ_j are respectively right and left Poisson actions of $(\mathfrak{g}^*, \delta_{\mathfrak{g}^*})$ and $(\mathfrak{g}, \delta_{\mathfrak{g}})$ on (Y_j, π_j) . Let again $\{x_i\}_{i=1}^m$ be any basis of \mathfrak{g} and $\{\xi_i\}_{i=1}^m$ the dual basis of \mathfrak{g}^* . Define the bi-vector field π_Y on the product manifold $Y = Y_1 \times Y_2 \times \dots \times Y_n$ by

$$(5.9) \quad \pi_Y = (\pi_1, \pi_2, \dots, \pi_n) - \sum_{1 \leq j < k \leq n} \sum_{i=1}^m (0, \dots, \underset{j^{\text{th}} \text{ entry}}{\rho_j(\xi_i)}, \dots, 0) \wedge (0, \dots, \underset{k^{\text{th}} \text{ entry}}{\lambda_k(x_i)}, \dots, 0),$$

where $(\pi_1, \pi_2, \dots, \pi_n)$ is the direct product Poisson structure on Y (see notation in §1.5). Let $\sigma : \mathfrak{d}^n \rightarrow \mathcal{V}^1(Y)$ be the direct product Lie algebra action, i.e.,

$$(5.10) \quad \sigma(a_1, a_2, \dots, a_n) = (\sigma_1(a_1), \sigma_2(a_2), \dots, \sigma_n(a_n)), \quad a_1, \dots, a_n \in \mathfrak{d}.$$

The following Proposition 5.7 is a direct consequence of Lemma 5.5 and Proposition 3.4.

Proposition 5.7. *The triple (Y, π_Y, σ) is a left $(\mathfrak{d}^n, \delta_{\mathfrak{d}^n})$ -Poisson space. Moreover, for $1 \leq j < k \leq n$, one has $p_{jk}(\pi_Y) = \pi_j \times_{(\rho_j, \lambda_k)} \pi_k$, where $p_{jk} : Y \rightarrow Y_j \times Y_k$ is the projection to the j 'th and the k 'th factor.*

Remark 5.8. If $n \geq 2$ and if we are given (Y_1, π_1, ρ_1) as a right $(\mathfrak{g}^*, \delta_{\mathfrak{g}^*})$ -Poisson space, (Y_n, π_n, λ_n) as a left $(\mathfrak{g}, \delta_{\mathfrak{g}})$ -Poisson space, and for $2 \leq j \leq n-1$, (Y_j, π_j, σ_j) as a left $(\mathfrak{d}, \delta_{\mathfrak{d}})$ -Poisson space, one can still define the bi-vector field π_Y on $Y = Y_1 \times Y_2 \times \dots \times Y_n$ as in (5.9), and Lemma 5.6 and Proposition 3.4 imply that (Y, π_Y, ρ) is a right $(\mathfrak{g}_{(2n-2)}^*, \delta_{\mathfrak{g}_{(2n-2)}^*})$ -Poisson space, where ρ is the right Lie algebra action $\rho : \mathfrak{g}_{(2n-2)}^* \rightarrow \mathcal{V}^1(Y)$ given by

$$(5.11) \quad \rho(\xi, a_2, a_2, \dots, a_{n-1}, a_{n-1}, x) = (\rho_1(\xi), -\sigma_2(a_2), \dots, -\sigma_{n-1}(a_{n-1}), -\lambda_n(x)),$$

generalizing the two-fold mixed product Poisson structure construction in Proposition 4.2. Using Lemma 5.6, one can similarly apply Proposition 3.4 to obtain mixed product Poisson structures with Poisson actions by the Lie bialgebras $(\mathfrak{g}_{(2n-1)}^*, \delta_{\mathfrak{g}_{(2n-1)}^*})$ and $(\mathfrak{g}_{(2n-1)}, \delta_{\mathfrak{g}_{(2n-1)}})$.

6. MIXED TWISTS OF PRODUCT QUASITRIANGULAR LIE BIALGEBRAS AND MIXED PRODUCT POISSON STRUCTURES

In §6, we give a construction of certain mixed twists of direct product quasitriangular Lie bialgebras, generalizing that of polyubles of Lie bialgebras in §5. Throughout §6, let r be a quasitriangular r -matrix on a Lie algebra \mathfrak{g} , and let $\delta_r : \mathfrak{g} \rightarrow \wedge^2 \mathfrak{g}$, $\delta_r(x) = \text{ad}_x r$ be the co-bracket on \mathfrak{g} (see (2.6)). The dual map of δ_r is the Lie bracket on \mathfrak{g}^* given by

$$(6.1) \quad [\xi, \eta] = \text{ad}_{r_-(\xi)}^* \eta - \text{ad}_{r_+(\eta)}^* \xi = \text{ad}_{r_+(\xi)}^* \eta - \text{ad}_{r_-(\eta)}^* \xi.$$

where $r_+, r_- : \mathfrak{g}^* \rightarrow \mathfrak{g}$ are given in (2.14). Recall that a (left or right) Poisson space of the Lie bialgebra (\mathfrak{g}, δ_r) is also called a Poisson space of the quasitriangular Lie bialgebra (\mathfrak{g}, r) .

6.1. The quasitriangular r -matrix $r^{(n)}$ on \mathfrak{g}^n . Let $n \geq 1$ and consider the direct product Lie algebra $\mathfrak{g}^n = \mathfrak{g} \oplus \cdots \oplus \mathfrak{g}$ (n -copies). For $X \in \mathfrak{g}^{\otimes k}$, $k \geq 1$, and $1 \leq j \leq n$, let $(X)_j \in (\mathfrak{g}^n)^{\otimes k}$ be the image of X under the embedding of \mathfrak{g} into \mathfrak{g}^n as the j 'th summand. If $r = \sum_i x_i \otimes y_i \in \mathfrak{g} \otimes \mathfrak{g}$, define $\text{Alt}^n(r) \in \mathfrak{g}^n \otimes \mathfrak{g}^n$ to be the alternating sum

$$(6.2) \quad \text{Alt}^n(r) = (r, -r^{21}, r, -r^{21}, \dots) = \sum_{1 \leq j \leq n, j \text{ is odd}} (r)_j + \sum_{1 \leq j \leq n, j \text{ is even}} (-r^{21})_j,$$

where recall that $r^{21} = \sum_i y_i \otimes x_i$, and define $\text{Mix}^n(r) \in \wedge^2(\mathfrak{g}^n)$

$$(6.3) \quad \text{Mix}^n(r) = \sum_{1 \leq j < k \leq n} (\text{Mix}^n(r))_{j,k},$$

where $(\text{Mix}^n(r))_{j,k} = \sum_i (y_i)_j \wedge (x_i)_k$ for $1 \leq j < k \leq n$. The elements $(\text{Mix}^n(r))_{j,k} \in \wedge^2(\mathfrak{g}^n)$, where $1 \leq j < k \leq n$, are indeed well-defined: the linear map $(\text{Mix}^n(r))_{j,k}^\# : (\mathfrak{g}^*)^n \cong (\mathfrak{g}^n)^* \rightarrow \mathfrak{g}^n$ is given by

$$(6.4) \quad (\text{Mix}^n(r))_{j,k}^\# (\xi_1, \dots, \xi_n) = (-r_+(\xi_k))_j + (-r_-(\xi_j))_k, \quad (\xi_1, \dots, \xi_n) \in (\mathfrak{g}^*)^n \cong (\mathfrak{g}^n)^*.$$

Define now $r^{(n)} \in \mathfrak{g}^n \otimes \mathfrak{g}^n$ by

$$(6.5) \quad r^{(n)} = \text{Alt}^n(r) - \text{Mix}^n(r) \in \mathfrak{g}^n \otimes \mathfrak{g}^n.$$

The next Theorem 6.2 says that $r^{(n)}$ is a quasitriangular r -matrix on \mathfrak{g}^n . For $n \geq 2$, $r^{(n)}$ is thus a mixed twist (see Definition 3.3) of the quasitriangular r -matrix $\text{Alt}^n(r)$ on \mathfrak{g}^n .

Remark 6.1. Consider the quasitriangular r -matrix $r_\mathfrak{d}$ on the double Lie algebra $(\mathfrak{d}, \langle, \rangle_\mathfrak{d})$ of an arbitrary Lie bialgebra $(\mathfrak{g}, \delta_\mathfrak{g})$ defined by the Lagrangian splitting $\mathfrak{d} = \mathfrak{g} + \mathfrak{g}^*$. By Lemma 5.5, $r_\mathfrak{d}^{(n)}$ coincides with the quasitriangular r -matrix on \mathfrak{d}^n defined by the Lagrangian splitting $\mathfrak{d}^n = \mathfrak{g}_{(n)} + \mathfrak{g}_{(n)}^*$ in (5.4). Note also that if $\phi : (\mathfrak{g}, r) \rightarrow (\mathfrak{g}, r')$ is a homomorphism of quasitriangular Lie bialgebras, then $\phi(r^{(n)}) = (r')^{(n)}$. \diamond

Theorem 6.2. For any integer $n \geq 1$, $r^{(n)}$ is a quasitriangular r -matrix on the direct product Lie algebra \mathfrak{g}^n , and the dual of $\delta_{r^{(n)}} : \mathfrak{g}^n \rightarrow \wedge^2 \mathfrak{g}^n$ is the Lie bracket $[\cdot, \cdot]_{r^{(n)}}$ on $(\mathfrak{g}^*)^n \cong (\mathfrak{g}^n)^*$ given by

$$[(\xi_1, \dots, \xi_n), (\eta_1, \dots, \eta_n)]_{r^{(n)}} = (\zeta_1, \dots, \zeta_n),$$

where $(\xi_1, \dots, \xi_n), (\eta_1, \dots, \eta_n) \in (\mathfrak{g}^*)^n$, and for each $1 \leq j \leq n$,

$$(6.6) \quad \zeta_j = [\xi_j, \eta_j] + \text{ad}_{r_-(\xi_1 + \dots + \xi_{j-1}) + r_+(\xi_{j+1} + \dots + \xi_n)}^* \eta_j - \text{ad}_{r_-(\eta_1 + \dots + \eta_{j-1}) + r_+(\eta_{j+1} + \dots + \eta_n)}^* \xi_j.$$

Moreover, for any $1 \leq k \leq m \leq n$, the map $\phi_{m,k} : \mathfrak{g}^m \rightarrow \mathfrak{g}^n$ given by

$$(6.7) \quad \phi_{m,k}(x_1, x_2, \dots, x_m) = (x_1, \dots, x_{k-1}, \overbrace{x_k, x_k, \dots, x_k}^{(n-m+1) \text{ terms}}, x_{k+1}, \dots, x_m), \quad x_j \in \mathfrak{g},$$

is a Lie bialgebra homomorphism from $(\mathfrak{g}^m, \delta_{r^{(m)}})$ to $(\mathfrak{g}^n, \delta_{r^{(n)}})$.

Proof. Note that the symmetric part of $r^{(n)}$ is the alternating sum $(s, -s, s, \dots, (-1)^n s)$, where s is the symmetric part of r . To show that $r^{(n)}$ is a quasitriangular r -matrix on \mathfrak{g}^n , we need to show that $\text{CYB}(r^{(n)}) = 0$. Let $(\mathfrak{d}, \delta_{\mathfrak{d}})$ be the double Lie bialgebra of (\mathfrak{g}, δ_r) , and consider the Lie algebra homomorphism $p_+ : \mathfrak{d} \rightarrow \mathfrak{g}$ given by $p_+(x + \xi) = x + r_+(\xi)$ for $x \in \mathfrak{g}$ and $\xi \in \mathfrak{g}^*$. By (2.16), $p_+(r_{\mathfrak{d}}) = r$. Thus $r^{(n)} = p_+(r_{\mathfrak{d}}^{(n)})$, where $p_+(a_1, \dots, a_n) = (p_+(a_1), \dots, p_+(a_n)) \in \mathfrak{g}^n$ for $(a_1, \dots, a_n) \in \mathfrak{d}^n$. As $r_{\mathfrak{d}}^{(n)}$ is a quasitriangular r -matrix on \mathfrak{d}^n by Remark 6.1,

$$\text{CYB}(r^{(n)}) = p_+^n(\text{CYB}(r_{\mathfrak{d}}^{(n)})) = 0.$$

To compute the Lie bracket $[\cdot, \cdot]_{r^{(n)}}$ on $(\mathfrak{g}^n)^* \cong (\mathfrak{g}^*)^n$, consider first the case of $n = 2$. It is easy to check that the two linear maps $(r^{(2)})_{\pm} : \mathfrak{g}^* \oplus \mathfrak{g}^* \rightarrow \mathfrak{g} \oplus \mathfrak{g}$ are respectively given by

$$(6.8) \quad (r^{(2)})_+(\xi_1, \xi_2) = (r_+(\xi_1 + \xi_2), r_-(\xi_1 + \xi_2)),$$

$$(6.9) \quad (r^{(2)})_-(\xi_1, \xi_2) = (r_-(\xi_1) + r_+(\xi_2), r_-(\xi_1) + r_+(\xi_2)), \quad \xi_1, \xi_2 \in \mathfrak{g}^*.$$

It follows that for $\xi_1, \xi_2, \eta_1, \eta_2 \in \mathfrak{g}^*$, one has

$$[(\xi_1, \xi_2), (\eta_1, \eta_2)]_{r^{(2)}} = ([\xi_1, \eta_1] + \text{ad}_{r_+(\xi_2)}^* \eta_1 - \text{ad}_{r_+(\eta_2)}^* \xi_1, [\xi_2, \eta_2] + \text{ad}_{r_-(\xi_1)}^* \eta_2 - \text{ad}_{r_-(\eta_1)}^* \xi_2).$$

The formula for $[\cdot, \cdot]_{r^{(n)}}$ now follows from the observation that, as $r^{(n)}$ is a mixed twist of $\text{Alt}^n(r)$, for each pair $1 \leq j < k \leq n$, the projection $p_{jk} : \mathfrak{g}^n \rightarrow \mathfrak{g} \oplus \mathfrak{g}$ given by $(x_1, \dots, x_n) \mapsto (x_j, x_k)$ is a Lie bialgebra homomorphism from $(\mathfrak{g}^n, \delta_{r^{(n)}})$ to $(\mathfrak{g}^2, \delta_{r^{(2)}})$.

