Yue's Solution of Classical Elasticity in *n*-Layered Solids: Part 1, Mathematical Formulation

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Abstract

This paper presents the exact and complete fundamental singular solutions for the boundary value problem of a *n*-layered elastic solid of either transverse isotropy or isotropy subject to body force vector at the interior of the solid. The layer number *n* is an arbitrary non-negative integer. The mathematical theory of linear elasticity is one of the most classical field theories in mechanics and physics. It was developed and established by many well-known scientists and mathematicians over 200 years from 1638 to 1838. For more than 150 years from 1838 to present, one of the remaining key tasks in classical elasticity has been the mathematical derivation and formulation of exact solutions for various boundary value problems of interesting in science and engineering. However, exact solutions and/or fundamental singular solutions in closed form are still very limited in literature. The boundary-value problems of classical elasticity in *n*-layered and graded solids are also one of the classical problems challenging many researchers. Since 1984, the author has analytically and rigorously examined the solutions of such classical problems using the classical mathematical tools such as Fourier integral transforms. In particular, he has derived the exact and complete fundamental singular solutions for elasticity of either isotropic or transversely isotropic layered solids subject to concentrated loadings. The solutions in *n*-layered or graded solids can be calculated with any controlled accuracy in association with classical numerical integration techniques. Findings of this solution formulation are further used in the companion paper for mathematical verification of the solutions and further applications for exact and complete solutions of other problems in elasticity, elastodynamics, poroelasticty and thermoelasticity. The mathematical formulations and solutions have been named by other researchers as Yue's approach, Yue's treatment, Yue's method and Yue's solution.

Keywords: elasticity, solution, layered solid, graded material

1 Introduction

1.1 Initiation of this study

The author has studied mathematics and mechanics since 1979 when he started his studies toward a BSc degree at Peking University. In 1983, he was admitted for post-graduate study by closed-book examination. Because of his strength at mathematics and mechanics, his MSc supervisor Professor Ren Wang [1, 2] assigned the project of ground subsidence due to underground coal mining as his MSc thesis project in 1984. This project was initiated by Mr. Zeng-qi Li who was a researcher at China Coal Research Institute and a graduate of the Department of Mathematics and Mechanics of Peking University in 1957. Mr. Li had investigated the ground subsidence problems for many years [3, 4]. In 1984, Mr. Li came to Peking University and sought advices from Professor Ren Wang about some mathematical issues associated with the mathematical formulation of analytical solution in multi-layered elastic solid. The multi-layered elastic solid was the model that Mr. Li used for prediction of the ground subsidence due to underground coal mining.

Under the supervision of his supervisor and others, the author quickly understood and mastered the key points of the mathematics and mechanics of the topic and entered into the frontiers of the mathematics and mechanics of elasticity in n-layered solids, where n is an arbitrary non-negative integer. He examined the topic with mind and derived the solution of elasticity in n-layered solids using the classical mathematical tool of Fourier integral transform and Laplace transform. His MSc thesis examination committee assessed his mathematical results and considered them of certain originality in June 1986 [5]. Consequently, this piece of MSc degree work was selected and published at *Acta Scientiarum Naturalium Universitatis Pekinensis* in 1988 [6] on the basis of the committee's recommendation. The committee members were Professor Ren Wang, Professor Zhong-yi Ding, Professor Min-zhong Wang [7, 8] and Professor Tianyou Fan [9, 10].

On the other hand, the MSc thesis examination committee also clearly pointed out the following questions. The solution given in the thesis was in the form of improper integrals of infinite intervals only. Does it converge? What is its singularity? Does it satisfy the governing partial differential equations and the boundary conditions? In other words, the solution given in the thesis was only the initial result of the mathematics and mechanics of elasticity in n-layered solids. Much more detailed and in-depth examinations of the mathematics of the solution of elasticity in n-layered solids had to be carried out, which are difficult.

1.2 Ten years effort and results

After his graduation from Peking University in July 1986, the author used much of his spare time to think and examine the questions. He further carried out careful and rigorous investigations on the mathematical formulation and properties of the solutions in the form of two-dimensional improper integrals of Fourier transforms. In 1995, he eventually made breakthroughs and gave rigorous mathematical answers to the questions raised by the committee. Subsequently, he published his findings in *Journal of Elasticity*, *Journal of Engineering Science*, *Quarterly Journal of Mechanics and Applied Mathematics*, *Journal of Engineering Mechanics* and *International Journal Solids and Structure* [11-17] etc.

Because of his mathematical developments, the author was selected by Mr. G. H. Argue [18], the then Chief Engineer of Civil Engineering of Transport Canada in the development of a layered elastic model and associated criteria for the structural design and evaluation of airport pavements in 1995 and 1996 [19, 20]. Two research contracts were signed with the author. Using his solution and computer program, the author undertook the tasks and completed the contracts [21]. On March 5, 1996, Mr. Argue wrote in a reference letter that "I selected Dr. Yue for the project because his qualifications in layered elastic theory are unique in Canada. He has published mathematical developments of the theory, and his computer program for the stress and strain analysis of layered elastic systems is the best available."

About twelve years later, i.e., in 2007 and 2008, four researchers at Research Centre Jülich and four researchers at Massachusetts Institute of Technology published their papers in *Biophysical Journal* [22] and *Physical Review E* [23], respectively. In their papers, they used the elastic solutions in layered solids to analyze the stresses and deformation of cells. In their papers, they made literature reviews on the analytical solutions of elasticity in homogeneous solid and layered solids. They found that since Boussinesq's solution given in 1885 for a homogeneous elastic halfspace, the solutions given by Yue in 1995 and 1996 for layered solids are concise and convenient, which were expressed in the form of matrices. They called them Yue's approach, Yue's treatment, Yue's method and Yue's solution. Their experimental results also supported Yue's solution.

1.3 Objectives and outlines

This paper and the companion paper [24] have three objectives: (1) to give a step by step mathematical formulation process of the approach, treatment, method and solutions developed by the author for elasticity in *n*-layered solids; (2) to present a detailed and rigorous mathematical verification to the questions on the convergence, singularity and satisfaction of the solution; (3) to show the approach, treatment and method applicable to transversely isotropic layered solids, mixed-boundary value problems, boundary element method, and initial-boundary value problems in the framework of elastodynamics, thermoelasticity and Biot's theory of poroelasticity.

To achieve the objectives, this paper has been outlined as follows. A comprehensive literature review on the history of elasticity since 1638 is presented to illustrate the importance of the mathematical theory of elasticity and difficulty and limitation of mathematical formulation of closed-form solutions for its boundary-value problems in *n*-layered or graded solids. Secondly, the matrix Fourier transform approach developed by the author is presented for boundary-value problems in *n*-layered solids of transverse isotropy.

Next, details of the treatment, method and solutions are presented for the mathematical formulation of the solutions in *n*-layered solids in both transform and physical domains.

In the companion paper [24], the fundamental singular solutions in exact closed-form are presented for the basic and classical boundary-value problems in either homogeneous or bi-homogeneous solids and show their mathematical properties and singularities. Furthermore, the mathematical properties of the solutions for elasticity in *n*-layered solids are examined and presented to analytically show their convergence, singularity and satisfaction. The singularity of the solution is given in exact closed-form. The applications of the approach, treatment and method to other problems are briefly presented. So, the solutions for other boundary-value problems, mixed boundary-value problems and initial-boundary value problems can be derived and formulated similarly and systematically in the form of matrix operations. Some concluding remarks are given at the end to summarize this study over the last 30 years and to recommend further studies and applications of interests in science and engineering.

2 Background (Fundamentals of Elasticity)

2.1 The mathematical theory of classical elasticity

2.1.1 The displacement vector, strain and stress tensors

The mathematical theory of classical elasticity is one of the essential foundations of continuum mechanics and advanced mathematics [1,7,8]. It is a classical field theory that deals with the fields of elastic displacements, strains and stresses in a continuous solid material subjected to external and/or internal loadings. It has a total of 15 field variables in a three-dimensional space occupied by the loaded solid materials. The 15 field variables at any point in the solid material include three displacements, six strains and six stresses which form a displacement vector $\mathbf{u}$, a strain tensor $\boldsymbol{\varepsilon}$ and a stress tensor $\boldsymbol{\sigma}$, respectively. In the Cartesian coordinate system (O*xyz*), they can be expressed as follows:

$$\mathbf{u} = \mathbf{u}(x, y, z) = \left(u_x, u_y, u_z\right) \tag{1a}$$

$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}(x, y, z) = \begin{pmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz} \end{pmatrix} \tag{1b}$$

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}(x, y, z) = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{pmatrix} \tag{1c}$$

The strain tensor $\boldsymbol{\varepsilon}$ is also called Cauchy's strain tensor in honour of the French mathematician Augustin-Louis Cauchy (1789 – 1857). It has three normal strain components (ε_{xx} , ε_{yy} , ε_{zz}) and six shear strain components (ε_{xy} , ε_{xz} , ε_{yz} , ε_{yx} , ε_{zx} , ε_{zy}). ε_{xx} , ε_{yy} and ε_{zz} respectively represent the infinitesimal extensions or shortening of the solid material along the *x*, *y* and *z* coordinate directions. ε_{xy} , ε_{xz} and ε_{yz} respectively represent the half infinitesimal angle changes of the solid material between the *x* and *y* coordinate directions, between the *x* and *z* coordinate directions, and between the *y* and *z* coordinate directions. For ease of understanding, this paper does not use the compacted tensor notations but use the specific expressions for the tensors and governing equations.

2.1.2 The geometric equations

Under the assumption of infinitesimal displacement and deformation, the strain tensor $\boldsymbol{\varepsilon}$ has the following linear partial differentiation relationship with the displacement vector $\mathbf{u}$, which are also called the geometric equations.

$$\varepsilon_{xx} = \frac{\partial u_x}{\partial x} \tag{2a}$$

$$\varepsilon_{yy} = \frac{\partial u_y}{\partial y} \tag{2b}$$

$$\varepsilon_{zz} = \frac{\partial u_z}{\partial z} \tag{2c}$$

$$\varepsilon_{xy} = \varepsilon_{yx} = \frac{1}{2}(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}) \tag{2d}$$

$$\varepsilon_{xz} = \varepsilon_{zx} = \frac{1}{2}(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}) \tag{2e}$$

$$\varepsilon_{yz} = \varepsilon_{zy} = \frac{1}{2}(\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}) \tag{2f}$$

2.1.3 The force and moment equilibriums

The stress tensor $\boldsymbol{\sigma}$ is also called Cauchy's stress tensor. It has three normal stress components (σ_{xx} , σ_{xy} , σ_{xz}) and six shear stress components (σ_{xy} , σ_{yx} , σ_{xz} , σ_{zx} , σ_{yz} , σ_{zy}). They are defined as follows.

$$\left(\sigma_{xx}, \sigma_{xy}, \sigma_{xz}\right) = \lim_{\Delta A_{yz} \to 0} \frac{\mathbf{F}_{yz}}{\Delta A_{yz}} \tag{3a}$$

$$\left(\sigma_{yx},\sigma_{yy},\sigma_{yz}\right)=\lim_{\Delta A_{xz}\to 0}\frac{\mathbf{F}_{xz}}{\Delta A_{xz}} \tag{3b}$$

$$\left(\sigma_{zx},\sigma_{zy},\sigma_{zz}\right)=\lim_{\Delta A_{xy}\to 0}\frac{\mathbf{F}_{xy}}{\Delta A_{xy}} \tag{3c}$$

where ΔA_{yz}, ΔA_{xz} and ΔA_{xy} are infinitesimal areas respectively perpendicular to the *x*, *y* and *z* coordinate directions at any point (*x*, *y*, *z*). $\mathbf{F}_{yz}$, $\mathbf{F}_{xz}$ and $\mathbf{F}_{xy}$ are the three force vectors acting on ΔA_{yz}, ΔA_{xz} and ΔA_{xy}, respectively.

Based on the Newton's second law of motion, the static equations of force equilibrium at any point in the solid material along the *x*, *y*, and *z* coordinate directions can be expressed in terms of the partial differentiations of the relevant stress tensor components. They are also called the equations of equilibrium and take the form

$$\frac{\partial\sigma_{xx}}{\partial x}+\frac{\partial\sigma_{xy}}{\partial y}+\frac{\partial\sigma_{xz}}{\partial z}+f_x=0 \tag{4a}$$

$$\frac{\partial\sigma_{yx}}{\partial x}+\frac{\partial\sigma_{yy}}{\partial y}+\frac{\partial\sigma_{yz}}{\partial z}+f_y=0 \tag{4b}$$

$$\frac{\partial\sigma_{zx}}{\partial x}+\frac{\partial\sigma_{zy}}{\partial y}+\frac{\partial\sigma_{zz}}{\partial z}+f_z=0 \tag{4c}$$

where $\left(f_x, f_y, f_z\right)=\mathbf{f}(x,y,z)$ is body force vector acting in the interior of the solid material along the *x*, *y*, and *z* coordinate directions.

Because of moment equilibrium at any point (*x*, *y*, *z*) about the *x*, *y*, and *z* coordinate directions, the six shear stresses at that point (*x*, *y*, *z*) have the following relations.

$$\sigma_{xy}=\sigma_{yx} \tag{5a}$$

$$\sigma_{yz}=\sigma_{zy} \tag{5b}$$

$$\sigma_{xz}=\sigma_{zx} \tag{5c}$$

2.1.4 The Hooke's law

The strains (or the deformation) and the stresses (or forces) induced by loading at any point (*x*, *y*, *z*) in the solid material are two completely different natural phenomena. However, researchers discovered that there are laws governing their relationship. Such laws are generally called constitutive relationships at present. The mathematical theory of classical elasticity adopts the linear relationship between the strains and stresses in elastic solids. It is called Hooke's law in honor of the English polymath and inventor Robert Hooke (1635-1703). Because transversely isotropic solid material is the main concern of this paper, its Hooke's law is specifically given. The six stresses in (4) can be expressed in terms of the six strains in the following six independent equations.

$$\sigma_{xx} = c_1\varepsilon_{xx} + (c_1 - 2c_5)\varepsilon_{yy} + c_2\varepsilon_{zz} \tag{6a}$$

$$\sigma_{yy} = c_1\varepsilon_{yy} + (c_1 - 2c_5)\varepsilon_{xx} + c_2\varepsilon_{zz} \tag{6b}$$

$$\sigma_{zz} = c_2\varepsilon_{xx} + c_2\varepsilon_{yy} + c_3\varepsilon_{zz} \tag{6c}$$

$$\sigma_{xz} = 2c_4\varepsilon_{xz} \tag{6d}$$

$$\sigma_{yz} = 2c_4\varepsilon_{yz} \tag{6e}$$

$$\sigma_{xy} = 2c_5\varepsilon_{xy} \tag{6f}$$

where $c_i (i = 1,2,3,4,5)$ are the five elastic stiffness parameters. They are independent to the levels of strains and stresses. For a positive definition of elastic stress-strain energy, the five elastic stiffness parameters shall have the following limits.

$$c_1 > c_5 > 0 \tag{7a}$$

$$c_3 > 0 \tag{7b}$$

$$c_4 > 0 \tag{7c}$$

$$c_5 > 0 \tag{7d}$$

$$\sqrt{c_1 \cdot c_3} > c_2 \tag{7e}$$

They have the following relationships with the Young's moduli, Poisson's ratios and the shear moduli.

$$c_1 = \frac{\mu(E' - \nu'^2 E)}{(1-\nu)E' - 2\nu'^2 E} \tag{8a}$$

$$c_2 = \frac{\nu' E E'}{(1-\nu)E' - 2\nu'^2 E} \tag{8b}$$

$$c_3 = \frac{(1-\nu)E'^2}{(1-\nu)E' - 2\nu'^2 E} \tag{8c}$$

$$c_4 = \mu' \tag{8d}$$

$$c_5 = \mu = \frac{E}{2(1+\nu)} \tag{8e}$$

where E and E' are the two Young's moduli in honor of the English polymath Thomas Young (1773 – 1829), ν and ν' the two Poisson's ratios in honor of the French mathematician and physicist Siméon Denis Poisson (1781 – 1840), μ and μ' the two shear moduli, respectively in the isotropic plane and along the z-axis direction.

The isotropic plane of a transversely isotropic solid in (6) is parallel to the *x*-*y* plane and perpendicular to the *z*-axis. The material property at any point is the same along any directions in the isotropic plane and can have different values along other directions. On the other hand, if the solid material property at any point can be the same along any directions, such material is called an isotropic material. The five elastic parameters degenerate into two elastic parameters as follows.

$$c_1 = c_3 = \lambda + 2\mu = 2(1-\nu)\mu/(1-2\nu) \tag{9a}$$

$$c_2 = \lambda = 2\nu\mu/(1-2\nu) > -2\mu/3 \tag{9b}$$

$$c_4 = c_5 = \mu = \mu' > 0 \tag{9c}$$

$$E = E' = 2(1+\nu)\mu > 0 \tag{9d}$$

$$-1 < \nu = \nu' < 1/2 \tag{9e}$$

where λ and μ are called Lamé constants in honor of the French mathematician Gabriel Léon Jean Baptiste Lamé (1795 – 1870).