Let $1 \leq k \leq m \leq n$. Using (6.6), one checks by direct computations (we omit the details) that the dual map $\phi_{m,k}^*$ of $\phi_{m,k}$, which is given by

$$\phi_{m,k}^* : (\mathfrak{g}^*)^n \longrightarrow (\mathfrak{g}^*)^m, \quad \phi_{m,k}^*(\xi_1, \dots, \xi_n) = (\xi_1, \dots, \xi_{k-1}, \xi_k + \dots + \xi_l, \xi_{l+1}, \dots, \xi_n),$$

where $l = n - m + k$, is a Lie algebra homomorphism from $((\mathfrak{g}^*)^n, [\cdot, \cdot]_{r^{(n)}})$ to $((\mathfrak{g}^*)^m, [\cdot, \cdot]_{r^{(m)}})$.

Q.E.D.

Remark 6.3. Starting from the quasitriangular Lie bialgebra (\mathfrak{g}, r) , P. Etingof and D. Kazhdan constructed in [16] a Lie bracket on $(\mathfrak{g}^*)^n = \mathfrak{g}^* \oplus \dots \oplus \mathfrak{g}^*$, $n \geq 2$, which, together with the co-bracket dual to the direct product Lie bracket on \mathfrak{g}^n , form a *locally factored Lie bialgebra with equal components*, i.e., for each $1 \leq j \leq n$, the j 'th summand $(\mathfrak{g}^*)_j$ is a sub-Lie bialgebra, isomorphic to the dual Lie bialgebra of (\mathfrak{g}, r) , and $[(\mathfrak{g}^*)_i, (\mathfrak{g}^*)_j] \subset (\mathfrak{g}^*)_j \oplus (\mathfrak{g}^*)_j$ for each pair $i \neq j$. One checks directly that the Lie bracket $[\cdot, \cdot]_{r^{(n)}}$ given in Theorem 6.2 is precisely the Lie bracket on $(\mathfrak{g}^*)^n$ in [16, Proposition 1.9]. \diamond

Remark 6.4. One may obtain other quasitriangular r -matrices on \mathfrak{g}^n from $r^{(n)}$. Indeed, let S_n be the permutation group of $\{1, 2, \dots, n\}$, and for $\tau \in S_n$, define $\phi_\tau \in \text{Aut}(\mathfrak{g}^n)$ by $\phi_\tau(x_1, \dots, x_n) = (x_{\tau^{-1}(1)}, \dots, x_{\tau^{-1}(n)})$ for $x_1, \dots, x_n \in \mathfrak{g}$. Then $\phi_\tau(r^{(n)})$ is a quasitriangular r -matrix on \mathfrak{g}^n for every $\tau \in S_n$. Write $r = \Lambda + s$, where $\Lambda \in \wedge^2 \mathfrak{g}$ and $s \in (S^2(\mathfrak{g}))^{\mathfrak{g}}$, so that

$$r^{(n)} = (\Lambda, \Lambda, \dots, \Lambda) - \text{Mix}^n(\Lambda) - \text{Mix}^n(s) + (s, -s, s, \dots, (-1)^{n-1} s),$$

where $\text{Mix}^n(\Lambda) \in \wedge^2 \mathfrak{g}^n$ and $\text{Mix}^n(s) \in \wedge^2 \mathfrak{g}^n$ are defined using (6.3) by replacing r by Λ and s . Then for every $\tau \in S_n$,

$$\phi_\tau(r^{(n)}) = (\Lambda, \Lambda, \dots, \Lambda) - \phi_\tau(\text{Mix}^n(\Lambda)) - \phi_\tau(\text{Mix}^n(s)) + (\varepsilon(1)s, \varepsilon(2)s, \dots, \varepsilon(n)s)$$

for some sign function $\varepsilon : \{1, 2, \dots, n\} \rightarrow \{1, -1\}$. It is easy to show that $\phi_\tau(\text{Mix}^n(\Lambda)) = \text{Mix}^n(\Lambda)$, while

$$\phi_\tau(\text{Mix}^n(s)) = \sum_{1 \leq j < k \leq n, \tau^{-1}(j) < \tau^{-1}(k)} (\text{Mix}^n(s))_{j,k} - \sum_{1 \leq j < k \leq n, \tau^{-1}(j) > \tau^{-1}(k)} (\text{Mix}^n(s))_{j,k}.$$

For an arbitrary function $\varepsilon : \{1, 2, \dots, n\} \rightarrow \{1, -1\}$ and any $\tau \in S_n$, define

$$(6.10) \quad r^{(\varepsilon, \tau, n)} = (\varepsilon(1)s, \varepsilon(2)s, \dots, \varepsilon(n)s) + (\Lambda, \Lambda, \dots, \Lambda) - \text{Mix}^n(\Lambda) - \phi_\tau(\text{Mix}^n(s)) \in \mathfrak{g}^n \otimes \mathfrak{g}^n.$$

By (2.8) - (2.10), $r^{(\varepsilon, \tau, n)}$ is a quasitriangular r -matrix on \mathfrak{g}^n . By Definition 3.1, the corresponding Lie bialgebra $(\mathfrak{g}^n, \delta_{r^{(\varepsilon, \tau, n)}})$ is a mixed twist of the n -fold direct product Lie bialgebra (\mathfrak{g}, δ_r) with itself. As a special case, for any $J = \{j_1, \dots, j_k\} \subset \{1, \dots, n\}$ with $j_1 < \dots < j_k$ and $p_J : \mathfrak{g}^n \rightarrow \mathfrak{g}^k$ the projection given by $(x_1, \dots, x_n) \mapsto (x_{j_1}, \dots, x_{j_k})$, one has

$$p_J(r^{(n)}) = r^{(\varepsilon, e, k)} = (\varepsilon(1)s, \varepsilon(2)s, \dots, \varepsilon(k)s) + (\Lambda, \Lambda, \dots, \Lambda) - \text{Mix}^k(r),$$

where e is the identity element of S_k , and $\varepsilon(i) = (-1)^{j_i-1}$ for $i = 1, \dots, k$. \diamond

For $n \geq 1$, we now compare the n -uple $(\mathfrak{g}_{(n)}, \delta_{\mathfrak{g}_{(n)}})$ of the Lie bialgebra (\mathfrak{g}, δ_r) with the Lie bialgebra $(\mathfrak{g}^n, \delta_{r^{(n)}})$. Consider first the $(2n)$ -uple $(\mathfrak{g}_{(2n)}, \delta_{\mathfrak{g}_{(2n)}})$ of (\mathfrak{g}, δ_r) , where $n \geq 1$. Recall from Lemma 5.5 that one has the Lie bialgebra isomorphism

$$(\mathfrak{g}_{(2n)}, \delta_{\mathfrak{g}_{(2n)}}) \longrightarrow (\mathfrak{d}^n, \delta_{r_{\mathfrak{d}}^{(n)}}) : (a_1, a_1, \dots, a_n, a_n) \longmapsto (a_1, \dots, a_n), \quad a_j \in \mathfrak{d}.$$

Let $p_{2n} : \mathfrak{d}^n \rightarrow \mathfrak{g}^{2n}$ be the Lie algebra homomorphism given by

$$p_{2n}(a_1, a_1, \dots, a_n) = (p_+(a_1), p_+(a_2), \dots, p_+(a_n), p_-(a_n), p_-(a_{n-1}), \dots, p_-(a_1)), \quad a_j \in \mathfrak{d},$$

where $p_{\pm} : \mathfrak{d} \rightarrow \mathfrak{g}$ are the Lie algebra homomorphisms given in (2.15).

Lemma 6.5. *One has $p_{2n}(r_{\mathfrak{d}}^{(n)}) = r^{(2n)}$. Consequently, $J_{2n} : (\mathfrak{g}_{(2n)}, \delta_{\mathfrak{g}_{(2n)}}) \rightarrow (\mathfrak{g}^{2n}, \delta_{r^{(2n)}})$ given by*

$$J_{2n}(a_1, a_1, \dots, a_n, a_n) = (p_+(a_1), p_+(a_2), \dots, p_+(a_n), p_-(a_n), p_-(a_{n-1}), \dots, p_-(a_1)), \quad a_j \in \mathfrak{d},$$

is a Lie bialgebra homomorphism.

Proof. Using (2.16), one proves that $p_{2n}(r_{\mathfrak{d}}^{(n)}) = r^{(2n)}$ by a straightforward calculation.

Q.E.D.

For the $(2n-1)$ -uple $(\mathfrak{g}_{(2n-1)}, \delta_{\mathfrak{g}_{(2n-1)}})$ of (\mathfrak{g}, δ_r) , where $n \geq 1$, define $J_{2n-1} : \mathfrak{g}_{(2n-1)} \rightarrow \mathfrak{g}^{2n-1}$ by

$$J_{2n-1}(a_1, a_1, \dots, a_{n-1}, a_{n-1}, x) = (p_+(a_1), \dots, p_+(a_{n-1}), x, p_-(a_{n-1}), \dots, p_-(a_1)),$$

where $a_1, \dots, a_{n-1} \in \mathfrak{d}$ and $x \in \mathfrak{g}$.

Lemma 6.6. *The map $J_{2n-1} : (\mathfrak{g}_{(2n-1)}, \delta_{\mathfrak{g}_{(2n-1)}}) \rightarrow (\mathfrak{g}^{2n-1}, \delta_{r^{(2n-1)}})$ is a Lie bialgebra homomorphism.*

Proof. One has $J_{2n-1} = p_{(2n)} \circ J_{2n} \circ I_{2n-1}$, where $I_{2n-1} : (\mathfrak{g}_{(2n-1)}, \delta_{\mathfrak{g}_{(2n-1)}}) \rightarrow (\mathfrak{g}_{(2n)}, \delta_{\mathfrak{g}_{(2n)}})$ is the Lie bialgebra embedding given by

$$I_{2n-1}(a_1, a_1, \dots, a_{n-1}, a_{n-1}, x) \longmapsto (a_1, a_1, \dots, a_{n-1}, a_{n-1}, x, x), \quad a_j \in \mathfrak{d}, x \in \mathfrak{g},$$

and $p_{(2n)} : (\mathfrak{g}^{2n}, \delta_{r^{(2n)}}) \rightarrow (\mathfrak{g}^{2n-1}, \delta_{r^{(2n-1)}})$ is the surjective Lie bialgebra homomorphism (see Remark 6.4)

$$p_{(2n)}(x_1, x_2, \dots, x_{2n}) = (x_1, x_2, \dots, x_n, x_{n+2}, \dots, x_{2n}), \quad x_j \in \mathfrak{g}.$$

Q.E.D.

Proposition 6.7. *When r factorizable, the Lie bialgebra homomorphisms $J_n : (\mathfrak{g}_{(n)}, \delta_{\mathfrak{g}_{(n)}}) \rightarrow (\mathfrak{g}^n, \delta_{r^{(n)}})$ in Lemma 6.5 and Lemma 6.6 are Lie bialgebra isomorphisms.*

Proof. When r is factorizable, the map $\mathfrak{d} \rightarrow \mathfrak{g} \oplus \mathfrak{g}, a \mapsto (p_+(a), p_-(a))$, $a \in \mathfrak{d}$, is a Lie algebra isomorphism. Thus $J_n : \mathfrak{g}_{(n)} \rightarrow \mathfrak{g}^n$ is a Lie algebra isomorphism for any $n \geq 1$.

Q.E.D.

6.2. Mixed product Poisson structures associated to quasitriangular r -matrices. Assume that (Y_j, π_j, λ_j) , $1 \leq j \leq n$, are left (\mathfrak{g}, r) -Poisson spaces. Let $Y = Y_1 \times \cdots \times Y_n$ be the product manifold, and let $\lambda = (\lambda_1, \dots, \lambda_n) : \mathfrak{g}^n \rightarrow \mathcal{V}^1(Y)$ be the direct product left Lie algebra action of \mathfrak{g}^n on Y given by

$$(6.11) \quad \lambda(x_1, \dots, x_n) = (\lambda_1(x_1), \dots, \lambda_n(x_n)), \quad x_j \in \mathfrak{g}.$$

If $r = \sum_i x_i \otimes y_i \in \mathfrak{g} \otimes \mathfrak{g}$, define the bi-vector field π_Y on Y by

$$(6.12) \quad \begin{aligned} \pi_Y &= (\pi_1, \dots, \pi_n) + \lambda(\text{Mix}^n(r)) \\ &= (\pi_1, \pi_2, \dots, \pi_n) + \sum_{1 \leq j < k \leq n} \sum_i (0, \dots, \lambda_j(y_i), \dots, 0) \wedge (0, \dots, \lambda_k(x_i), \dots, 0), \end{aligned}$$

where again $(\pi_1, \pi_2, \dots, \pi_n)$ is the direct product Poisson structure on $Y = Y_1 \times \cdots \times Y_n$.

Theorem 6.8. *The triple (Y, π_Y, λ) is a left $(\mathfrak{g}^n, r^{(n)})$ -Poisson space, and $(Y, \pi_Y, \lambda_{\text{diag}})$ is a left (\mathfrak{g}, r) -Poisson space, where λ_{diag} is the diagonal action*

$$(6.13) \quad \lambda_{\text{diag}} : \mathfrak{g} \longrightarrow \mathcal{V}^1(Y), \quad \lambda_{\text{diag}}(x) = (\lambda_1(x), \dots, \lambda_n(x)), \quad x \in \mathfrak{g},$$

of \mathfrak{g} on Y . If, in addition, $\pi_j = -\lambda_j(r)$ for each $1 \leq j \leq n$ (see Proposition 2.18), then $\pi_Y = -\lambda(r^{(n)})$.