2.1.5 The boundary-value problems

The above equations (2), (4) and (6) govern the displacement field, strain field and the stress field within an elastic solid material induced by an external and/or internal loading. To obtain a meaningful boundary value problem, the external and internal loading has to be properly prescribed on the boundaries and in the interior of the elastic solid material. They can be classified as the traction (or stress) boundary conditions, the displacement boundary conditions and the stress-displacement mixed boundary conditions. The system of linear partial differential equations (2), (4) and (6) has to be solved for a solution of the displacement, strain and stress field for the solid material under the prescribed boundary conditions. This solution is unique. In other words, there is one and only one solution for a properly given boundary value problem.

2.2 Establishment history from 1638 to 1838

The development of the mathematical theory of classical elasticity was started by Galileo Galilei (1564 – 1642). In his Discourses published in 1638, Galileo made a scientific description of the strength of solids although solids were treated as inextensible [25]. About 22 years later in 1660, Robert Hooke (1635-1703) discovered the first rough law of proportionality between the forces and displacements, a revolutionary idea in science. He wrote in a Latin anagram that "the extension is proportional to the force", which forms the foundation of the constitutive relation equations (6). It can be expressed as follows.

$$F = K\Delta L \tag{10}$$

where F is the force, ΔL is the extension and K is an elastic parameter depending on the size of the solid material.

About 49 years later, the English physicist and mathematician Isaac Newton (1642 – 1726) published his book *Philosophiæ Naturalis Principia Mathematica* ("*Mathematical Principles of Natural Philosophy*") in 1687 and gave the second law of motion as follows.

$$F = ma \tag{11}$$

where a is the acceleration and m is the mass of the solid material. This law laid the foundations for the governing equations of force and moment equilibriums in (4) and (5).

During this 150-year period from 1660 to 1821, many mathematician and physicists worked on the science of elasticity with some special problems of beams, torsion, columns and plates. They included Edme Mariotte (1620–1684), Jacob Bernoulli (1655 – 1705), Daniel Bernoulli FRS (1700 – 1782), Leonhard Euler (1707 – 1783), Charles-Augustin de Coulomb (1736 – 1806), Thomas Young (1773 – 1829), and Siméon Denis Poisson (1781 – 1840). For example, Thomas Young sharpened the first rough law (10) in 1807 by giving a clear formulation of the modulus of elasticity in tension as follows.

$$\sigma = \frac{F}{A} = \frac{KL}{A}\frac{\Delta L}{L} = E\varepsilon \tag{12}$$

where σ is the tensile stress and equals the ratio of the tension force F over its applied cross-section area A. ε is the tensile strain and equals the ratio of the extension ΔL over its total length L. The elastic modulus (or Young's modulus) ($E = KL/A$) becomes a real elastic parameter independent to the size of the solid material.

The mathematical theory of classical elasticity presented in above sub-section in equations (1) to (9) was formerly established from 1821 to 1838 by the French engineer and physicist Claude-Louis Navier (1785 – 1836), the French mathematician Augustin-Louis Cauchy (1789 – 1857), and the British mathematical physicist George Green (1793 – 1841). In 1838, George Green developed the revolutionary principle of conservation of elastic energy and showed that the number of elastic parameters necessary to characterize the elastic solid of general anisotropy turns out to be 21. In particular, Green's functions are used to name the fundamental singular solutions of boundary-value problems in physics and elasticity. Using "Green's function" as the topic, a result of 21,574 papers is shown up in the database of Web of Science on April 6, 2015. More details of the establishment history of classical elasticity can be found in the textbooks and monographs [26-39].

2.3 The key task of solution from 1838 to present

As shown in (1) to (9), the mathematical theory of classical elasticity comprises a complete set of linear partial differential equations governing the fields of displacement vector, strain tensor and stress tensor in a solid material subject to external and internal loadings. Since its former establishment in 1838, the key task in the mathematical theory of classical elasticity has become to derive, formulate and find the solution of the elastic fields for specific types of the boundary-value problems [26-40].

The mathematical formulation and derivation of solutions of boundary-value problems within the framework of classical elasticity, however, are definitely not a routine task and have intrinsic difficulties. It has been always a difficult task to find a solution for a boundary-value problem in classical elasticity. Many mechanicians, mathematicians and elasticians devoted their time and efforts to attack the key task in elasticity because of its wide applications in engineering and science. Construction of the mathematical tools for formulation of stresses and deformations in strained elastic bodies are their dominant concerns. The common methods include serial expansions, potential methods, complex variables, Fourier transforms and integral transforms. In particular, Barré de Saint-Venant (1797-1886) made monumental contributions to torsion and flexure of cylinders with an approximation principle (i.e., the Saint-Venant's principle) of statically equivalent systems of load. Gustav Robert Kirchhoff (1824 – 1887) initiated the study of the deflection of plates. George Biddell Airy (1801 – 1892) solved two-dimensional plane stress problems.

Exact, complete and closed-form solutions are limited in the literature. They have been derived only for some special cases of homogeneous isotropic (or transversely isotropic) elastic solids with some regular and special geometries and loadings. These regular geometries include beams, columns, cylinders, plates, shells and sphere. The solutions can be found in classical textbooks and monographs [26-40] including Love (1927) and Poulos & Davis (1974) and many journal papers. Exact, complete and closed-form solutions are not available for many problems of scientific and technological importance although there are ten thousand papers related to the topic of linear elasticity in the database of web of science journals at present.

Because of the intrinsic difficulties in deriving analytical and mathematical solutions, more and more researchers have turned their efforts in developing numerical methods and software for numerical solutions of various boundary-value problems in elasticity since the introduction of digital electronic programmable computers in 1950s. These numerical methods include finite element methods, finite difference methods, boundary element methods and discrete element methods.

2.4 Closed-form fundamental singular solutions

Fundamental singular solutions are of the most importance in the mathematical theory of classical elasticity. They are the solutions of boundary-value problems due to the action of force loading concentrated at a point or a curve in the interior or on the boundary of an elastic solid. Furthermore, a closed-form solution is a peculiar solution that can be exactly expressed in the forms of elementary or special functions with known and accurately evaluable singularities. Consequently, the closed-form fundamental singular solutions are extremely limited and useful in classical elasticity or in the much wide areas of continuum mechanics and applied mathematics. They can be used to formulate solutions of various distributed loadings. They have become much more powerful in solving various boundary-value problems due to the development of boundary element methods since 1960s.

The first complete and closed-form fundamental singular solution in elasticity is Kelvin's solution [41]. It was given by the British mathematical physicist William Thomson (or Lord Kelvin) (1824–1907) in 1848. He also did important work in the formulation of the first and second laws of thermodynamics. Kelvin's solution gives the complete elastic field in a homogeneous and isotropic medium of infinite extent $(-\infty < x, y, z < +\infty)$ induced by an internal body force concentrated at a point. It forms the core basis of the modern boundary element methods. The second complete and closed-form singular solution is Boussinesq's solution. It was given by the French mathematician and physicist Joseph Valentin Boussinesq (1842-1929) in 1885 [42]. Boussinesq's solution is also a fundamental singular solution and describes the complete elastic field in a homogeneous and isotropic medium of semi-infinite extent $(-\infty < x, y < +\infty, 0 \le z < +\infty)$ induced by a normal traction concentrated at a point on the boundary surface. The third complete and closed-form solution is Mindlin's solution [43]. It was given by the American mechanician Raymond David Mindlin (1906-1987) in 1936. Mindlin's solution is a fundamental singular solution and describes the complete elastic field in a homogeneous and isotropic medium of semi-infinite extent $(-\infty < x, y < +\infty, 0 \le z < +\infty)$ induced by an internal body force concentrated at a point.

The above three classical solutions have become the theoretical basis of many engineering sciences. Other closed-form fundamental singular solutions available in open literature [44-47] for boundary-value problems in classical elasticity are some logical extensions of the above three solutions to transversely isotropic solid and bi-materials. They include (1) solution of a point force in the interior of a homogeneous and transversely isotropic elastic solid of infinite extent; (2) solution of a point force on the boundary of a homogeneous and transversely isotropic elastic solid of a half-space extent; (3) solution of a point force in the interior of a homogeneous and transversely isotropic elastic solid of infinite extent, where the isotropic plane is parallel to the boundary surface; (4) solution of a point force in the interior of two perfectly bonded homogeneous and isotropic elastic solids of infinite extent; (5) solution of a point force in the interior of two perfectly bonded homogeneous and transversely isotropic elastic solids of infinite extent, where the two isotropic planes are parallel to the interface plane. Various solution methods were used for the formulations of the above closed-form fundamental singular solutions.

2.5 Solutions in non-homogeneous and/or anisotropic materials

Homogeneous and isotropic solids are an idealized model of actual materials. Actual materials are usually and commonly non-homogeneous and anisotropic and their properties are variable spatially and directionally. They can be observed in many natural and engineered materials. The heterogeneity and anisotropy can have significant effects on the elastic responses of materials under loadings [48]. Literature reviews over the past 30 years by the author have shown that there are no closed-form fundamental singular solutions for general non-homogeneous and/or anisotropic elastic solids [49-76].