Proof. The first and the third statements of Theorem 6.8 follow directly from Proposition 3.4. By Theorem 6.2, for any $1 \leq k \leq m \leq n$, $(Y, \pi_Y, \lambda \circ \phi_{m,k})$ is a left $(\mathfrak{g}^m, r^{(m)})$ -Poisson space, where $\phi_{m,k} : \mathfrak{g}^m \rightarrow \mathfrak{g}^n$ is given in (6.7). In particular, the triple $(Y, \pi_Y, \lambda_{\text{diag}})$ is a left (\mathfrak{g}, r) -Poisson space.

Q.E.D.

Definition 6.9. Given (\mathfrak{g}, r) -Poisson spaces (Y_j, π_j, λ_j) for $1 \leq j \leq n$ with $n \geq 2$, we also denote the Poisson structure π_Y on $Y = Y_1 \times \cdots \times Y_n$ given in (6.12) by

$$\pi_Y = (\pi_1 \times \cdots \times \pi_n)_{(\lambda_1, \dots, \lambda_n)},$$

and we will refer to the left (\mathfrak{g}, r) -Poisson space $(Y, \pi_Y, \lambda_{\text{diag}})$, where λ_{diag} is the diagonal action of \mathfrak{g} on Y given in (6.13), as the *fusion product* of (Y_j, π_j, λ_j) , $1 \leq j \leq n$.

Remark 6.10. Recall from Lemma 2.7 the dual pair of Lie bialgebras $(\mathfrak{f}_-, \delta_r|_{\mathfrak{f}_-})$ and $(\mathfrak{f}_+, -\delta_r|_{\mathfrak{f}_+})$. Let $\{x_i\}_{i=1}^m$ be a basis of \mathfrak{g} such that $\{x_i\}_{i=1}^l$ is a basis of \mathfrak{f}_- , and let $\{\xi_i\}_{i=1}^m$ be the dual basis of \mathfrak{g}^* . Then $\{r_+(\xi_i)\}_{i=1}^l$ is a basis of \mathfrak{f}_+ , dual to the basis $\{x_i\}_{i=1}^l$ of \mathfrak{f}_- under the pairing $\langle, \rangle_{(\mathfrak{f}_-, \mathfrak{f}_+)}$ in (2.18), and

$$r = \sum_{i=1}^m x_i \otimes r_+(\xi_i) = \sum_{i=1}^l x_i \otimes r_+(\xi_i) \in \mathfrak{f}_- \otimes \mathfrak{f}_+ \subset \mathfrak{g} \otimes \mathfrak{g}$$

(see Remark 2.8). In the context of Theorem 6.8, it then follows that for any $1 \leq j < k \leq n$, the projection of π_Y to $Y_j \times Y_k$ is precisely the mixed product Poisson structure $\pi_j \times_{(-\lambda_j|_{\mathfrak{f}_+}, \lambda_k|_{\mathfrak{f}_-})} \pi_k$ formed out of the right $(\mathfrak{f}_+, -\delta_r|_{\mathfrak{f}_+})$ -Poisson space $(Y_j, \pi_j, -\lambda_j|_{\mathfrak{f}_+})$ and the left $(\mathfrak{f}_-, \delta_r|_{\mathfrak{f}_-})$ -Poisson space $(Y_k, \pi_k, \lambda_k|_{\mathfrak{f}_-})$. Furthermore, let $1 \leq j \leq n$, and let $Y_{(j)} = Y_1 \times \cdots \times Y_j$ and $Y_{[j]} = Y_{j+1} \times \cdots \times Y_n$. It then follows from

$$\begin{aligned} \text{Mix}^n(r) &= (\text{Mix}^j(r), 0) + (0, \text{Mix}^{n-j}(r)) \\ &+ \sum_{i=1}^l \underbrace{(r_+(\xi_i), r_+(\xi_i), \dots, r_+(\xi_i))}_{j \text{ terms}} \wedge (0, 0, \dots, 0) \wedge (0, 0, \dots, 0, \underbrace{x_i, x_i, \dots, x_i}_{n-j \text{ terms}}) \end{aligned}$$

that under the obvious identification $Y = Y_{(j)} \times Y_{[j]}$, π_Y is also a two-fold mixed product, namely

$$\pi_Y = \pi_{(j)} \times_{(-\lambda_{(j)}|_{\mathfrak{f}_+}, \lambda_{[j]}|_{\mathfrak{f}_-})} \pi_{[j]},$$

where $\pi_{(j)} = (\pi_1 \times \cdots \times \pi_j)_{(\lambda_1, \dots, \lambda_j)}$, $\pi_{[j]} = (\pi_{j+1} \times \cdots \times \pi_n)_{(\lambda_{j+1}, \dots, \lambda_n)}$, and

$$\begin{aligned} \lambda_{(j)} : \mathfrak{g} &\longrightarrow V^1(Y_{(j)}), \quad \lambda_{(j)}(x) = (\lambda_1(x), \dots, \lambda_j(x)), \quad x \in \mathfrak{g}, \\ \lambda_{[j]} : \mathfrak{g} &\longrightarrow V^1(Y_{[j]}), \quad \lambda_{[j]}(x) = (\lambda_{j+1}(x), \dots, \lambda_n(x)), \quad x \in \mathfrak{g}. \end{aligned}$$

Set, for notational simplicity,

$$\begin{aligned} \lambda_j^+ &= \lambda_j|_{\mathfrak{f}_+} : \mathfrak{f}_+ \longrightarrow \mathcal{V}^1(Y_j), & \lambda_{(j)}^+ &= \lambda_{(j)}|_{\mathfrak{f}_+} : \mathfrak{f}_+ \longrightarrow \mathcal{V}^1(Y_{(j)}), \\ \lambda_j^- &= \lambda_j|_{\mathfrak{f}_-} : \mathfrak{f}_- \longrightarrow \mathcal{V}^1(Y_j), & \lambda_{[j]}^- &= \lambda_{[j]}|_{\mathfrak{f}_-} : \mathfrak{f}_- \longrightarrow \mathcal{V}^1(Y_{[j]}). \end{aligned}$$

Then π_Y can also be realized as the successive 2-fold mixed products

$$\begin{aligned} \pi_Y &= \left(\left(\pi_1 \times_{(-\lambda_1^+, \lambda_2^-)} \pi_2 \right) \times_{(-\lambda_{(2)}^+, \lambda_3^-)} \cdots \times_{(-\lambda_{(n-2)}^+, \lambda_{n-1}^-)} \pi_{n-1} \right) \times_{(-\lambda_{(n-1)}^+, \lambda_n^-)} \pi_n \\ &= \pi_1 \times_{(-\lambda_1^+, \lambda_{[2]}^-)} \left(\pi_2 \times_{(-\lambda_2^+, \lambda_{[3]}^-)} \cdots \times_{(-\lambda_{n-2}^+, \lambda_{[n-1]}^-)} \left(\pi_{n-1} \times_{(-\lambda_{n-1}^+, \lambda_n^-)} \pi_n \right) \right). \end{aligned}$$

◇

We now give a typical example of the construction in Theorem 6.8.

Example 6.11. Suppose that G is a connected Lie group with Lie algebra \mathfrak{g} and let $s \in (S^2\mathfrak{g})^{\mathfrak{g}}$ be the symmetric part of r . Suppose that Q_1, \dots, Q_n are closed Lie subgroups of G such that the Lie algebra \mathfrak{q}_j of Q_j for each j is coisotropic with respect to s (see §2.6). Let λ_j be the left action of \mathfrak{g} on G/Q_j induced by the left action of G on G/Q by left translation. By Proposition 2.18, $\lambda_j(r)$ is a Poisson structure on G/Q_j for each j , and by Theorem 6.8, $\lambda(r^{(n)})$ is a mixed product Poisson structure on the product manifold $G/Q_1 \times \cdots \times G/Q_n$, where $\lambda = (\lambda_1, \dots, \lambda_n)$ is the direct product action of \mathfrak{g}^n on $G/Q_1 \times \cdots \times G/Q_n$. Note that any twist r' of r , having the same symmetric part as r , also gives rise to the Poisson structure $\lambda((r')^{(n)})$ on $G/Q_1 \times \cdots \times G/Q_n$. We will return to these examples in §8.6 for the case when G is a complex semi-simple Lie group. ◇

Let G be again any connected Lie group with Lie algebra \mathfrak{g} . We end §6.2 by a discussion on the Poisson Lie group $(G^n, \pi_G^{(n)})$, where $n \geq 1$, and

$$(6.14) \quad \pi_G^{(n)} = \left(r^{(n)} \right)^L - \left(r^{(n)} \right)^R,$$

with $(r^{(n)})^L$ and $(r^{(n)})^R$ respectively denoting the left and right invariant 2-tensor fields on G^n with value $r^{(n)}$ at the identity element. Here $r^{(1)} = r$ by convention, and we also write $\pi_G^{(1)} = \pi_G$. By Proposition 6.7, when r is factorizable, $(G \times G, \pi_G^{(2)})$ is a Drinfeld double of the Poisson Lie group (G, π_G) .

Consider again the pair $((\mathfrak{f}_+, -\delta_r|_{\mathfrak{f}_+}), (\mathfrak{f}_-, \delta_r|_{\mathfrak{f}_-}))$ of dual Lie bialgebras, where $\mathfrak{f}_{\pm} = \text{Im}(r_{\pm}) \subset \mathfrak{g}$. Let F_+ and F_- be the connected subgroups of G with Lie algebras \mathfrak{f}_+ and \mathfrak{f}_- respectively, and denote by the same symbol the restrictions of π_G to F_+ and F_- . Then the Poisson Lie groups $(F_+, -\pi_G)$ and (F_-, π_G) form a dual pair, and so do (F_+, π_G) and (F_-^{op}, π_G) , where F_-^{op} denotes the manifold F_- with the opposite group structure. Consider now the dual pair of Poisson Lie groups

$$(6.15) \quad (F, \pi_F) = (F_-, \pi_G) \times (F_-^{\text{op}}, \pi_G) \quad \text{and} \quad (F^*, \pi_{F^*}) = (F_+, -\pi_G) \times (F_+, \pi_G).$$

By the multiplicativity of the Poisson structure π_G , one has the right and left Poisson actions

$$(6.16) \quad \rho : (G, \pi_G) \times (F^*, \pi_{F^*}) \longrightarrow (G, \pi_G), \quad (g, (f_1, f_2)) \longmapsto f_1^{-1} g f_2, \quad g \in G, f_1, f_2 \in F_+,$$

$$(6.17) \quad \lambda : (F, \pi_F) \times (G, \pi_G) \longrightarrow (G, \pi_G), \quad ((f_{-1}, f_{-2}), g) \longmapsto f_{-1} g f_{-2}, \quad g \in G, f_{-1}, f_{-2} \in F_-.$$

For $1 \leq j < k \leq n$, let $p_{jk} : G^n \rightarrow G \times G$ be the projection to the j 'th and the k 'th factor.

Proposition 6.12. *For any $n \geq 2$, $\pi_G^{(n)}$ is a mixed product Poisson structure on G^n , and $p_{jk}(\pi_G^{(n)}) = \pi_G^{(2)}$ for every $1 \leq j < k \leq n$; Moreover, $\pi_G^{(2)} = \pi_G \times_{(\rho, \lambda)} \pi_G$.*

Proof. The first statement is clear from the definition of $r^{(n)}$. To prove the second one, let $\{x_i\}_{i=1}^m$ be a basis of \mathfrak{g} such that $\{x_i\}_{i=1}^k$ is a basis of \mathfrak{f}_- , and let $\{\xi_i\}_{i=1}^m$ be the dual basis of \mathfrak{g}^* . By (2.19), one has

$$\pi_G^{(2)} = (\pi_G, \pi_G) - \sum_{i=1}^k ((r_+(\xi_i)^L, 0) \wedge (0, x_i^L) + (r_+(\xi_i)^R, 0) \wedge (0, x_i^R)) = \pi_G \times_{(\rho, \lambda)} \pi_G.$$

Q.E.D.

6.3. Fusion products of Poisson spaces and quasi-Poisson spaces. Write $r = s + \Lambda$, where $s \in (S^2 \mathfrak{g})^{\mathfrak{g}}$ and $\Lambda \in \wedge^2 \mathfrak{g}$, and let $\phi_s = -2\text{CYB}(s) \in (\wedge^3 \mathfrak{g})^{\mathfrak{g}}$. By (2.8) - (2.10), $[\Lambda, \Lambda] = \phi_s$, and

$$\phi_s(\xi, \eta, \zeta) = -2\langle \xi, [s^\#(\eta), s^\#(\zeta)] \rangle, \quad \xi, \eta, \zeta \in \mathfrak{g}^*.$$

Recall from [2] that a (\mathfrak{g}, ϕ_s) -quasi-Poisson space is a triple (Y, Q_Y, λ) , where Y is a manifold, $\lambda : \mathfrak{g} \rightarrow \mathcal{V}^1(Y)$ a left Lie algebra action, and Q_Y a \mathfrak{g} -invariant bi-vector field on Y such that

$$[Q_Y, Q_Y] = \lambda(\phi_s).$$

Let $\mathcal{P}(\mathfrak{g}, \delta_r)$ be the category of left (\mathfrak{g}, δ_r) -Poisson spaces with \mathfrak{g} -equivariant Poisson morphisms as morphisms, and let $\mathcal{QP}(\mathfrak{g}, \phi_s)$ be the category of (\mathfrak{g}, ϕ_s) -quasi-Poisson spaces whose morphisms are the \mathfrak{g} -equivariant morphisms respecting the quasi-Poisson bi-vector fields. The following Lemma 6.13 is proved in [1, 27]. Due to our sign conventions, we include a proof for the convenience of the reader.

Lemma 6.13. [1, 27] *One has the equivalence of categories*

$$(6.18) \quad \mathcal{P}(\mathfrak{g}, \delta_r) \longleftrightarrow \mathcal{QP}(\mathfrak{g}, \phi_s), \quad (Y, \pi_Y, \lambda) \longleftrightarrow (Y, Q_Y, \lambda),$$

where $Q_Y = \pi_Y + \lambda(\Lambda)$, and the map on morphisms is the identity map.

Proof. Let Y be a manifold with a left action λ of \mathfrak{g} , and let π_Y and Q_Y be bi-vector fields on Y related by $Q_Y = \pi_Y + \lambda(\Lambda)$. Assume first that (Y, Q_Y, λ) is (\mathfrak{g}, ϕ_s) -quasi-Poisson. Then

$$[\pi_Y, \pi_Y] = [Q_Y, Q_Y] - \lambda([\Lambda, \Lambda]) = \lambda(\phi_s - [\Lambda, \Lambda]) = 0,$$

and for $x \in \mathfrak{g}$, $[\lambda(x), \pi_Y] = -[\lambda(x), \lambda(\Lambda)] = \lambda([x, \Lambda]) = \lambda(\delta_r(x))$. Thus $(Y, \pi_Y, \lambda) \in \mathcal{P}(\mathfrak{g}, \delta_r)$. Conversely, assume that $(Y, \pi_Y, \lambda) \in \mathcal{P}(\mathfrak{g}, \delta_r)$. Then $[\lambda(x), Q_Y] = [\lambda(x), \pi_Y] - \lambda([x, \Lambda]) = [\lambda(x), \pi_Y] - \lambda(\delta_r(x)) = 0$ for all $x \in \mathfrak{g}$, and

$$[Q_Y, Q_Y] = 2[\lambda(\Lambda), \pi_Y] - \lambda([\Lambda, \Lambda]) = \lambda(2[\Lambda, \Lambda] - [\Lambda, \Lambda]) = \lambda([\Lambda, \Lambda]) = \lambda(\phi_s).$$

Thus $(Y, Q_Y, \lambda) \in \mathcal{QP}(\mathfrak{g}, \phi_s)$. Clearly the two functors in (6.18) are inverse functors of each other.

Q.E.D.

Example 6.14. If $\lambda : \mathfrak{g} \rightarrow \mathcal{V}^1(Y)$ is a left action of \mathfrak{g} on a manifold Y such that $\lambda(s) = 0$ (see §2.6), then $\lambda(\phi_s) = 0$ by the proof of Proposition 2.18, so $(Y, Q_Y = 0, \lambda)$ is a left (\mathfrak{g}, ϕ_s) -quasi-Poisson space, which, under the equivalence of categories in Lemma 6.13, corresponds to the left (\mathfrak{g}, δ_r) -Poisson space $(Y, -\lambda(r), \lambda)$ in Proposition 2.18. \diamond

Remark 6.15. Note that if (Y, Q_Y, λ) is a (\mathfrak{g}, ϕ_s) -quasi-Poisson space, so is $(Y, -Q_Y, \lambda)$. If (Y, Q_Y, λ) corresponds to the (\mathfrak{g}, δ_r) -Poisson space (Y, π_Y, λ) as in Lemma 6.13, then $(Y, -Q_Y, \lambda)$ corresponds to $(Y, \pi_Y - 2Q_Y, \lambda)$. \diamond

Remark 6.16. The category $\mathcal{QP}(\mathfrak{g}, \phi_s)$ depends only on $s \in (S^2\mathfrak{g})^{\mathfrak{g}}$, while $\mathcal{P}(\mathfrak{g}, \delta_r)$ depends only on $\Lambda \in \wedge^2\mathfrak{g}$ subject to the condition $[\Lambda, \Lambda] = \phi_s$. If $r' = r - t$ is a twist of r (see Definition 2.11), the composition functor

$$\mathcal{P}(\mathfrak{g}, \delta_r) \longrightarrow \mathcal{QP}(\mathfrak{g}, \phi_s) \longrightarrow \mathcal{P}(\mathfrak{g}, \delta_{r'}) : (Y, \pi_Y, \lambda) \longmapsto (Y, \pi_Y + \lambda(t), \lambda)$$

is the equivalence of categories given in Lemma 2.16.

Let $n \geq 2$ be any integer and let $\phi_s^n = (\phi_s, \phi_s, \dots, \phi_s) \in \wedge^3(\mathfrak{g}^n)$. Applying Lemma 6.13 to the quasitriangular r -matrix $r^{(n)}$ on \mathfrak{g}^n defined in §6.1, one has the equivalence of categories

$$(6.19) \quad \mathcal{P}(\mathfrak{g}^n, \delta_{r^{(n)}}) \longleftrightarrow \mathcal{QP}(\mathfrak{g}^n, \phi_s^n),$$

$$(6.20) \quad (Y, \pi_Y, \lambda) \longleftrightarrow (Y, Q_Y, \lambda), \quad \text{where } Q_Y = \pi_Y + \lambda(\Lambda, \dots, \Lambda) - \lambda(\text{Mix}^n(r)).$$

By Theorem 6.2, one has the restriction functor

$$(6.21) \quad \mathcal{P}(\mathfrak{g}^n, \delta_{r^{(n)}}) \xrightarrow{\text{Res}} \mathcal{P}(\mathfrak{g}, \delta_r), \quad (Y, \pi_Y, \lambda) \longmapsto (Y, \pi_Y, \lambda \circ (\text{diag})_n),$$

where $(\text{diag})_n : \mathfrak{g} \longrightarrow \mathfrak{g}^n, (\text{diag})_n(x) = (x, x, \dots, x)$ for $x \in \mathfrak{g}$. On the other hand, by [1, 2] (see also [28]), one has the fusion functor

$$(6.22) \quad \mathcal{QP}(\mathfrak{g}^n, \phi_s^n) \xrightarrow{\text{Fus}} \mathcal{QP}(\mathfrak{g}, \phi_s), \quad (Y, Q_Y, \lambda) \longmapsto (Y, Q_Y + \lambda(\text{Mix}^n(s)), \lambda \circ (\text{diag})_n).$$

Proposition 6.17. *One has the commutative diagram*

$$\begin{array}{ccc} \mathcal{P}(\mathfrak{g}^n, \delta_{r^{(n)}}) & \longleftrightarrow & \mathcal{QP}(\mathfrak{g}^n, \phi_s^n) \\ \text{Res} \downarrow & & \downarrow \text{Fus} \\ \mathcal{P}(\mathfrak{g}, \delta_r) & \longleftrightarrow & \mathcal{QP}(\mathfrak{g}, \phi_s), \end{array}$$

where the functors represented by the top and bottom horizontal arrows are respectively given in (6.19) and (6.18).

Proof. Using the definitions of the functors, it is enough to show that

$$(6.23) \quad (\text{diag})_n(\Lambda) = (\Lambda, \dots, \Lambda) - \text{Mix}^n(\Lambda),$$

which is straightforward to check.

Q.E.D.

Corollary 6.18. *Under the equivalence in (6.18), the fusion product in $\mathcal{P}(\mathfrak{g}, \delta_r)$ defined in Definition 6.9 corresponds to the fusion product in $\mathcal{QP}(\mathfrak{g}, \phi_s)$.*

Remark 6.19. For each permutation $\tau \in S_n$, the element

$$\phi_\tau(\text{Mix}^n(r)) = \text{Mix}^n(\Lambda) + \phi_\tau(\text{Mix}^n(s)) \in \wedge^2(\mathfrak{g}^n)$$

is also a mixed twisting element of the n -fold direct product of the Lie bialgebra (\mathfrak{g}, δ_r) with itself. Given (\mathfrak{g}, δ_r) -Poisson spaces (Y_j, π_j, λ_j) for $1 \leq j \leq n$ with $n \geq 2$, we then have the left (\mathfrak{g}, δ_r) -Poisson space $(Y, \pi_Y^\tau, \lambda_{\text{diag}})$, called the τ -fusion product, of $\{(Y_j, \pi_j, \lambda_j)\}_j$, where π_Y^τ is now the mixed product Poisson structure on $Y = Y_1 \times \dots \times Y_n$ defined by

$$\pi_Y^\tau = (\pi_1, \dots, \pi_n) + \lambda(\phi_\tau(\text{Mix}^n(r))).$$

Replacing the fusion functor Fus in (6.22) by

$$(6.24) \quad \mathcal{QP}(\mathfrak{g}^n, \phi_s^n) \xrightarrow{\text{Fus}^\tau} \mathcal{QP}(\mathfrak{g}, \phi_s), \quad (Y, Q_Y, \lambda) \longmapsto (Y, Q_Y + \lambda(\phi_\tau(\text{Mix}^n(s))), \lambda \circ (\text{diag})_n),$$

it follows from (6.23) again that under the functor in Lemma 6.13, τ -fusion products of (\mathfrak{g}, δ_r) -Poisson spaces correspond to τ -fusion products of (\mathfrak{g}, ϕ_s) -quasi-Poisson spaces. See [28] for examples of τ -fusion products of (\mathfrak{g}, ϕ_s) -quasi-Poisson spaces for non-trivial $\tau \in S_n$. \diamond

7. QUOTIENT POISSON STRUCTURES AS MIXED PRODUCTS

In §7, we present examples that have motivated our introduction to the notion of mixed product Poisson structures. Namely, we identify a class of quotient Poisson manifolds of product Poisson Lie groups as mixed product Poisson manifolds (see Definition 1.1).

7.1. Quotient Poisson structures. Let (G, π_G) be a Poisson Lie group. Recall that a *coisotropic subgroup* of (G, π_G) is, by definition, a Lie subgroup Q of G which is also a coisotropic submanifold with respect to π_G , i.e., $\pi_G(q) \in T_q Q \otimes T_q G + T_q G \otimes T_q Q$ for every $q \in Q$.

Lemma 7.1. [38] *If (Y, π_Y, ρ) is a right (G, π_G) -Poisson space and Q a closed coisotropic subgroup of (G, π_G) such that Y/Q is a smooth manifold, then the Poisson structure π_Y projects to a well-defined Poisson structure on Y/Q .*

We will refer to the projection of π_Y to Y/Q as the *quotient Poisson structure* on Y/Q . As an example, assume that (Y, π_Y) is a Poisson manifold with a left Poisson action of a closed Poisson subgroup $(Q, \pi_Q = \pi_G|_Q)$ of (G, π_G) . Then one has the right Poisson action of the product Poisson Lie group $(Q \times Q, \pi_Q \times (-\pi_Q))$ on the product Poisson manifold $(G \times Y, \pi_G \times \pi_Y)$ by

$$(g, y) \cdot (q_1, q_2) = (gq_1, q_2^{-1}y), \quad g \in G, y \in Y, q_1, q_2 \in Q.$$

As $Q_{\text{diag}} = \{(q, q) : q \in Q\}$ is a coisotropic subgroup of $(Q \times Q, \pi_Q \times (-\pi_Q))$, the Poisson structure $\pi_G \times \pi_Y$ on $G \times Y$ projects to a well-defined Poisson structure on the quotient of $G \times Y$ by Q_{diag} , denoted by $G \times_Q Y$. Repeating the construction, one has the following Lemma-Definition 7.2.

Lemma-Definition 7.2. *If Q_1, \dots, Q_n are closed Poisson subgroups of (G, π_G) , the quotient manifold*

$$Z = G \times_{Q_1} \cdots \times_{Q_{n-1}} G/Q_n$$

in (1.2) has the well-defined Poisson structure $\pi_Z = \varpi_Z(\pi_G^n)$, where $\varpi_Z : G^n \rightarrow Z$ is the projection. We will refer to (Z, π_Z) as a quotient Poisson manifold of the product Poisson Lie group (G^n, π_G^n) .

We now look at the case when $n = 1$, leaving the case of an arbitrary n to §7.2. Assume thus Q is a closed Poisson Lie subgroup of (G, π_G) and consider the Poisson structure

$$\pi_{G/Q} := \varpi_{G/Q}(\pi_G)$$

on G/Q , where $\varpi_{G/Q} : G \rightarrow G/Q$ is the projection. Let $(\mathfrak{g}, \delta_{\mathfrak{g}})$ be the Lie bialgebra of (G, π_G) , and let $(\mathfrak{g}^*, \delta_{\mathfrak{g}^*})$ and $(\mathfrak{d}, \delta_{\mathfrak{d}})$ be respectively the dual and double Lie bialgebras of $(\mathfrak{g}, \delta_{\mathfrak{g}})$. Recall from (2.29) the left Poisson action ς of $(\mathfrak{d}, \delta_{\mathfrak{d}})$ on (G, π_G) , i.e.,

$$\varsigma : \mathfrak{d} \longrightarrow \mathcal{V}^1(G), \quad \varsigma(x + \xi) = x^R - \pi_G^\#(\xi^R), \quad x \in \mathfrak{g}, \xi \in \mathfrak{g}^*.$$

As Q is a Poisson subgroup, it follows from (2.28) that

$$(7.1) \quad \varrho_{G/Q} : \mathfrak{g}^* \longrightarrow \mathcal{V}^1(G/Q), \quad \varrho_{G/Q}(\xi) \stackrel{\text{def}}{=} \varpi_{G/Q}(\pi_G^\#(\xi^R)), \quad \xi \in \mathfrak{g}^*,$$

is a well-defined Poisson action, called the *right dressing action*, of $(\mathfrak{g}^*, \delta_{\mathfrak{g}^*})$ on $(G/Q, \pi_{G/Q})$. As

$$(7.2) \quad \lambda_{G/Q} : \mathfrak{g} \longrightarrow \mathcal{V}^1(G/Q), \quad \lambda_{G/Q}(x) \stackrel{\text{def}}{=} \varpi_{G/Q}(x^R), \quad x \in \mathfrak{g},$$

is also well-defined, one has the well-defined left Lie algebra action

$$(7.3) \quad \varsigma_{G/Q} = \varpi_{G/Q} \circ \varsigma : \mathfrak{d} \longrightarrow \mathcal{V}^1(G/Q), \quad \varsigma_{G/Q}(x + \xi) = \varpi_{G/Q}(x^R - \pi_G^\#(\xi^R)), \quad x \in \mathfrak{g}, \xi \in \mathfrak{g}^*,$$

of \mathfrak{d} on G/Q , making $(G/Q, \pi_{G/Q}, \varsigma_{G/Q})$ into a left $(\mathfrak{d}, \delta_{\mathfrak{d}})$ -Poisson space. Recall from (2.12) the quasitriangular r -matrix $r_{\mathfrak{d}}$ on \mathfrak{d} associated to the Lagrangian splitting $\mathfrak{d} = \mathfrak{g} + \mathfrak{g}^*$.