In order to solve the boundary-value problems in heterogeneous and anisotropic materials, many researchers have concentrated their attentions on the solutions of boundary-value problems in elastic solids whose properties vary with depth z only since 1940. The elastic solutions for the depth variation models can be classified into two categories. In the first category, it is assumed that the Poisson's ratio keeps constant and the shear modulus varies continuously with depth in a certain simple manner in the material region. Elementary functions including power law, linear, hyperbolic and exponential functions have been adopted to represent the depth variations of the shear modulus. Studies on this category of the boundary-value problems can be found in Holl [49], Gibson [54], Ozturk and Erdogan [62] and Selvadurai [63, 66].

In the second category, it is assumed that elastic materials are piece-wise homogeneous and consist of a limited number of distinctive finite elastic layers of an infinite lateral extent. The elastic properties (i.e., Poisson's ratio and shear modulus) keep constant within each elastic layer and are different for any two connected layers. Inter-facial conditions, such as a fully bonded interface, are imposed to connect different layers together into a layered elastic material system. Studies on this topic can be found in Burminster [50], Lemcoe [51], Schiffman [52], Michelow [53], Bufler [55], Small & Booker [56], Wang [57], Benitez and Rosakis [58], Kausel and Seale [59], Pindera [60], Conte and Dente [61], Ta and Small [64], Cheung and Tham [65].

Furthermore, the studies on homogeneous and/or layered solids of transverse isotropy can be found in Huber [67], Elliott and Mott [68], Hu [69, 70], Pan and Chou [71], Ding and Xu [72, 73], Pan [74], Lin and Keer [75] and Ding et al. [76].

2.6 The common issues

The formulation and solutions of elastic problems in the first category are certainly non-routine. Many cases cannot provide a complete set of the solutions for the displacement vector, strain and stress tensors. In the second category, many analytical or semi-analytical methods are developed for deriving and formulating solutions of similar boundary-value problems in layered elastic solids with isotropic and/or transversely isotropic properties. The methods include forward transfer matrix, flexibility matrix, stiffness matrix, finite layer, finite strip and thin layer methods. These solutions are usually expressed in very complicated forms involving improper integrals and/or approximated forms. There is almost no systematical and rigorous mathematical examination of these solutions in terms of their convergence and singularities. Most importantly, closed-form fundamental singular solutions were also not available at least in these cited literatures [25-76] on the boundary-value problems with depth variations of either the two isotropic or the five transversely isotropic elastic parameters.

2.7 The author's work

From 1984 to present, the author has examined the three-dimensional boundary-value problems of the classical elasticity in nonhomogeneous and transversely isotropic solid [5, 6, 11-17, 19, 20, 77-81]. The solid occupies a three-dimensional space of an infinite lateral extent $(-\infty < x, y < +\infty)$ and a finite thickness $a \le z \le b)$, a semi-infinite $a \le z < +\infty)$, and/or an infinite extent $(-\infty < z < +\infty)$. The five elastic material parameters $(c_1, c_2, c_3, c_4, c_5)$ or (E, E', ν, ν', μ) are arbitrary piece-wise functions of the *z*–axis. The solid is subjected to various loadings at the external boundary surfaces $z = a$ and $z = b$ as well as the internal body force. His key task is to rigorously derive and show the complete and closed-form solution of the elastic fields from the set of partial differential equations given in (1) to (9) for each of the boundary-value problems. Moreover, the author has given the closed-form fundamental singular solutions induced by loadings concentrated at a point, a circular ring and a rectangular area whose normal direction is parallel to the vertical *z*-axis in the interior or on the boundary of the solid. His closed-form fundamental singular solutions can automatically and analytically degenerate as Kelvin solution, Boussinesq solution and Mindlin's solution once the material properties become homogeneous and isotropic. Details of his mathematical approach, treatment, method and solutions are presented in this paper and the companion paper [24] using the model of *n*-layered solid with both transverse isotropy and isotropy.

3 The Matrix Fourier Integral Approach

3.1 General

The author used the Fourier integral transforms to rigorously derive and formulate the general algebraic solution for the set of linear partial differential equations (2) to (6). The

Fourier integral transforms are one of the classical mathematical tools for solutions of initial and/or boundary value problems in physics and mechanics [82-90]. The concept of Fourier integral transforms was originated by the French mathematician and physicist Jean-Baptiste Joseph Fourier (1768 – 1830) in his monumental treatise entitled *La Théorie Analytique de la Chaleur* (*The Analytical Theory of Heat*) in 1822 [82]. He stated the Fourier Integral Theorem and used it to solve problems of heat transfer and vibrations. In 1843, A. L. Cauchy (1789 – 1857) gave the exponential form of the Fourier Integral Theorem [90].

On the basis of the classical Fourier integral transforms, the author [5, 6, 11-17] developed a matrix approach to solve the set of fifteen linear partial differential equations (2) to (6) and derived a general solution in symmetrical matrix form. Details of this approach and results are presented below using the transversely isotropic material model.

3.2 The matrix solution representation

The strain and stress tensors in (1) to (6) can be re-expressed as the vertical stress vector $\mathbf{T}_z(x,y,z)$, the plane stresses $\mathbf{T}_p(x,y,z)$, the vertical strains $\mathbf{\Gamma}_z(x,y,z)$ and the plane strains $\mathbf{\Gamma}_p(x,y,z)$. As a result, the fifteen field variables can be grouped into five vectors as follows,

$$\mathbf{u}=\begin{pmatrix}u_x\\u_y\\u_z\end{pmatrix},\ \mathbf{T}_z=\begin{pmatrix}\sigma_{xz}\\\sigma_{yz}\\\sigma_{zz}\end{pmatrix},\ \mathbf{T}_p=\begin{pmatrix}\sigma_{xx}\\\sigma_{xy}\\\sigma_{yy}\end{pmatrix},\ \mathbf{\Gamma}_z=\begin{pmatrix}\varepsilon_{xz}\\\varepsilon_{yz}\\\varepsilon_{zz}\end{pmatrix},\ \mathbf{\Gamma}_p=\begin{pmatrix}\varepsilon_{xx}\\\varepsilon_{xy}\\\varepsilon_{yy}\end{pmatrix}. \tag{13}$$

Yue [5, 6, 11-17] has shown that the above five sets of vectors for the fifteen field variables in the physical domain can be represented by two unknown vectors $\mathbf{w}(\xi,\eta,z)$ and $\mathbf{Y}_z(\xi,\eta,z)$ in the transform domain for all the boundary-value problems of a solid occupying the layer region of $-\infty < x, y < +\infty$ and $a \le z \le b$. In particular, the solution representation can be expressed as follows in the Cartesian coordinate system,

$$\mathbf{u}(x,y,z)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\frac{1}{\rho}\mathbf{\Pi}\mathbf{w}(\xi,\eta,z)Kd\xi d\eta \tag{14a}$$

$$\mathbf{T}_z(x,y,z)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\mathbf{\Pi}\mathbf{Y}_z(\xi,\eta,z)Kd\xi d\eta \tag{14b}$$

$$\mathbf{\Gamma}_p(x,y,z)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\mathbf{\Pi}_p\mathbf{w}(\xi,\eta,z)Kd\xi d\eta \tag{14c}$$

$$\mathbf{T}_p(x,y,z)=\mathbf{C}_{pz}\mathbf{T}_z(x,y,z)+\mathbf{C}_{pp}\mathbf{\Gamma}_p(x,y,z) \tag{14d}$$

$$\mathbf{\Gamma}_z(x,y,z) = \mathbf{C}_{zz}\mathbf{T}_z(x,y,z) + \mathbf{C}_{zp}\mathbf{\Gamma}_p(x,y,z). \tag{14e}$$

where $\rho = \sqrt{\xi^2+\eta^2}$; $K = e^{i(\xi x+\eta y)}$; $i=\sqrt{-1}$; The coordinate coefficient matrices $\mathbf{\Pi}$ and $\mathbf{\Pi}_p$ are defined by

$$\mathbf{\Pi} = \frac{1}{\rho}\begin{pmatrix} i\xi & i\eta & 0 \\ i\eta & -i\xi & 0 \\ 0 & 0 & \rho \end{pmatrix} \tag{14f}$$

$$\mathbf{\Pi}_p = -\frac{1}{\rho^2}\begin{pmatrix} \xi^2 & \xi\eta & 0 \\ \xi\eta & \frac{1}{2}(\eta^2-\xi^2) & 0 \\ \eta^2 & -\xi\eta & 0 \end{pmatrix} \tag{14g}$$

The four elastic parameter matrices $\mathbf{C}_{pz}, \mathbf{C}_{pp}$, $\mathbf{C}_{zz}$ and $\mathbf{C}_{zp}$ can vary with the depth z and are defined by

$$\mathbf{C}_{pz} = \frac{1}{c_3}\begin{pmatrix} 0 & 0 & c_2 \\ 0 & 0 & 0 \\ 0 & 0 & c_2 \end{pmatrix};\ \mathbf{C}_{pp} = \begin{pmatrix} c_p & 0 & c_p - 2c_5 \\ 0 & 2c_4 & 0 \\ c_p - 2c_5 & 0 & c_p \end{pmatrix}; \tag{14h}$$

$$\mathbf{C}_{zz} = \frac{1}{2c_4}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2c_4/c_2 \end{pmatrix};\quad \mathbf{C}_{zp} = -\frac{c_2}{c_3}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{14i}$$

$$c_p = c_1 - \frac{c_2^2}{c_3} \tag{14j}$$

The two unknown field variable vectors $\mathbf{w}(\xi,\eta,z)$ and $\mathbf{Y}_z(\xi,\eta,z)$ in the transform domain are defined as follows.

$$\mathbf{w} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix},\ \mathbf{Y}_z = \begin{pmatrix} \tau_1 \\ \tau_2 \\ \tau_3 \end{pmatrix}. \tag{15}$$

They can be re-expressed by $\mathbf{u}(x,y,z)$ and $\mathbf{T}_z(x,y,z)$ in Cartesian coordinate system as follows.

$$\mathbf{w}(\xi,\eta,z) = \mathbf{\Pi}^* \frac{\rho}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\mathbf{u}(x,y,z)K^* dxdy = \frac{\rho}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\mathbf{\Pi}^*\mathbf{u}(x,y,z)K^* dxdy \tag{16a}$$

$$\mathbf{Y}_z(\xi,\eta,z) = \mathbf{\Pi}^* \frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\mathbf{T}_z(x,y,z)K^* dxdy = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\mathbf{\Pi}^*\mathbf{T}_z(x,y,z)K^* dxdy \quad (16b)$$

where $K^* = e^{-i(\xi x+\eta y)}$; the coordinate coefficient matrix $\mathbf{\Pi}^*$ is defined by

$$\mathbf{\Pi}^* = \frac{1}{\rho}\begin{pmatrix} -i\xi & -i\eta & 0 \\ -i\eta & i\xi & 0 \\ 0 & 0 & \rho \end{pmatrix} \quad (16c)$$

Correspondingly, the body force vector $\mathbf{f}(x,y,z)$ and its counterpart $\mathbf{g}(\xi,\eta,z)$ in the transform domain have the following relations.

$$\mathbf{f}(x,y,z) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\mathbf{\Pi}\mathbf{g}(\xi,\eta,z)K d\xi d\eta \quad (17a)$$

$$\mathbf{g}(\xi,\eta,z) = \frac{\mathbf{\Pi}^*}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\mathbf{f}(x,y,z)K^* dxdy = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\mathbf{\Pi}^*\mathbf{f}(x,y,z)K^* dxdy \quad (17b)$$

where

$$\mathbf{f} = \begin{pmatrix} f_x \\ f_y \\ f_z \end{pmatrix}, \quad \mathbf{g} = \begin{pmatrix} g_1 \\ g_2 \\ g_3 \end{pmatrix}. \quad (17c)$$

3.3 Two sets of governing ordinary differential equations

The solution representation can be applied to the system of fifteen partial differential equations (2) to (6). The fifteen field variables in the physical domain can be replaced by the six field variables in the transform domain. The system of the fifteen linear partial differential equations can be then degenerated and reduced to a set of six first-order linear ordinary differential equations in terms of the six field variables in the transform domain. Due to the symmetry and anti-symmetry of the elastic solid of transverse isotropy in the *x*-*y* plane about the *z*-axis, the set of six first-order ordinary differential equations can be decoupled into the two sets of first order ordinary linear differential equations.

3.3.1 The first set for anti-axial-symmetry of materials

The first set is due to the anti-symmetry about the *z*-axis and has two linear ordinary differential equations with two field variables and variable coefficients with *z*. It can be expressed as follows.

$$\frac{d}{dz}\mathbf{V}(z) = \rho\mathbf{C}_v(z)\mathbf{V}(z) + \mathbf{G}_v(z) \tag{18a}$$

where $a \le z \le b$, $0 \le \rho < +\infty$, and

$$\mathbf{V}(\mathrm{z}) = \begin{pmatrix} w_2 \\ \tau_2 \end{pmatrix},\ \mathbf{G}_v = \begin{pmatrix} 0 \\ g_2 \end{pmatrix},\ \mathbf{C}_v(z) = \begin{bmatrix} 0 & 1/c_4 \\ c_5 & 0 \end{bmatrix} \tag{18b}$$

3.3.2 The second set for axial-symmetry of materials

The second set is due to the axial symmetry about the *z*-axis and has four linear ordinary differential equations with four field variables and variable coefficients with *z*. It can be expressed as follows.

$$\frac{d}{dz}\mathbf{U}(z) = \rho\mathbf{C}_u(z)\mathbf{U}(z) + \mathbf{G}_u(z) \tag{19a}$$

where $a \le z \le b$, $0 \le \rho < +\infty$, and

$$\mathbf{U}(z) = \begin{pmatrix} w_1 \\ w_3 \\ \tau_3 \\ \tau_1 \end{pmatrix},\ \mathbf{G}_u = \begin{pmatrix} 0 \\ 0 \\ g_3 \\ g_1 \end{pmatrix},\ \mathbf{C}_u(z) = \begin{bmatrix} 0 & -1 & 0 & 1/c_4 \\ c_2/c_3 & 0 & 1/c_3 & 0 \\ 0 & 0 & 0 & 1 \\ c_p & 0 & -c_2/c_3 & 0 \end{bmatrix} \tag{19b}$$

Most importantly, the matrix approach eliminates the two independent variables ξ and η in the six governing ordinary differential equations and preserves only the radial distance ρ of the material axial symmetry about the *z*-axis. The two coefficient matrices $\mathbf{C}_v(z)$ and $\mathbf{C}_u(z)$ contain only the five material parameters $(c_i = c_i(z), i = 1,2,3,4,5)$ and do not have the radial distance variable ρ. The five elastic parameters in $\mathbf{C}_v(z)$ (18b) and $\mathbf{C}_u(z)$ (19b) can be arbitrary functions of the depth *z*, i.e., $c_i = c_i(z), i = 1,2,3,4,5$.

3.4 The general solution of $\mathbf{V}(\mathrm{z})$ and $\mathbf{U}(z)$ for homogeneous materials

To solve a specific type of the boundary-value problem, the specific depth variation functions $c_i(z)(i = 1,2,3,4,5)$ have to be provided for the five elastic parameters. Consequently, a general solution can be derived for the two sets of ordinary differential equations with variable coefficients (18-19). The general solution can then be used to derive and formulate specific solutions for various boundary-value problems imposed on the non-homogeneous solid in the transform domain. The solutions in the physical domain can subsequently obtained by applying the solution representations (14) to the specific solutions in the transform domain.

If the five elastic parameters $(c_i, i=1,2,3,4,5)$ in (18-19) do not vary with the depth z (in other words, the solid is homogeneous), the two coefficient matrices $\mathbf{C}_v(z)$ and $\mathbf{C}_u(z)$ become constant coefficient matrices. General solutions in matrix forms can be found for the two sets of two and four linear ordinary differential equations with constant coefficient matrices. They are given in the ensuing derivations.

3.4.1 The general solution of $\mathbf{V}(z)$

The basic solution for the first set of two linear ordinary differential equations with constant elastic parameters (18) can be obtained as follows,

$$\mathbf{V}(z) = \mathbf{A}(z-z_1)\mathbf{V}(z_1) - \int_{z_1}^{z} \mathbf{A}(z-\varsigma)\mathbf{G}_v(\varsigma)d\varsigma \tag{20a}$$

where $z \geq z_1$ or $z \leq z_1$. The first basic square matrix $\mathbf{A}(z)$ is defined as follow.

$$\mathbf{A}(z) = \mathbf{B}(\gamma_0)e^{\gamma_0 \rho z} + \mathbf{B}(-\gamma_0)e^{-\gamma_0 \rho z} \tag{20b}$$

where the material characteristic root $\gamma_0 = \sqrt{c_5/c_4} > 0$. The material constant square matrix $\mathbf{B}(\chi)$ is defined as follows.

$$\mathbf{B}(\chi) = \frac{1}{2}\begin{bmatrix} 1 & \dfrac{1}{c_4\chi} \\ c_4\chi & 1 \end{bmatrix} \tag{20c}$$

The above two basic solution matrices have the following properties.