Lemma 7.3. *For any closed Poisson Lie subgroup Q of (G, π_G) , one has $\pi_{G/Q} = -\varsigma_{G/Q}(r_{\mathfrak{d}})$.*

Proof. Let $\{x_i\}_{i=1}^m$ be any basis of \mathfrak{g} and $\{\xi_i\}_{i=1}^m$ the dual basis of \mathfrak{g}^* . By the definition of the bundle map $\pi_G^\# : T^*G \rightarrow TG$, one has

$$\pi_G = \sum_{i=1}^m x_i^R \otimes \pi_G^\#(\xi_i^R) = - \sum_{i=1}^m \varsigma(x_i) \otimes \varsigma(\xi_i) = -\varsigma(r_{\mathfrak{d}}).$$

Composing with $\varpi_{G/Q} : G \rightarrow G/Q$ gives

$$(7.4) \quad \pi_{G/Q} = \sum_{i=1}^m \varpi_{G/Q}(x_i^R) \otimes \varpi_{G/Q}(\pi_G^\#(\xi_i^R)) = -\varsigma_{G/Q}(r_{\mathfrak{d}}).$$

Q.E.D.

For the remainder of §7.1, let (\mathfrak{g}, r) be quasitriangular Lie bialgebra, and let G be any connected Lie group with Lie algebra \mathfrak{g} . One then has the Poisson Lie group (G, π_G) , where $\pi_G = r^L - r^R$. We now consider some particular examples of Poisson Lie subgroups of (G, π_G) . We first compute the right dressing action ϱ of $(\mathfrak{g}^*, \delta_{\mathfrak{g}^*})$ on (G, π_G) . Recall from §2.3 that $r_+ = r^\#$ and $r_- = -(r^{21})^\# = -r_+^*$ are Lie bialgebra homomorphisms from $(\mathfrak{g}^*, -\delta_{\mathfrak{g}^*})$ to (\mathfrak{g}, δ_r) . For $g \in G$, let L_g and R_g denote again the respective left and right translations on G by g .

Lemma 7.4. *For $\xi \in \mathfrak{g}^*$ and $g \in G$, one has*

$$(7.5) \quad \pi_G^\#(\xi^R)(g) = L_g r_+(\text{Ad}_g^* \xi) - R_g r_+(\xi) = L_g r_-(\text{Ad}_g^* \xi) - R_g r_-(\xi).$$

Proof. Writing $r = \sum_j x_j \otimes y_j$ where $x_j, y_j \in \mathfrak{g}$, one has

$$\begin{aligned} \pi_G^\#(\xi^R)(g) &= L_g \left(\sum_j \langle R_{g^{-1}}^* \xi, L_g x_j \rangle y_j \right) - R_g \left(\sum_j \langle R_{g^{-1}}^* \xi, R_g x_j \rangle y_j \right) \\ &= L_g r_+(\text{Ad}_g^* \xi) - R_g r_+(\xi). \end{aligned}$$

The second identity in (7.5) is proved using $\pi_G(g) = L_g(-r^{21}) - R_g(-r^{21})$.

Q.E.D.

Consider again (see §2.3) the sub-Lie bialgebras $(\mathfrak{f}_-, \delta_r|_{\mathfrak{f}_-})$ and $(\mathfrak{f}_+, \delta_r|_{\mathfrak{f}_+})$ of (\mathfrak{g}, δ_r) , where $\mathfrak{f}_\pm = \text{Im}(r_\pm)$.

Proposition 7.5. *Let Q be a Lie subgroup of G with Lie algebra $\mathfrak{q} \subset \mathfrak{g}$ such that $\mathfrak{f}_+ \subset \mathfrak{q}$ or $\mathfrak{f}_- \subset \mathfrak{q}$. Then Q is a Poisson Lie subgroup of (G, π_G) . Moreover, if $Q \subset G$ is closed, then*

$$(7.6) \quad \varrho_{G/Q}(\xi) = -\lambda_{G/Q}(r_+(\xi)), \quad \xi \in \mathfrak{g}^*, \quad \text{if } \mathfrak{f}_+ \subset \mathfrak{q};$$

$$(7.7) \quad \varrho_{G/Q}(\xi) = -\lambda_{G/Q}(r_-(\xi)), \quad \xi \in \mathfrak{g}^*, \quad \text{if } \mathfrak{f}_- \subset \mathfrak{q}.$$

In both cases, one has $\pi_{G/Q} = -\lambda_{G/Q}(r)$.

Proof. As $\mathfrak{q} \supset \mathfrak{f}_+$ or $\mathfrak{q} \supset \mathfrak{f}_-$, it follows from Lemma 7.4 that $\pi_G^\#(\xi^R)(g) \in T_g Q$ for every $\xi \in \mathfrak{g}^*$ and $g \in Q$. Thus Q is a Poisson Lie subgroup of (G, π_G) .

Assume that Q is closed in G . Then (7.6) and (7.7) follow from (7.5). Assume that $\mathfrak{f}_+ \subset \mathfrak{q}$. Let $\{x_i\}_{i=1}^m$ be basis of \mathfrak{g} and $\{\xi_i\}_{i=1}^m$ the dual basis of \mathfrak{g}^* such that $\{x_i\}_{i=1}^k$ is a basis of \mathfrak{f}_- . Then $\text{Span}\{\xi_{k+1}, \dots, \xi_m\} = \mathfrak{f}_-^0 = \ker r_+$. By (7.4) and (7.6),

$$\pi_{G/Q} = \sum_{i=1}^k \lambda_{G/Q}(x_i) \otimes \varrho_{G/Q}(\xi_i) = -\lambda_{G/Q} \left(\sum_{i=1}^k x_i \otimes r_+(\xi_i) \right) = -\lambda_{G/Q}(r).$$

If $\mathfrak{f}_- \subset \mathfrak{q}$, one has $\pi_{G/Q} = -\lambda_{G/Q}(-r^{21}) = -\lambda_{G/Q}(r)$.

Q.E.D.

7.2. Quotient Poisson structures as mixed products. In §7.2, fix a connected Poisson Lie group (G, π_G) and let $(\mathfrak{g}, \delta_{\mathfrak{g}})$ be its Lie bialgebra. Let $(\mathfrak{g}^*, \delta_{\mathfrak{g}^*})$ and $(\mathfrak{d}, \delta_{\mathfrak{d}})$ be respectively the dual and the double Lie bialgebra of $(\mathfrak{g}, \delta_{\mathfrak{g}})$. Recall the right dressing action ϱ of $(\mathfrak{g}^*, \delta_{\mathfrak{g}^*})$ on (G, π_G) given in (2.27).

Let (Y, π_Y, λ) be a left (G, π_G) -Poisson space. Let J be the diffeomorphism

$$J : G \times Y \longrightarrow G \times Y, \quad (g, y) \longmapsto (g, gy), \quad g \in G, y \in Y.$$

Recall from §4.1 that one has the mixed product Poisson structure $\pi_G \times_{(\varrho, \lambda)} \pi_Y$ on $G \times Y$.

Lemma 7.6. *One has $J(\pi_G \times \pi_Y) = \pi_G \times_{(\varrho, \lambda)} \pi_Y$.*

Proof. Let $\pi = J(\pi_G \times \pi_Y)$. With $\kappa_G : G \times Y \rightarrow G$ and $\kappa_Y : G \times Y \rightarrow Y$ denoting the two projections, it is clear that $\kappa_G : (G \times Y, \pi) \rightarrow (G, \pi_G)$ and $\kappa_Y : (G \times Y, \pi) \rightarrow (Y, \pi_Y)$ are Poisson maps. Let μ be the mixed term of $\pi_G \times_{(\varrho, \lambda)} \pi_Y$ and let $g \in G, y \in Y$. It remains to show that

$$\pi(g, y)(\kappa_G^* \theta_g, \kappa_Y^* \zeta_y) = \mu(g, y)(\kappa_G^* \theta_g, \kappa_Y^* \zeta_y), \quad \theta_g \in T_g^* G, \zeta_y \in T_y^* Y.$$

Let $y_1 = g^{-1}y \in Y$, and let $\lambda_{y_1} : G \rightarrow Y, \lambda_{y_1}(h) = hy_1$ for $h \in G$. Let $\iota_g : Y \rightarrow G \times Y$ and $\iota_{y_1} : G \rightarrow G \times Y$ be respectively given by $\iota_g(y') = (g, y')$ and $\iota_{y_1}(g') = (g', y_1)$. Then

$$\begin{aligned} \pi(g, y)(\kappa_G^* \theta_g, \kappa_Y^* \zeta_y) &= \pi_G(g)(\iota_{y_1}^* J^* \kappa_G^* \theta_g, \iota_{y_1}^* J^* \kappa_Y^* \zeta_y) + \pi_Y(y_1)(\iota_g^* J^* \kappa_G^* \theta_g, \iota_g^* J^* \kappa_Y^* \zeta_y) \\ &= \pi_G(g)(\theta_g, \lambda_{y_1}^* \zeta_y) - \langle \theta_g, \varrho(\pi_g^* \lambda_{y_1}^* \zeta_y) \rangle = \mu(g, y)(\kappa_G^* \theta_g, \kappa_Y^* \zeta_y). \end{aligned}$$

This proves that $\pi = \pi_G \times_{(\varrho, \lambda)} \pi_Y$.

Q.E.D.

Assume now that Q is a closed Poisson Lie subgroup of (G, π_G) , and consider the quotient manifold $G \times_Q Y$ with the quotient Poisson structure $\pi = \varpi(\pi_G \times \pi_Y)$, where $\varpi : G \times Y \rightarrow G \times_Q Y$ is the projection. Let J_Q be the diffeomorphism

$$J_Q : G \times_Q Y \longrightarrow (G/Q) \times Y, \quad J_Q(\varpi(g, y)) = (gQ, gy), \quad g \in G, y \in Y.$$

Recall the right Poisson action $\varrho_{G/Q}$ of the Lie bialgebra $(\mathfrak{g}^*, \delta_{\mathfrak{g}^*})$ on $(G/Q, \pi_{G/Q})$ given in (7.1).

Lemma 7.7. *As Poisson structures on $(G/Q) \times Y$, one has*

$$(7.8) \quad J_Q(\pi) = \pi_{G/Q} \times_{(\varrho_{G/Q}, \lambda)} \pi_Y.$$

Proof. Lemma 7.7 follows from Lemma 7.6 and the commutative diagram

$$\begin{array}{ccc} G \times Y & \xrightarrow{J} & G \times Y \\ \varpi \downarrow & & \downarrow \varpi_{G/Q} \times \text{Id}_Y \\ G \times_Q Y & \xrightarrow{J_Q} & G/Q \times Y. \end{array}$$

Q.E.D.

Let now Q_1, \dots, Q_n be closed Poisson Lie subgroups of (G, π_G) and consider the quotient

$$Z = G \times_{Q_1} G \times_{Q_2} \cdots \times_{Q_{n-1}} G/Q_n$$

defined in (1.2) with the quotient Poisson structure $\pi_Z = \varpi_Z(\pi_G^n)$, where $\varpi_Z : G^n \rightarrow Z$ is the projection. Set $[g_1, \dots, g_n]_Z = \varpi_Z(g_1, \dots, g_n) \in Z$ for $(g_1, \dots, g_n) \in G^n$ and recall the diffeomorphism $J_Z : Z \rightarrow G/Q_1 \times \cdots \times G/Q_n$ given in (1.3), i.e.,

$$J_Z[g_1, g_2, \dots, g_n]_Z = (g_1 Q_1, g_1 g_2 Q_2, \dots, g_1 g_2 \cdots g_n Q_n), \quad g_j \in G.$$

By Lemma 7.3, for each $1 \leq j \leq n$, one has the $(\mathfrak{d}, \delta_{\mathfrak{d}})$ -Poisson space $(G/Q_j, \pi_{G/Q_j}, \varsigma_j)$, where $\varsigma_j = \varsigma_{G/Q_j}$ (see (7.3)), and $\pi_{G/Q_j} = -\varsigma_j(r_{\mathfrak{d}})$, where $r_{\mathfrak{d}} \in \mathfrak{d} \otimes \mathfrak{d}$ is the quasitriangular r -matrix on \mathfrak{d} defined by the Lagrangian splitting $\mathfrak{d} = \mathfrak{g} + \mathfrak{g}^*$. Consider the direct product left Lie algebra action $\sigma = (\varsigma_1, \dots, \varsigma_n)$ of \mathfrak{d}^n on the product manifold $G/Q_1 \times \cdots \times G/Q_n$.

Theorem 7.8. *One has (see notation in Definition 6.9)*

$$(7.9) \quad J_Z(\pi_Z) = (\pi_{G/Q_1} \times \cdots \times \pi_{G/Q_n})_{(\varsigma_1, \dots, \varsigma_n)} = -\sigma \left(r_{\mathfrak{d}}^{(n)} \right).$$

Proof. When $n = 1$, (7.9) holds by Lemma 7.3. Assume that $n > 1$ and that (7.9) holds for $n - 1$. By taking $Q = Q_1$ and $Y = G \times_{Q_2} \cdots \times_{Q_{n-1}} G/Q_n$ in Lemma 7.7 and by noting that $\text{Im}((r_{\mathfrak{d}})_+) = \mathfrak{g}^* \subset \mathfrak{d}$ and $\text{Im}((r_{\mathfrak{d}})_-) = \mathfrak{g} \subset \mathfrak{d}$, (7.9) holds by Lemma 7.7 and Remark 6.10.

Q.E.D.