$$\det \mathbf{A}(z) = 1 \tag{21a}$$

$$\mathbf{A}(0) = \mathbf{I}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{21b}$$

$$\mathbf{A}(z)\mathbf{A}(z_1) = \mathbf{A}(z+z_1) \tag{21c}$$

$$\mathbf{A}(z)^{-1} = \mathbf{A}(-z) \tag{21d}$$

3.4.2 The general solution of $\mathbf{U}(z)$

Similarly, the general matrix solution for the second set of four linear ordinary differential equations with constant coefficients (19) can be obtained as follows,

$$\mathbf{U}(z) = \mathbf{Q}(z - z_1)\mathbf{U}(z_1) - \int_{z_1}^{z} \mathbf{Q}(z - \varsigma)\mathbf{G}_u(\varsigma)d\varsigma \tag{22a}$$

where $z \geq z_1$ or $z \leq z_1$. The second basic square matrix $\mathbf{Q}(z)$ is defined as follow.

$$\mathbf{Q}(z) = \begin{cases} \mathbf{C}(\gamma_1)e^{\gamma_1 \rho z} - \mathbf{C}(\gamma_2)e^{\gamma_2 \rho z} + \mathbf{C}(-\gamma_1)e^{-\gamma_1 \rho z} - \mathbf{C}(-\gamma_2)e^{-\gamma_2 \rho z} & \text{for } \Delta \neq 0 \\ \mathbf{D}(\gamma_3)e^{\gamma_3 \rho z} + \mathbf{D}(-\gamma_3)e^{-\gamma_3 \rho z} + \gamma_3 \rho z\left(\mathbf{E}(\gamma_3)e^{\gamma_3 \rho z} + \mathbf{E}(-\gamma_3)e^{-\gamma_3 \rho z}\right) & \text{for } \Delta = 0 \end{cases} \tag{22b}$$

where the material characteristic roots γ_1, γ_2 and γ_3 are defined as follows,

$$\gamma_1 = c_a + c_b > 0 \text{ and } \gamma_2 = c_a - c_b > 0 \qquad \text{for } \Delta > 0 \tag{23a}$$

$$\gamma_1 = c_a + i\,|c_b| \text{ and } \gamma_2 = c_a - i\,|c_b| \qquad \text{for } \Delta < 0 \tag{23b}$$

$$\gamma_3 = \left(c_1 / c_3\right)^{\frac{1}{4}} > 0 \qquad \text{for } \Delta = 0 \tag{23c}$$

where

$$c_a = \frac{\sqrt{(\sqrt{c_1 c_3} + c_2 + 2c_4)(\sqrt{c_1 c_3} - c_2)}}{2\sqrt{c_3 c_4}} > 0 \tag{24a}$$

$$c_b = \frac{\sqrt{(\sqrt{c_1 c_3} + c_2)\Delta}}{2\sqrt{c_3 c_4}} \tag{24b}$$

$$\Delta = \sqrt{c_1 c_3} - c_2 - 2c_4 \tag{24c}$$

where $\pm\gamma_1$ and $\pm\gamma_2$ are the four roots for $\Delta \neq 0$ and $\pm\gamma_3$ are the two equal roots for $\Delta = 0$ of the following material characteristic equation.

$$\gamma^4 + \frac{c_2^2 + 2c_2 c_4 - c_1 c_3}{c_3 c_4}\gamma^2 + \frac{c_1}{c_3} = 0 \tag{24d}$$

The three material constant square matrices $\mathbf{C}(\chi)$, $\mathbf{D}(\chi)$ and $\mathbf{E}(\chi)$ are defined as follows,

$$\mathbf{C}(\chi)=\frac{1}{2(\gamma_1^2-\gamma_2^2)}\begin{bmatrix}\chi^2+\frac{c_2}{c_3} & -(\chi+\frac{c_2}{c_3\chi}) & -\frac{c_2+c_4}{c_3c_4} & \frac{\chi}{c_4}-\frac{1}{c_3\chi} \\ \frac{c_2}{c_3}\chi+\frac{c_1}{c_3\chi} & \chi^2+c_q & \frac{\chi}{c_3}-\frac{c_1}{c_3c_4\chi} & \frac{c_2+c_4}{c_3c_4} \\ c_p & -\frac{c_p}{\chi} & \chi^2+c_q & (\chi+\frac{c_2}{c_3\chi}) \\ c_p\chi & -c_p & -(\frac{c_2}{c_3}\chi+\frac{c_1}{c_3\chi}) & \chi^2+\frac{c_2}{c_3}\end{bmatrix} \tag{25a}$$

$$\mathbf{D}(\chi)=\frac{1}{2}\begin{bmatrix}1 & -\frac{c_4}{c_3\chi^3} & 0 & \frac{c_2+3c_4}{2c_3c_4\chi^3} \\ -\frac{c_4}{c_3\chi} & 1 & \frac{c_2+3c_4}{2c_3c_4\chi} & 0 \\ 0 & \frac{2c_4(c_2+c_4)}{c_3\chi^3} & 1 & \frac{c_4}{c_3\chi^3} \\ \frac{2c_4(c_2+c_4)}{c_3\chi} & 0 & \frac{c_4}{c_3\chi} & 1\end{bmatrix} \tag{25b}$$

$$\mathbf{E}(\chi)=\frac{c_2+c_4}{2c_3\chi^2}\begin{bmatrix}1 & -\frac{1}{\chi} & -\frac{1}{2c_4} & \frac{1}{2c_4\chi} \\ \chi & -1 & -\frac{\chi}{2c_4} & \frac{1}{2c_4} \\ 2c_4 & -\frac{2c_4}{\chi} & -1 & \frac{1}{\chi} \\ 2c_4\chi & -2c_4 & -\chi & 1\end{bmatrix} \tag{25c}$$

$$c_p=c_1-\frac{c_2^2}{c_3};\quad c_q=\frac{c_2^2+c_2c_4-c_1c_3}{2c_3c_4} \tag{25d}$$

The above basic solution matrices have the following properties.

$$\det\mathbf{Q}(z)=1 \tag{26a}$$

$$\mathbf{Q}(0)=\mathbf{I}_4 \tag{26b}$$

$$\mathbf{Q}(z)\mathbf{Q}(z_1)=\mathbf{Q}(z+z_1) \tag{26c}$$

$$\mathbf{Q}(z)^{-1}=\mathbf{Q}(-z) \tag{26d}$$

$$\begin{aligned}&\lim_{\gamma_1\to\gamma_2=\gamma_3}\left[\mathbf{C}(\gamma_1)e^{\gamma_1\rho z}-\mathbf{C}(\gamma_2)e^{\gamma_2\rho z}+\mathbf{C}(-\gamma_1)e^{-\gamma_1\rho z}-\mathbf{C}(-\gamma_2)e^{-\gamma_2\rho z}\right]\\&=\mathbf{D}(\gamma_3)e^{\gamma_3\rho z}+\mathbf{D}(-\gamma_3)e^{-\gamma_3\rho z}+\gamma_3\rho z\left(\mathbf{E}(\gamma_3)e^{\gamma_3\rho z}+\mathbf{E}(-\gamma_3)e^{-\gamma_3\rho z}\right)\end{aligned} \tag{26e}$$

where $\mathbf{I}_4$ is the unit square matrix of the dimension 4.

3.4.3 The three constant matrices for isotropic solids

For isotropic solids (9), $\Delta = 0$ and the material characteristic roots $\gamma_0 = 1$ and $\gamma_3 = 1$. The three material constant square matrices $\mathbf{B}(\chi)$, $\mathbf{D}(\chi)$ and $\mathbf{E}(\chi)$ (20c, 25b, 25c) can be simplified with the two material parameters μ and α for isotropic solid as follows.

$$\mathbf{B}(\chi) = \frac{1}{2}\begin{bmatrix} 1 & \dfrac{1}{\mu\chi} \\ \mu\chi & 1 \end{bmatrix} \tag{27a}$$

$$\mathbf{D}(\chi) = \frac{1}{2\chi^3}\begin{bmatrix} \chi^3 & -\alpha & 0 & \dfrac{1+\alpha}{2\mu} \\ -\alpha\chi^2 & \chi^3 & \dfrac{1+\alpha}{2\mu}\chi^2 & 0 \\ 0 & 2\mu(1-\alpha) & \chi^3 & \alpha \\ 2\mu(1-\alpha)\chi^2 & 0 & \alpha\chi^2 & \chi^3 \end{bmatrix} \tag{27b}$$

$$\mathbf{E}(\chi) = \frac{(1-\alpha)}{2\chi^3}\begin{bmatrix} \chi & -1 & -\dfrac{\chi}{2\mu} & \dfrac{1}{2\mu} \\ \chi^2 & -\chi & -\dfrac{\chi^2}{2\mu} & \dfrac{\chi}{2\mu} \\ 2\mu\chi & -2\mu & -\chi & 1 \\ 2\mu\chi^2 & -\mu\chi & -\chi^2 & \chi \end{bmatrix} \tag{27c}$$

$$\alpha = \frac{\mu}{\lambda + 2\mu} = \frac{1-2\nu}{2(1-\nu)} \tag{27d}$$

3.5 The $\mathbf{V}(z)$ and $\mathbf{U}(z)$ of a homogeneous layer $a \le z \le b$

The general matrix solutions for one homogeneous elastic layer can be re-expressed in terms of the six boundary variables at $z = a$ as follows.