Remark 7.9. (Reduction to sub-Lie bialgebras, II.) For a Poisson Lie group (G, π_G) with Lie bialgebra $(\mathfrak{g}, \delta_{\mathfrak{g}})$, a pair $((\mathfrak{p}, \delta_{\mathfrak{p}}), Q)$, where $(\mathfrak{p}, \delta_{\mathfrak{p}} = \delta_{\mathfrak{g}}|_{\mathfrak{p}})$ is a sub-Lie bialgebra of $(\mathfrak{g}, \delta_{\mathfrak{g}})$ and Q is a closed Poisson Lie subgroup of (G, π_G) , is said to be *special* if $\varrho_{G/Q}(\xi) = 0$ for all $\xi \in \mathfrak{p}^0$, where $\varrho_{G/Q}$ is given in (7.1). In this case, the left Poisson action $\varsigma_{G/Q}$ of $(\mathfrak{d}, \delta_{\mathfrak{d}})$ on $(G/Q, \pi_{G/Q})$ induces a left action, denoted by $\varsigma_{\mathfrak{p}}$, of the double Lie bialgebra of $(\mathfrak{p}, \delta_{\mathfrak{p}})$ on $(G/Q, \pi_{G/Q})$ by $\varsigma_{\mathfrak{p}}(x + \xi + \mathfrak{p}^0) = \varsigma_{G/Q}(x + \xi)$, $x \in \mathfrak{p}, \xi \in \mathfrak{g}^*$ (see Example 2.2 and Remark 4.4), and in the setting of Theorem 7.8 with $Q_j = Q$ for each j , one checks directly that the Poisson structure $J_Z(\pi_Z)$ on $G/Q \times \cdots \times G/Q$ is also given by

$$J_Z(\pi_Z) = (\pi_{G/Q} \times \cdots \times \pi_{G/Q})_{(\varsigma_{\mathfrak{p}}, \dots, \varsigma_{\mathfrak{p}})} = -\sigma_{\mathfrak{p}} \left(r_{\mathfrak{d}_{\mathfrak{p}}}^{(n)} \right),$$

where $\sigma_{\mathfrak{p}} = (\varsigma_{\mathfrak{p}}, \dots, \varsigma_{\mathfrak{p}})$ is the direct product action of $\mathfrak{d}_{\mathfrak{p}}$ on Z and $r_{\mathfrak{d}_{\mathfrak{p}}}$ is the quasitriangular r -matrix on $\mathfrak{d}_{\mathfrak{p}}$ associated to the Lagrangian splitting $\mathfrak{d}_{\mathfrak{p}} = \mathfrak{p} + \mathfrak{p}^*$. \diamond

8. PRIMARY EXAMPLES

In §8, we identify a specific class of mixed Poisson structures, which are the main motivation for the work in the present paper. Examples include those defined on products of flag varieties and related spaces that are mentioned in the Introduction.

8.1. The set-up. Throughout §8, we fix a quasitriangular Lie bialgebra (\mathfrak{g}, r) , and let G be any connected Poisson Lie group with Lie algebra \mathfrak{g} . One then has the Poisson Lie groups (G, π_G) and $(G \times G, \pi_G^{(2)})$, where $\pi_G = r^L - r^R$, and $\pi_G^{(2)} = (r^{(2)})^L - (r^{(2)})^R$ (see §6.2).

Recall the two Lie subalgebras $\mathfrak{f}_{\pm} = \text{Im}(r_{\pm})$ of \mathfrak{g} . Let (Q_+, Q_-) be a pair of connected and closed Lie subgroups of G such that their respective Lie algebras \mathfrak{q}_+ and \mathfrak{q}_- satisfy

$$(8.1) \quad \mathfrak{f}_+ \subset \mathfrak{q}_+ \quad \text{and} \quad \mathfrak{f}_- \subset \mathfrak{q}_-.$$

By Proposition 7.5, both Q_+ and Q_- are Poisson Lie subgroups of (G, π_G) . By (6.8), $\mathfrak{q}_+ \oplus \mathfrak{q}_- \supset \text{Im}((r^{(2)})_+)$, so $Q_+ \times Q_-$ is a Poisson Lie subgroup of $(G \times G, \pi_G^{(2)})$ again by Proposition 7.5. One thus has the following six series of quotient Poisson manifolds (see Lemma-Definition 7.2)

$$(8.2) \quad (X_n, \pi_{X_n}), \quad (X_{-n}, \pi_{X_{-n}}), \quad (Y_n, \pi_{Y_n}), \quad (Y_{-n}, \pi_{Y_{-n}}), \quad (\mathbb{X}_n, \pi_{\mathbb{X}_n}), \quad (\mathbb{Y}_n, \pi_{\mathbb{Y}_n})$$

of the product Poisson Lie groups (G^n, π_G^n) and $((G \times G)^n, (\pi_G^{(2)})^n)$, where $n \geq 2$ is any integer, and

$$(8.3) \quad X_n = G \times_{Q_+} \cdots \times_{Q_+} G, \quad X_{-n} = G \times_{Q_-} \cdots \times_{Q_-} G,$$

$$(8.4) \quad Y_n = G \times_{Q_+} \cdots \times_{Q_+} G/Q_+, \quad Y_{-n} = G \times_{Q_-} \cdots \times_{Q_-} G/Q_-,$$

$$(8.5) \quad \mathbb{X}_n = (G \times G) \times_{(Q_+ \times Q_-)} \cdots \times_{(Q_+ \times Q_-)} (G \times G),$$

$$(8.6) \quad \mathbb{Y}_n = (G \times G) \times_{(Q_+ \times Q_-)} \cdots \times_{(Q_+ \times Q_-)} (G \times G)/(Q_+ \times Q_-).$$

Theorem 7.8 applies to any (Z, π_Z) in the series in (8.2) and identifies the Poisson structures π_Z as a mixed product Poisson structure defined by a quasitriangular r -matrix. We show in §8 that the assumptions on \mathfrak{q}_+ and \mathfrak{q}_- in (8.1) allow one to “reduce” the quasitriangular r -matrix in Theorem 7.8 in the same spirit as the reduction of the r -matrix in Lemma 7.3 to that in Proposition 7.5 for the case of $n = 1$.

8.2. The Poisson manifolds $(X_n, \pi_{X_n}), (X_{-n}, \pi_{X_{-n}})$. Let $n \geq 1$ and consider first the Poisson manifold (X_n, π_{X_n}) . Define the direct sum quasitriangular r -matrix $r^{\langle n+1 \rangle}$ on $\mathfrak{g}^{n+1} = \mathfrak{g}^n \oplus \mathfrak{g}$ by

$$(8.7) \quad r^{\langle n+1 \rangle} = \begin{cases} (r^{(n)}, 0) + (0, -r), & \text{if } n = 2m + 1 \text{ is odd,} \\ (r^{(n)}, 0) + (0, r^{21}), & \text{if } n = 2m \text{ is even,} \end{cases}$$

and let λ be the left Lie algebra action of \mathfrak{g}^{n+1} on the product manifold $(G/Q_+)^{n-1} \times G$ by

$$\lambda(x_1, \dots, x_n, x_{n+1}) = (\lambda_{G/Q_+}(x_1), \dots, \lambda_{G/Q_+}(x_n), x_n^R - x_{n+1}^L), \quad x_j \in \mathfrak{g}.$$

Let $J_{X_n} : X_n \rightarrow (G/Q_+)^{n-1} \times G$ be the diffeomorphism given by

$$J_{X_n}([g_1, g_2, \dots, g_n]_{X_n}) = (g_1 Q_+, g_1 g_2 Q_+, \dots, g_1 g_2 \cdots g_{n-1} Q_+, g_1 g_2 \cdots g_n), \quad g_j \in G.$$

Proposition 8.1. *As Poisson structures on $(G/Q_+)^{n-1} \times G$, one has $J_{X_n}(\pi_{X_n}) = -\lambda(r^{\langle n+1 \rangle})$.*

Proof. We apply Theorem 7.8 by taking $Q_1 = \cdots = Q_{n-1} = Q_+$ and $Q_n = \{e\}$. Let the notation be as in Theorem 7.8. Let $\{x_i\}_{i=1}^m$ be a basis of \mathfrak{g} such that $\{x_i\}_{i=1}^k$ is a basis of \mathfrak{f}_- , and let $\{\xi_i\}_{i=1}^m$ be the dual basis of \mathfrak{g}^* . Then $r_{\mathfrak{d}} = \sum_{i=1}^m x_i \otimes \xi_i$. By Theorem 7.8,

$$J_{X_n}(\pi_{X_n}) = (\pi_{G/Q_+}, \dots, \pi_{G/Q_+}, \pi_G) + \sigma(\text{Mix}^n(r_{\mathfrak{d}})).$$

Let $1 \leq j < k \leq n$ and consider (see (6.3))

$$\mu_{jk} = \sigma\left((\text{Mix}^n(r_{\mathfrak{d}}))_{jk}\right) = \sum_{i=1}^m (0, \dots, \underset{j^{\text{th}} \text{ entry}}{\sigma_{G/Q_+}(\xi_i)}, \dots, 0) \wedge (0, \dots, \underset{k^{\text{th}} \text{ entry}}{\sigma_{G/Q_+}(x_i)}, \dots, 0).$$

Using the fact that $\mathfrak{f}_+ \subset \mathfrak{q}_+$ and by Proposition 7.5, one has

$$\mu_{jk} = \sum_{i=1}^k (0, \dots, \underset{j^{\text{th}} \text{ entry}}{\lambda_{G/Q_+}(r_+(\xi_i))}, \dots, 0) \wedge (0, \dots, \underset{k^{\text{th}} \text{ entry}}{\lambda_{G/Q_+}(x_i)}, \dots, 0) = \lambda\left((\text{Mix}^n(r))_{jk}\right).$$

Thus

$$J_{X_n}(\pi_{X_n}) = (\pi_{G/Q_+}, \dots, \pi_{G/Q_+}, \pi_G) + \lambda((\text{Mix}^n(r), 0)).$$

On the other hand, again by Proposition 7.5, one has

$$-\lambda\left((r^{(n)}, 0)\right) = \begin{cases} (\pi_{G/Q_+}, \dots, \pi_{G/Q_+}, -r^R) + \lambda((\text{Mix}^n(r), 0)), & \text{if } n \text{ is odd,} \\ (\pi_{G/Q_+}, \dots, \pi_{G/Q_+}, (r^{21})^R) + \lambda((\text{Mix}^n(r), 0)), & \text{if } n \text{ is even,} \end{cases}$$

Let $r' = -r$ if n is odd and $r' = r^{21}$ if n is even. Then $-\lambda(0, r') = r^L$ if n is odd and $-\lambda(0, r') = -(r^{21})^L$ if n is even. Since $\pi_G = r^L - r^R = (r^{21})^R - (r^{21})^L$, one has

$$-\lambda\left(r^{\langle n+1 \rangle}\right) = -\lambda\left(\left(r^{(n)}, 0\right)\right) - \lambda(0, r') = J_{X_n}(\pi_{X_n}).$$

Q.E.D.

Replacing r by $-r^{21}$, we have

$$J_{X_{-n}}(\pi_{X_{-n}}) = -\lambda\left((-r^{21})^{\langle n+1 \rangle}\right),$$

where $J_{X_{-n}}$ is the diffeomorphism $X_{-n} \rightarrow (G/Q_-)^{n-1} \times G$ given by

$$J_{X_{-n}}([g_1, g_2, \dots, g_n]_{X_{-n}}) = (g_1 Q_-, g_1 g_2 Q_-, \dots, g_1 g_2 \cdots g_{n-1} Q_-, g_1 g_2 \cdots g_n), \quad g_j \in G.$$

8.3. The Poisson manifolds (Y_n, π_{Y_n}) and $(Y_{-n}, \pi_{Y_{-n}})$. Let $n \geq 1$, and let $J_{Y_n} : Y_n \rightarrow (G/Q_+)^n$ and $J_{Y_{-n}} : Y_{-n} \rightarrow (G/Q_-)^n$ be the diffeomorphisms, respectively given by

$$\begin{aligned} J_{Y_n}([g_1, g_2, \dots, g_n]_{Y_n}) &= (g_1 Q_+, g_1 g_2 Q_+, \dots, g_1 g_2 \cdots g_n Q_+), \quad g_j \in G, \\ J_{Y_{-n}}([g_1, g_2, \dots, g_n]_{Y_{-n}}) &= (g_1 Q_-, g_1 g_2 Q_-, \dots, g_1 g_2 \cdots g_n Q_-), \quad g_j \in G. \end{aligned}$$

Using the surjective Poisson morphisms

$$(8.8) \quad (X_n, \pi_{X_n}) \longrightarrow (Y_n, \pi_{Y_n}), \quad [g_1, g_2, \dots, g_n]_{X_n} \longmapsto [g_1, g_2, \dots, g_n]_{Y_n}, \quad g_j \in G,$$

$$(8.9) \quad (X_{-n}, \pi_{X_{-n}}) \longrightarrow (Y_{-n}, \pi_{Y_{-n}}), \quad [g_1, g_2, \dots, g_n]_{X_{-n}} \longmapsto [g_1, g_2, \dots, g_n]_{Y_{-n}}, \quad g_j \in G,$$

one has the following direct consequence of Proposition 8.1.

Theorem 8.2. *As Poisson structures on $(G/Q_+)^n$ and $(G/Q_-)^n$ respectively, one has*

$$J_{Y_n}(\pi_{Y_n}) = -\lambda\left(r^{(n)}\right) \quad \text{and} \quad J_{Y_{-n}}(\pi_{Y_{-n}}) = -\lambda\left((-r^{21})^{(n)}\right),$$

where the action λ of \mathfrak{g}^n on $(G/Q_\pm)^n$ is the direct product of λ_{G/Q_\pm} of \mathfrak{g} on each factor.

8.4. The Poisson structures π_{X_n} and $\pi_{X_{-n}}$ as two-fold mixed products. Let $n \geq 1$, and consider the diffeomorphism $S_{X_n} : X_n \rightarrow X_n \times X_{-n}$ given by

$$(8.10) \quad S_{X_n}([g_1, k_2, g_2, k_2, \dots, g_n, k_n]_{X_n}) = ([g_1, g_2, \dots, g_n]_{X_n}, [k_1, k_2, \dots, k_n]_{X_{-n}}), \quad g_j, k_j \in G.$$

We now express the Poisson structure $S_{X_n}(\pi_{X_n})$ on $X_n \times X_{-n}$ as a mixed product. To this end, recall from (6.15) the pair of dual Poisson Lie groups

$$(F, \pi_F) = (F_-, \pi_G) \times (F_-^{\text{op}}, \pi_G) \quad \text{and} \quad (F^*, \pi_{F^*}) = (F_+, -\pi_G) \times (F_+, \pi_G),$$

where F_+ and F_- are the connected Lie subgroup of G with Lie algebras \mathfrak{f}_+ and \mathfrak{f}_- respectively. Consider the right Poisson action ρ_{X_n} of (F^*, π_{F^*}) on (X_n, π_{X_n}) and the left Poisson action $\lambda_{X_{-n}}$ of (F, π_F) on $(X_{-n}, \pi_{X_{-n}})$, respectively given by

$$\begin{aligned} \rho_{X_n}([g_1, g_2, \dots, g_n]_{X_n}, (f_1, f_2)) &= [f_1^{-1} g_1, g_2, \dots, g_n f_2]_{X_n}, \quad g_j \in G, f_1, f_2 \in F_+, \\ \lambda_{X_{-n}}((f_{-1}, f_{-2}), [g_1, g_2, \dots, g_n]_{X_{-n}}) &= [f_{-1} g_1, g_2, \dots, g_n f_{-2}]_{X_{-n}}, \quad g_j \in G, f_{-1}, f_{-2} \in F_-. \end{aligned}$$

Proposition 8.3. *One has $S_{X_n}(\pi_{X_n}) = \pi_{X_n} \times_{(\rho_{X_n}, \lambda_{X_{-n}})} \pi_{X_{-n}}$.*

Proof. Let S be the diffeomorphism $(G \times G)^n \longrightarrow G^n \times G^n$ given by

$$S(g_1, k_2, g_2, k_2, \dots, g_n, k_n) = ((g_1, g_2, \dots, g_n), (k_1, k_2, \dots, k_n)), \quad g_j, k_j \in G.$$

By Proposition 6.12, $S\left(\left(\pi_G^{(2)}\right)^n\right) = \pi_G^n \times_{(\rho^n, \lambda^n)} \pi_G^n$, where ρ is the right Poisson action of (F^*, π_{F^*}) on (G, π_G) given in (6.16), λ is the left Poisson action of (F, π_F) on (G, π_G) given in (6.17), and ρ^n and λ^n

denote respectively the direct product actions of the direct product Poisson Lie groups $((F^*)^n, \pi_{F^*}^n)$ and (F^n, π_F^n) on (G^n, π_G^n) . Let ϖ_n and ϖ_{-n} be respectively the projections from G^n to X_n and X_{-n} . Then

$$S_{x_n}(\pi_{x_n}) = (\varpi_n \times \varpi_{-n}) \left(S \left(\left(\pi_G^{(2)} \right)^n \right) \right) = (\varpi_n \times \varpi_{-n}) (\pi_G^n \times_{(\rho^n, \lambda^n)} \pi_G^n).$$

Consider the product Poisson Lie group $(M, \pi_M) = (F_-^{\text{op}}, \pi_G) \times (F_-, \pi_G)$ and its dual Poisson Lie group $(M^*, \pi_{M^*}) = (F_+, \pi_G) \times (F_+, -\pi_G)$, and note the direct product decompositions

$$(8.11) \quad (F^n, \pi_F^n) = (F_-, \pi_G) \times (M^{n-1}, \pi_M^{n-1}) \times (F_-^{\text{op}}, \pi_G),$$

$$(8.12) \quad ((F^*)^n, \pi_{F^*}^n) = (F_+, -\pi_G) \times ((M^*)^{n-1}, \pi_{M^*}^{n-1}) \times (F_+, \pi_G).$$

The action λ^n of (F^n, π_F^n) on (G^n, π_G^n) restricts to left Poisson actions by the Poisson subgroups

$$(M^{n-1}, \pi_M^{n-1}) \cong \{e\} \times (M^{n-1}, \pi_M^{n-1}) \times \{e\} \subset (F^n, \pi_F^n),$$

$$(F, \pi_F) \cong (F_-, \pi_G) \times \{e\} \times (F_-^{\text{op}}, \pi_G) \subset (F^n, \pi_F^n),$$

which will be respectively denoted by $\lambda^{(n-1)}$ and λ' . Similarly, ρ^n restricts to right Poisson actions $\rho^{(n-1)}$ of $((M^*)^{n-1}, \pi_{M^*}^{n-1})$ on (G^n, π_G^n) and ρ' of $(F^*, \pi_{F^*}) = (F_+, -\pi_G) \times (F_+, \pi_G)$ on (G^n, π_G^n) via (8.12). We then have $\pi_G^n \times_{(\rho^n, \lambda^n)} \pi_G^n = \pi_G^n \times_{(\rho', \lambda')} \pi_G^n + \mu$, where μ is the mixed term of the mixed product Poisson structure $\pi_G^n \times_{(\rho^{(n-1)}, \lambda^{(n-1)})} \pi_G^n$. Thus

$$S_{x_n}(\pi_{x_n}) = (\varpi_n \times \varpi_{-n}) (\pi_G^n \times_{(\rho', \lambda')} \pi_G^n) + (\varpi_n \times \varpi_{-n})(\mu) = \pi_{x_n} \times_{(\rho_{x_n}, \lambda_{x_{-n}})} \pi_{x_{-n}} + (\varpi_n \times \varpi_{-n})(\mu).$$

It remains to show that $(\varpi_n \times \varpi_{-n})(\mu) = 0$.

Consider the coisotropic subgroup $A = \{(f_-, f_-^{-1}) : f_- \in F_-\}$ of (M, π_M) and the coisotropic subgroup $B = \{(f, f) : f \in F_+\}$ of (M^*, π_{M^*}) . It is easy to see that the annihilator of the Lie algebra of A in the Lie algebra of M^* is precisely the Lie algebra of B . Thus the annihilator of the Lie algebra of A^{n-1} in the Lie algebra of $(M^*)^{n-1}$ is precisely the Lie algebra of B^{n-1} . Since $X_n = G^n/B^{n-1}$, where B^{n-1} acts on G^n by $\rho^{(n-1)}$ via $B^{n-1} \subset (M^*)^{n-1}$, and $X_{-n} = A^{n-1} \backslash G^n$, where A^{n-1} acts on G^n by $\lambda^{(n-1)}$ via $A^{n-1} \subset M^{n-1}$, it follows from Lemma 4.10 that $(\varpi_n \times \varpi_{-n})(\mu) = 0$.

Q.E.D.

Similarly, define the diffeomorphism $S_{y_n} : \mathbb{Y}_n \rightarrow Y_n \times Y_{-n}$ by

$$(8.13) \quad S_{y_n}([g_1, k_2, g_2, k_2, \dots, g_n, k_n]_{y_n}) = ([g_1, g_2, \dots, g_n]_{y_n}, [k_1, k_2, \dots, k_n]_{y_{-n}}), \quad g_j, k_j \in G.$$

Define the right Poisson action ρ_{y_n} of $(F_+, -\pi_G)$ on (Y_n, π_{y_n}) and the left Poisson action $\lambda_{y_{-n}}$ of (F_-, π_G) on $(Y_{-n}, \pi_{y_{-n}})$, respectively given by

$$\rho_{y_n}([g_1, g_2, \dots, g_n]_{y_n}, f) = [f^{-1}g_1, g_2, \dots, g_n]_{y_n}, \quad g_j \in G, f \in F_+,$$

$$\lambda_{y_{-n}}(f_-, [g_1, g_2, \dots, g_n]_{y_{-n}}) = [f_-g_1, g_2, \dots, g_n]_{y_{-n}}, \quad g_j \in G, f_- \in F_-.$$

Using again the Poisson morphisms in (8.8) - (8.9), one has the following consequence of Proposition 8.3.

Proposition 8.4. *One has $S_{y_n}(\pi_{y_n}) = \pi_{y_n} \times_{(\rho_{y_n}, \lambda_{y_{-n}})} \pi_{y_{-n}}$.*

8.5. A weak Poisson dual pair.

Definition 8.5. A *weak Poisson dual pair* is a pair of surjective Poisson submersions

$$(8.14) \quad \Phi_X : (Z, \pi_Z) \longrightarrow (X, \pi_X) \quad \text{and} \quad \Phi_Y : (Z, \pi_Z) \longrightarrow (Y, \pi_Y),$$

between Poisson manifolds such that the map

$$(\Phi_X, \Phi_Y) : (Z, \pi_Z) \longrightarrow (X \times Y, \pi_X \times \pi_Y), \quad z \longmapsto (\Phi_X(z), \Phi_Y(z)), \quad z \in Z,$$

is Poisson, where $\pi_X \times \pi_Y$ is the product Poisson structure on $X \times Y$.

Remark 8.6. When (Z, π_Z) is symplectic and when the tangent spaces to the fibers of Φ_X and Φ_Y are the symplectic orthogonals of each other, the pair (Φ_X, Φ_Y) is called a *symplectic dual pair*. Our notion of *weak Poisson dual pairs* thus generalizes that of symplectic dual pairs. \diamond

Let the setting, in particular, the pair of subgroups (Q_+, Q_-) of G , be as in §8.1. Similar to the Poisson manifold $(Y_{-n}, \pi_{Y_{-n}})$, where $n \geq 1$, introduce the left action of Q_-^n on G^n by

$$(q_{-1}, q_{-2}, \dots, q_{-n}) \cdot (g_1, g_2, \dots, g_n) = (q_{-1}g_1q_{-2}^{-1}, q_{-2}g_2q_{-3}^{-1}, \dots, q_{-n}g_n), \quad q_{-j} \in Q_-, g_j \in G,$$

and denote by

$$(8.15) \quad Y'_{-n} = Q_- \backslash G \times_{Q_-} G \times_{Q_-} \dots \times_{Q_-} G$$

the corresponding quotient of G^n by Q_-^n , and by

$$\Phi_- : G^n \longrightarrow Y'_{-n}, \quad (g_1, g_2, \dots, g_n) \longmapsto [g_1, g_2, \dots, g_n]_{Y'_{-n}}, \quad g_j \in G,$$

the corresponding quotient map. Then

$$\pi_{Y'_{-n}} := \Phi_-(\pi_G^n)$$

is a well-defined Poisson structure on Y'_{-n} . Let $\Phi_+ : G^n \rightarrow X_n$ be the projection from G^n to X_n , i.e., $\Phi_+(g_1, \dots, g_n) = [g_1, \dots, g_n]_{X_n}$ for $(g_1, \dots, g_n) \in G^n$.

Proposition 8.7. *For any $n \geq 1$, the pair*

$$\Phi_+ : (G^n, \pi_G^n) \longrightarrow (X_n, \pi_{X_n}) \quad \text{and} \quad \Phi_- : (G^n, \pi_G^n) \longrightarrow (Y'_{-n}, \pi_{Y'_{-n}})$$

is a weak Poisson dual pair.

Proof. By definition, both Φ_+ and Φ_- are surjective Poisson submersions. Denote the Poisson submersion in (8.8) by $\phi_+ : (X_n, \pi_{X_n}) \rightarrow (Y_n, \pi_{Y_n})$, and consider the Poisson submersion

$$\phi_- : (X_{-n}, \pi_{X_{-n}}) \longrightarrow (Y'_{-n}, \pi_{Y'_{-n}}), \quad [g_1, \dots, g_n]_{X_{-n}} \longmapsto [g_1, \dots, g_n]_{Y'_{-n}}, \quad g_j \in G.$$

By Proposition 8.3, one has the Poisson morphism

$$S_{\mathbb{X}_n} : (\mathbb{X}_n, \pi_{\mathbb{X}_n}) \longrightarrow (X_n \times X_{-n}, \pi_{X_n} \times_{(\rho_{X_n}, \lambda_{X_{-n}})} \pi_{X_{-n}}).$$

By the definition the the projections ϕ_+ and ϕ_- , the mixed term in the mixed product Poisson structure $\pi_{X_n} \times_{(\rho_{X_n}, \lambda_{X_{-n}})} \pi_{X_{-n}}$ on $X_n \times X_{-n}$ vanishes under $\phi_+ \times \phi_- : X_n \times X_{-n} \rightarrow Y_n \times Y'_{-n}$. Thus one has the Poisson morphism

$$(\phi_+ \times \phi_-) \circ S_{\mathbb{X}_n} : (\mathbb{X}_n, \pi_{\mathbb{X}_n}) \longrightarrow (Y_n \times Y'_{-n}, \pi_{Y_n} \times \pi_{Y'_{-n}}).$$

As $(G, \pi_G) \rightarrow (G \times G, \pi_G^{(2)})$, $g \mapsto (g, g)$ is Poisson, one has the Poisson embedding

$$I_n : (G, \pi_G^n) \longrightarrow ((G \times G)^n, (\pi_G^{(2)})^n), \quad (g_1, g_2, \dots, g_n) \longmapsto (g_1, g_1, g_2, g_2, \dots, g_n, g_n), \quad g_j \in G.$$

With $\Phi : ((G \times G)^n, (\pi_G^{(2)})^n) \rightarrow (\mathbb{X}_n, \pi_{\mathbb{X}_n})$ denoting the projection, one sees that

$$(\Phi_+, \Phi_-) = (\phi_+ \times \phi_-) \circ S_{\mathbb{X}_n} \circ \Phi \circ I_n : (G^n, \pi_G^n) \longrightarrow (Y_n \times Y'_{-n}, \pi_{Y_n} \times \pi_{Y'_{-n}})$$

is a Poisson morphism.

Q.E.D.

8.6. Holomorphic Poisson structures on flag varieties. Let G be a connected complex semi-simple Lie group with Lie algebra \mathfrak{g} . Recall that a flag variety of G is a (left) G -homogeneous space with parabolic stabilizer subalgebras. Let $\mathcal{F}_1, \dots, \mathcal{F}_n$ be any finite collection of flag varieties of G , and for each $1 \leq j \leq n$, let λ_j be the corresponding (left) Lie algebra action of \mathfrak{g} on \mathcal{F}_j .