$$\mathbf{V}(z) = e^{\gamma_0\rho(z-a)}\mathbf{A}^p(z-a)\mathbf{V}(a) - \int_a^z e^{\gamma_0\rho(z-\varsigma)}\mathbf{A}^p(z-\varsigma)\mathbf{G}_v(\varsigma)d\varsigma \tag{28a}$$

$$\mathbf{U}(z) = e^{\gamma_a\rho(z-a)}\mathbf{Q}^p(z-a)\mathbf{U}(a) - \int_a^z e^{\gamma_a\rho(z-\varsigma)}\mathbf{Q}^p(z-\varsigma)\mathbf{G}_u(\varsigma)d\varsigma \tag{28b}$$

where the two solution square matrices $\mathbf{A}^p(s)$ and $\mathbf{Q}^p(s)$ have only the exponential functions with negative independent variable because $s = z - a \geq 0$ and/or $s = z - \varsigma \geq 0$

$$\mathbf{A}^p(s) = e^{-\gamma_0 \rho s}\mathbf{A}(s) = \mathbf{B}(\gamma_0) + e^{-2\gamma_0 \rho s}\mathbf{B}(-\gamma_0) \tag{28c}$$

$$\begin{aligned}\mathbf{Q}^p(s) &= e^{-\gamma_a \rho s}\mathbf{Q}(s) = \begin{cases} e^{-\gamma_1 \rho s}\mathbf{Q}(s) & \text{for } \Delta \neq 0 \\ e^{-\gamma_3 \rho s}\mathbf{Q}(s) & \text{for } \Delta = 0 \end{cases} \\ &= \begin{cases} \mathbf{C}(\gamma_1) - e^{-(\gamma_1-\gamma_2)\rho s}\mathbf{C}(\gamma_{2j}) + e^{-2\gamma_1 \rho s}\mathbf{C}(-\gamma_1) - e^{-(\gamma_1+\gamma_2)\rho s}\mathbf{C}(-\gamma_2) & \text{for } \Delta \neq 0 \\ \mathbf{D}(\gamma_3) + \gamma_3 \rho s \mathbf{E}(\gamma_3) + e^{-2\gamma_3 \rho s}\left[\mathbf{D}(-\gamma_3) + \gamma_3 \rho s \mathbf{E}(-\gamma_3)\right] & \text{for } \Delta = 0 \end{cases}\end{aligned} \tag{28d}$$

Secondly, the general matrix solutions for one homogeneous elastic layer can be re-expressed in terms of the six boundary variables at $z = b$ as follows.

$$\mathbf{V}(z) = e^{-\gamma_0 \rho (z-b)}\mathbf{A}^q(z-b)\mathbf{V}(b) - \int_b^z e^{-\gamma_0 \rho (z-\varsigma)}\mathbf{A}^q(z-\varsigma)\mathbf{G}_v(\varsigma)d\varsigma \tag{29a}$$

$$\mathbf{U}(z) = e^{-\gamma_a \rho (z-b)}\mathbf{Q}^q(z-b)\mathbf{U}(b) - \int_b^z e^{-\gamma_a \rho (z-\varsigma)}\mathbf{Q}^q(z-\varsigma)\mathbf{G}_u(\varsigma)d\varsigma \tag{29b}$$

where the two solution square matrices $\mathbf{A}^q(s)$ and $\mathbf{Q}^q(s)$ have only the exponential functions with negative independent variable because $s = z - b \leq 0$ and/or $s = z - \varsigma \leq 0$.

$$\mathbf{A}^q(s) = e^{\gamma_0 \rho s}\mathbf{A}(s) = e^{2\gamma_0 \rho s}\mathbf{B}(\gamma_0) + \mathbf{B}(-\gamma_0) \tag{29c}$$

$$\begin{aligned}\mathbf{Q}^q(s) &= e^{\gamma_a \rho s}\mathbf{Q}(s) = \begin{cases} e^{\gamma_1 \rho s}\mathbf{Q}(s) & \text{for } \Delta \neq 0 \\ e^{\gamma_3 \rho s}\mathbf{Q}(s) & \text{for } \Delta = 0 \end{cases} \\ &= \begin{cases} e^{2\gamma_1 \rho s}\mathbf{C}(\gamma_1) + \mathbf{C}(-\gamma_1) - e^{(\gamma_1+\gamma_2)\rho s}\mathbf{C}(\gamma_2) - e^{(\gamma_1-\gamma_2)\rho s}\mathbf{C}(-\gamma_2) & \text{for } \Delta \neq 0 \\ e^{2\gamma_3 \rho s}\left[\mathbf{D}(\gamma_3) + \gamma_3 \rho s \mathbf{E}(\gamma_3)\right] + \mathbf{D}(-\gamma_3) + \gamma_3 \rho s \mathbf{E}(-\gamma_3) & \text{for } \Delta = 0 \end{cases}\end{aligned} \tag{29d}$$

The four algebraic boundary equations governing the relationship of the four field variables $\mathbf{V}(a)$ and $\mathbf{V}(b)$ on the upper and lower boundaries $z = a$ and $z = b$ can be expressed as follows. It can be shown that they have only two independent equations.

$$\mathbf{V}(b) = e^{\gamma_0 \rho (b-a)}\mathbf{A}^p(b-a)\mathbf{V}(a) - \int_a^b e^{\gamma_0 \rho (b-\varsigma)}\mathbf{A}^p(b-\varsigma)\mathbf{G}_v(\varsigma)d\varsigma \tag{30a}$$

or

$$\mathbf{V}(a)=e^{-\gamma_0\rho(a-b)}\mathbf{A}^q(a-b)\mathbf{V}(b)-\int_b^a e^{-\gamma_0\rho(a-\varsigma)}\mathbf{A}^q(a-\varsigma)\mathbf{G}_v(\varsigma)d\varsigma \tag{30b}$$

The eight algebraic boundary equations governing the relationship of the eight field variables $\mathbf{U}(a)$ and $\mathbf{U}(b)$ on the upper and lower boundaries $z=a$ and $z=b$ can be expressed as follows. It can be shown that they have only four independent equations.

$$\mathbf{U}(b)=e^{\gamma_a\rho(b-a)}\mathbf{Q}^p(b-a)\mathbf{U}(a)-\int_a^b e^{\gamma_a\rho(b-\varsigma)}\mathbf{Q}^p(b-\varsigma)\mathbf{G}_u(\varsigma)d\varsigma \tag{31a}$$

or

$$\mathbf{U}(a)=e^{-\gamma_a\rho(a-b)}\mathbf{Q}^q(a-b)\mathbf{U}(b)-\int_b^a e^{-\gamma_a\rho(a-\varsigma)}\mathbf{Q}^q(a-\varsigma)\mathbf{G}_u(\varsigma)d\varsigma \tag{31b}$$

3.6 The $\mathbf{V}(z)$ and $\mathbf{U}(z)$ of a homogeneous upper halfspace $-\infty<z\le a$

The general matrix solutions for the upper homogeneous elastic half space are a special case of the general matrix solutions. They can be obtained as follows using the natural regularity conditions (i.e., the displacements shall be reduced to zero and the stresses shall be bounded as $z\to-\infty$).

$$\mathbf{V}(z)=e^{-\gamma_0\rho(a-z)}\mathbf{B}(\gamma_0)\mathbf{V}(a)+\int_{-\infty}^a \tfrac{z-\varsigma}{|z-\varsigma|}e^{-\gamma_0\rho|z-\varsigma|}\mathbf{B}(\tfrac{z-\varsigma}{|z-\varsigma|}\gamma_0)\mathbf{G}_v(\varsigma)d\varsigma \tag{32a}$$

$$\mathbf{U}(z)=\begin{cases}\left[e^{-\gamma_1\rho(a-z)}\mathbf{C}(\gamma_1)-e^{-\gamma_2\rho(a-z)}\mathbf{C}(\gamma_2)\right]\mathbf{U}(a) & \\ +\int_{-\infty}^a \tfrac{z-\varsigma}{|z-\varsigma|}\left[e^{-\gamma_1\rho|z-\varsigma|}\mathbf{C}(\tfrac{z-\varsigma}{|z-\varsigma|}\gamma_1)-e^{-\gamma_2\rho|z-\varsigma|}\mathbf{C}(\tfrac{z-\varsigma}{|z-\varsigma|}\gamma_2)\right]\mathbf{G}_u(\varsigma)d\varsigma & \text{for } \Delta\neq 0\\ e^{-\gamma_3\rho(a-z)}\left[\mathbf{D}(\gamma_3)+\gamma_3\rho(z-a)\mathbf{E}(\gamma_3)\right]\mathbf{U}(a) & \\ +\int_{-\infty}^a \tfrac{z-\varsigma}{|z-\varsigma|}e^{-\gamma_3\rho|z-\varsigma|}\left[\mathbf{D}(\tfrac{z-\varsigma}{|z-\varsigma|}\gamma_3)+\gamma_3\rho(z-\varsigma)\mathbf{E}(\tfrac{z-\varsigma}{|z-\varsigma|}\gamma_3)\right]\mathbf{G}_u(\varsigma)d\varsigma & \text{for } \Delta=0\end{cases} \tag{32b}$$

where $\tfrac{z-\varsigma}{|z-\varsigma|}=1$ if $z>\varsigma$, and $\tfrac{z-\varsigma}{|z-\varsigma|}=-1$ if $z<\varsigma$.