Proposition 8.8. *For any quasitriangular r -matrix r on \mathfrak{g} , $\lambda(r^{(n)})$ is a mixed product Poisson structure on the product manifold $\mathcal{F} = \mathcal{F}_1 \times \dots \times \mathcal{F}_n$, where $\lambda = (\lambda_1, \dots, \lambda_n)$ is the direct product action of \mathfrak{g}^n on \mathcal{F} .*

Proof. Let $s \in (S^2\mathfrak{g})^{\mathfrak{g}}$ be the symmetric part of r . If $\mathfrak{g} = \mathfrak{g}_1 \oplus \dots \oplus \mathfrak{g}_k$ is the decomposition of \mathfrak{g} into simple factors, then [7, Lemma 1], $s = (s_1, \dots, s_k)$, where, $s_i \in S^2(\mathfrak{g}_i)^{\mathfrak{g}_i}$, $1 \leq i \leq k$, is a scalar multiple of the element in $S^2(\mathfrak{g}_i)^{\mathfrak{g}_i}$ corresponding to the Killing form of \mathfrak{g}_i . Thus every parabolic subalgebra of \mathfrak{g} , being a direct sum of parabolic subalgebras of the \mathfrak{g}_i 's, is coisotropic with respect to s . Proposition 8.8 now follows from Theorem 6.8 (see also Example 6.11).

Q.E.D.

We remark that all factorizable quasitriangular r -matrices on a complex simple Lie algebra have been classified by Belavin-Drinfeld in [4]. Assume again that \mathfrak{g} is semi-simple. To give more concrete examples of the Poisson structures on flag varieties and related spaces, fix a symmetric non-degenerate invariant bilinear form $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$ on \mathfrak{g} , and fix also a pair $(\mathfrak{b}, \mathfrak{b}_-)$ of opposite Borel subalgebras of \mathfrak{g} . Let $\mathfrak{h} = \mathfrak{b} \cap \mathfrak{b}_-$, a Cartan subalgebra of \mathfrak{g} . Let Δ and $\Delta_+ \subset \Delta$ be respectively the set of roots for the pair $(\mathfrak{g}, \mathfrak{h})$ and $(\mathfrak{b}, \mathfrak{h})$, and let $\mathfrak{g} = \mathfrak{h} + \sum_{\alpha \in \Delta_+} \mathfrak{g}_{\alpha} + \sum_{\alpha \in \Delta_+} \mathfrak{g}_{-\alpha}$ be the corresponding root decomposition. Let $\{h_i\}_{i=1}^r$ be a basis of \mathfrak{h} such that $\langle h_i, h_j \rangle_{\mathfrak{g}} = \delta_{ij}$ for $1 \leq i, j \leq \dim \mathfrak{h}$, and let $E_{\alpha} \in \mathfrak{g}_{\alpha}$ and $E_{-\alpha} \in \mathfrak{g}_{-\alpha}$ be root vectors for $\alpha \in \Delta_+$ such that $\langle E_{\alpha}, E_{-\alpha} \rangle_{\mathfrak{g}} = 1$. The *standard quasitriangular r -matrix* associated to the choice of the triple $(\mathfrak{b}, \mathfrak{b}_-, \langle \cdot, \cdot \rangle_{\mathfrak{g}})$ is the element $r_{\text{st}} \in \mathfrak{g} \otimes \mathfrak{g}$ given by

$$(8.16) \quad r_{\text{st}} = \frac{1}{2} \sum_{i=1}^{\dim \mathfrak{h}} h_i \otimes h_i + \sum_{\alpha \in \Delta_+} E_{-\alpha} \otimes E_{\alpha}.$$

The Poisson Lie group (G, π_{st}) , where $\pi_{\text{st}} = r_{\text{st}}^L - r_{\text{st}}^R$, is called a *standard complex semi-simple Lie group*, and it has $(G \times G, \Pi_{\text{st}})$ as a Drinfeld double Poisson Lie group, where Π_{st} is the Poisson structure on the product group $G \times G$ given by $\Pi_{\text{st}} = (r_{\text{st}}^{(2)})^L - (r_{\text{st}}^{(2)})^R$.

It is clear from the definition in (2.17) that Lie subalgebras \mathfrak{f}_- and \mathfrak{f}_+ of \mathfrak{g} associated to r_{st} are respectively given by $\mathfrak{f}_- = \mathfrak{b}_-$ and $\mathfrak{f}_+ = \mathfrak{b}$. Let B and B_- be the subgroups of G with Lie algebras \mathfrak{b} and \mathfrak{b}_- respectively. Taking $Q_+ = B$ and $Q_- = B_-$ and applying the constructions in §8.1, one arrives at the four series of complex manifolds in §1.3, respectively denoted as

$$(F_n, \pi_{F_n} = \pi_n), \quad (\mathbb{F}_n, \pi_{\mathbb{F}_n} = \Pi_n), \quad (\tilde{F}_n, \pi_{\tilde{F}_n} = \tilde{\pi}_n), \quad (\tilde{\mathbb{F}}_n, \pi_{\tilde{\mathbb{F}}_n} = \tilde{\Pi}_n).$$

Using results in §8.2 and §8.3, we can thus identify the four series of Poisson manifolds with mixed product Poisson manifolds, but more importantly, their mixed product Poisson structures are now defined by quasitriangular r -matrices. In [30], a sequel to the current paper, we first develop a general theory on torus orbits of symplectic leaves for Poisson structures defined by quasitriangular r -matrices and then apply the general theory to obtain descriptions of the T -orbits of symplectic leaves for (F_n, π_n) , (\mathbb{F}_n, Π_n) , $(\tilde{F}_n, \tilde{\pi}_n)$ and $(\tilde{\mathbb{F}}_n, \tilde{\Pi}_n)$, where $T = B \cap B_-$, a maximal torus of G . See [30] for detail.

For $n \geq 1$, introduce also the quotient manifold

$$F'_{-n} = B_- \backslash G \times_{B_-} G \times_{B_-} \dots \times_{B_-} G$$

of G^n as in (8.15) and denote by π'_{-n} projection of π^n_{st} from G^n to F'_{-n} . Let Φ_+ and Φ_- be the projections

$$\begin{aligned}\Phi_+ : G^n &\longrightarrow F_n, \quad (g_1, g_2, \dots, g_n) \longmapsto [g_1, g_2, \dots, g_n]_{F_n}, \\ \Phi_- : G^n &\longrightarrow F'_{-n}, \quad (g_1, g_2, \dots, g_n) \longmapsto [g_1, g_2, \dots, g_n]_{F'_{-n}}, \quad g_j \in G.\end{aligned}$$

The following Proposition 8.9 is a special case of Proposition 8.7 and is used in [14].

Proposition 8.9. *For any $n \geq 1$, the pair*

$$\Phi_+ : (G^n, \pi^n_{\text{st}}) \longrightarrow (F_n, \pi_n) \quad \text{and} \quad \Phi_- : (G^n, \pi^n_{\text{st}}) \longrightarrow (F'_{-n}, \pi'_{-n})$$

is a weak Poisson dual pair.

REFERENCES

- [1] A. Alekseev and Y. Kosmann-Schwarzbach, Manin pairs and moment maps, *J. Diff. Geom.*, **56**(1) (2000), 133 - 165.
- [2] A. Alekseev, Y. Kosmann-Schwarzbach, and E. Meinrenken, Quasi-Poisson Manifolds, *Canadian J. Math.*, **54** (1) (2002), 3 - 29.
- [3] A. Alekseev, E. Meinrenken, Linearization of Poisson Lie group structures, arXiv:1312.1223v1 [math.DG].
- [4] A. Belavin and V. Drinfeld, Triangle equations and simple Lie algebras, *Soviet Sci. Rev. Sect. C: Math. Phys. Rev.*, **4** (1984) 93 - 165.
- [5] V. Chari and A. Pressley, A guide to quantum groups, *Cambridge University press*, 1994.
- [6] C. DeConcini and C. Procesi, Quantum Schubert cells and representations at roots of 1, *Algebraic groups and Lie groups*, 127 - 160, Austral. Math. Soc. Lect. Ser. 9, Cambridge Univ. Press, Cambridge, 1997.
- [7] P. Delorme, Classification des triples de Manin pour les algèbres de Lie réductives complexes, *J. Alg.*, **246** (2001), 97 - 174.
- [8] J. Donin, D. Gurevich, and S. Majid, R -matrix brackets and their quantization, *Ann. Inst. H. Poincaré Phys. Théor.*, **58** (2) (1993), 235 - 246.
- [9] V. G. Drinfeld, Hamiltonian structures on Lie groups, Lie bialgebras and the geometric meaning of the classical Yang-Baxter equations, *Soviet Math. Dokl.*, **27** (1) (1982), 68 - 71.
- [10] V. G. Drinfeld, Quantum groups, *Proceedings of the International Congress of Mathematicians*, **1** (2) (Berkeley, Calif., 1986), 798 - 820, Amer. Math. Soc., Providence, RI, 1987.
- [11] V. G. Drinfeld, Quasi-Hopf algebras, *Algebra i Analiz*, 1:6 (1989), 114 - 148; English version: *Leningrad Math. J.*, **1:6** (1990), 1419 - 1457
- [12] V. G. Drinfeld, Almost co-commutative Hopf algebras, *Leningrad Math. J.*, **1** (1990), 321 - 342.
- [13] V. G. Drinfeld, On Poisson homogeneous spaces of Poisson-Lie groups, *Theor. Math. Phys.*, **95** (2) (1993), 226 - 227.
- [14] B. Elek and J.-H. Lu, On a Poisson structure on Bott-Samelson varieties: computations in coordinates, arXiv:1601.00047.
- [15] B. Enriquez and Y. Kosmann-Schwarzbach, Quantum homogeneous spaces and quasi-Hopf algebras, Confrence Moshé Flato 1999, *Mathematical Physics Studies*, **21 - 22** (2000), 105 - 121.
- [16] P. Etingof and D. Kazhdan, Quantization of Lie bialgebras, III, *Sel. Math., New Ser.*, **4** (1998), 233 - 269.
- [17] P. Etingof and D. Kazhdan, Quantization of Poisson algebraic groups and Poisson homogeneous spaces, *Symétries quantiques (Les Houches 1995)*, A. Connes, K. Gawędzki, and J. Zinn-Justin, (eds.), North-Holland, Amsterdam (1998), 935 - 946.
- [18] P. Etingof and O. Schiffmann, *Lectures on quantum groups*, 2nd edition, international press, 2002.
- [19] V.V. Fock and A.A. Rosly, Flat connections and polyubles, *Theor. Math. Phys.*, **95** (1993), 526 - 534.
- [20] V.V. Fock and A.A. Rosly, Poisson structure on moduli of flat connections on Riemann surfaces and the r -matrix, *Amer. Math. Soc. Transl.*, **191** (2) (1999), 67 - 86
- [21] K. Goodearl and M. Yakimov, Quantum cluster algebras and quantum nilpotent algebras, *Proc. Natl. Acad. Sci. USA* **111** (27) (2014), 9696 - 9703.
- [22] J. E. Grabowski, A triple construction for Lie bialgebras, *Pacific J. Math.*, **221** (2005), no. 2, 281 - 301.
- [23] T. Hodges and M. Yakimov, The doubles and dual of a quasitriangular Lie bialgebra, *Math. Res. Letters*, **8** (2001), 91 - 105.
- [24] Y. Kosmann-Schwarzbach, Lie bialgebras, Poisson Lie groups and dressing transformations, *Integrability of nonlinear systems*, second edition, Lecture Notes in Physics, **638**, Springer-Verlag, (2004), 107 - 173.
- [25] L. Korogodski and Y. Soibelman, *Algebras of functions on quantum groups, part I*, AMS, Mathematical surveys and monographs, **56** (1998).
- [26] D. Li-Bland and E. Meinrenken, Courant algebroids and Poisson geometry, *Int. Math. Res. Notices*, **2009** (11), 2106 - 2145.
- [27] D. Li-Bland and P. Severa, Quasi-Hamiltonian groupoids and multiplicative manin pairs, *Int. Math. Res. Notices*, **2011** (20), 2295 - 2350.
- [28] D. Li-Bland and P. Severa, Moduli spaces for quilted surfaces and Poisson structures, *Doc. Math.*, **20** (2015), 1071 - 1135.
- [29] J.-H. Lu, *Multiplicative and Affine Poisson Structures on Lie Groups*, Berkeley thesis, 1990.

- [30] J.-H. Lu and V. Mouquin, On the T -leaves of some Poisson structures related to products of flag varieties, arXiv:1511.02559.
- [31] J.-H. Lu and A. Weinstein, Poisson Lie groups, dressing transformations, and Bruhat decomposition, *J. Diff. Geom.*, **31** (1990), 501 - 526.
- [32] J.-H. Lu and M. Yakimov, Group orbits and regular partitions of Poisson manifolds, *Comm. Math. Phys.*, **283** (3) (2008), 729 - 748.
- [33] S. Majid, *Foundations of quantum group theory*, Cambridge University Press, Cambridge, 1995.
- [34] V. Mouquin, Cell decompositions of double Bott-Samelson varieties, *Int. Math. Res. Notices*, **2015** (18), 8372 - 8410.
- [35] V. Mouquin, *On a Deodhar-type decomposition and a Poisson structure on double Bott-Samelson varieties*, PhD thesis of the University of Hong Kong, 2013.
- [36] N. Reshetikhin and M. Semenov-Tian-Shansky, Quantum R -matrices and factorization problems, *J. Geom. Phys.*, **5** (1988), 533 - 550.
- [37] M. Semenov-Tian-Shansky, What is a classical r -matrix?, *Funct. Anal. Appl.*, **17** (4) (1983), 259 - 272.
- [38] M. Semenov-Tian-Shansky, Dressing transformation and Poisson group actions, *Publ. Res. Inst. Math. Sci.*, **21** (6) (1985), 1237 - 1260.
- [39] M. Semenov-Tian-Shansky, *Integrable systems: the r -matrix approach*, RIMS, Kyoto University (2008).
- [40] A. Weinstein, Some remarks on dressing transformations, *J. Fac. Sci. Univ. Tokyo Sect. IA Math.*, **35** (1) (1988), 163 - 167.
- [41] M. Yakimov, Symplectic leaves of complex reductive Poisson Lie groups, *Duke Math. J.*, **112** (3) (2002), 453 - 509.
- [42] S. Zwirnagl, R -matrix Poisson algebras and their deformations, *Adv. in Math.*, **220** (1) (2009), 1 - 58

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