Putting $z=a$ into (32a), two algebraic boundary equations can be obtained as follows. It can be shown that they have only one independent boundary equation governing the relationship of the two field variables $\mathbf{V}(a)$ on the lower boundary $z=a$.

$$\mathbf{B}(-\gamma_0)\mathbf{V}(a)=-\mathbf{B}(-\gamma_0)\int_{-\infty}^a e^{-\gamma_0\rho(a-\varsigma)}\mathbf{G}_v(\varsigma)d\varsigma \tag{33}$$

Putting $z = a$ into (32b), four algebraic boundary equations can be obtained. It can be shown that they have only two independent equations governing the relationship of the four field variables $\mathbf{U}(a)$ on the lower boundary $z = a$.

$$\begin{cases} \left[\mathbf{C}(-\gamma_1) - \mathbf{C}(-\gamma_2)\right]\mathbf{U}(a) = -\mathbf{C}(-\gamma_1)\int\limits_{-\infty}^{a} e^{-\gamma_1\rho(a-\varsigma)}\mathbf{G}_u(\varsigma)d\varsigma + \mathbf{C}(-\gamma_2)\int\limits_{-\infty}^{a} e^{-\gamma_2\rho(a-\varsigma)}\mathbf{G}_u(\varsigma)d\varsigma & \text{for } \Delta \neq 0 \\ \mathbf{D}(-\gamma_3)\mathbf{U}(a) = -\mathbf{D}(-\gamma_3)\int\limits_{-\infty}^{a} e^{-\gamma_3\rho(a-\varsigma)}\mathbf{G}_u(\varsigma)d\varsigma - \mathbf{E}(-\gamma_3)\int\limits_{-\infty}^{a} \gamma_3\rho(a-\varsigma)e^{-\gamma_3\rho(a-\varsigma)}\mathbf{G}_u(\varsigma)d\varsigma & \text{for } \Delta = 0 \end{cases} \tag{34}$$

3.7 The $\mathbf{V}(z)$ and $\mathbf{U}(z)$ of a homogeneous lower halfspace $b \le z < +\infty$

Similarly, the general matrix solutions for the lower homogeneous elastic layer can be obtained as follows, using the three regularity conditions as $z \to +\infty$.

$$\mathbf{V}(z) = e^{-\gamma_0\rho(z-b)}\mathbf{B}(-\gamma_0)\mathbf{V}(b) + \int\limits_{b}^{+\infty} \tfrac{z-\varsigma}{|z-\varsigma|} e^{-\gamma_0\rho|z-\varsigma|}\mathbf{B}(\tfrac{z-\varsigma}{|z-\varsigma|}\gamma_0)\mathbf{G}_v(\varsigma)d\varsigma \tag{35a}$$

$$\mathbf{U}(z) = \begin{cases} \left[e^{-\gamma_1\rho(z-b)}\mathbf{C}(-\gamma_1) - e^{-\gamma_2\rho(z-b)}\mathbf{C}(-\gamma_2)\right]\mathbf{U}(b) & \\ + \int\limits_{b}^{+\infty} \tfrac{z-\varsigma}{|z-\varsigma|}\left[e^{-\gamma_1\rho|z-\varsigma|}\mathbf{C}(\tfrac{z-\varsigma}{|z-\varsigma|}\gamma_1) - e^{-\gamma_2\rho|z-\varsigma|}\mathbf{C}(\tfrac{z-\varsigma}{|z-\varsigma|}\gamma_2)\right]\mathbf{G}_u(\varsigma)d\varsigma & \text{for } \Delta \neq 0 \\ e^{-\gamma_3\rho(z-b)}\left[\mathbf{D}(-\gamma_3) + \gamma_3\rho(z-b)\mathbf{E}(-\gamma_3)\right]\mathbf{U}(b) & \\ + \int\limits_{b}^{+\infty} \tfrac{z-\varsigma}{|z-\varsigma|} e^{-\gamma_3\rho|z-\varsigma|}\left[\mathbf{D}(\tfrac{z-\varsigma}{|z-\varsigma|}\gamma_3) + \gamma_3\rho(z-\varsigma)\mathbf{E}(\tfrac{z-\varsigma}{|z-\varsigma|}\gamma_3)\right]\mathbf{G}_u(\varsigma)d\varsigma & \text{for } \Delta = 0 \end{cases} \tag{35b}$$

Putting $z = b$ into (35a), two algebraic boundary equations can be obtained as follows. It can be shown that they have only one independent boundary equation governing the relationship of the two field variables $\mathbf{V}(b)$ on the lower boundary $z = b$.

$$\mathbf{B}(\gamma_0)\mathbf{V}(b) = \mathbf{B}(\gamma_0)\int\limits_{b}^{+\infty} e^{-\gamma_0\rho(\varsigma-b)}\mathbf{G}_v(\varsigma)d\varsigma \tag{36}$$

Putting $z = b$ into (35b), four algebraic boundary equations can be obtained as follows. It can be shown that they have only two independent equations governing the relationship of the four field variables $\mathbf{U}(b)$ on the lower boundary $z = b$.

$$\begin{cases} \left[\mathbf{C}(\gamma_1) - \mathbf{C}(\gamma_2)\right]\mathbf{U}(b) = \mathbf{C}(\gamma_1)\int\limits_{b}^{+\infty} e^{-\gamma_1\rho(\varsigma-b)}\mathbf{G}_u(\varsigma)d\varsigma + \mathbf{C}(\gamma_2)\int\limits_{b}^{+\infty} e^{-\gamma_2\rho(\varsigma-b)}\mathbf{G}_u(\varsigma)d\varsigma & \text{for } \Delta \neq 0 \\ \mathbf{D}(\gamma_3)\mathbf{U}(b) = \mathbf{D}(\gamma_3)\int\limits_{b}^{+\infty} e^{-\gamma_3\rho(\varsigma-b)}\mathbf{G}_u(\varsigma)d\varsigma - \mathbf{E}(\gamma_3)\int\limits_{b}^{+\infty} \gamma_3\rho(\varsigma-b)e^{-\gamma_3\rho(\varsigma-b)}\mathbf{G}_u(\varsigma)d\varsigma & \text{for } \Delta = 0 \end{cases} \tag{37}$$

3.8 The $\mathbf{V}(z)$ and $\mathbf{U}(z)$ of a homogeneous infinite space $-\infty < z < +\infty$

The general matrix solutions for a homogeneous elastic solid of infinite space can be similarly obtained as follows, using the six regularity conditions as $z \to -\infty$ and $z \to +\infty$.

$$\mathbf{V}(z) = \int_{-\infty}^{+\infty} \tfrac{z-\varsigma}{|z-\varsigma|} e^{-\gamma_0 \rho |z-\varsigma|} \mathbf{B}(\tfrac{z-\varsigma}{|z-\varsigma|}\gamma_0) \mathbf{G}_v(\varsigma) d\varsigma \tag{38a}$$

$$\mathbf{U}(z) = \begin{cases} \int_{-\infty}^{+\infty} \tfrac{z-\varsigma}{|z-\varsigma|} \left[e^{-\gamma_1 \rho |z-\varsigma|} \mathbf{C}(\tfrac{z-\varsigma}{|z-\varsigma|}\gamma_1) - e^{-\gamma_2 \rho |z-\varsigma|} \mathbf{C}(\tfrac{z-\varsigma}{|z-\varsigma|}\gamma_2) \right] \mathbf{G}_u(\varsigma) d\varsigma & \text{for } \Delta \neq 0 \\ \int_{-\infty}^{+\infty} \tfrac{z-\varsigma}{|z-\varsigma|} e^{-\gamma_3 \rho |z-\varsigma|} \left[\mathbf{D}(\tfrac{z-\varsigma}{|z-\varsigma|}\gamma_3) + \gamma_3 \rho (z-\varsigma) \mathbf{E}(\tfrac{z-\varsigma}{|z-\varsigma|}\gamma_3) \right] \mathbf{G}_u(\varsigma) d\varsigma & \text{for } \Delta = 0 \end{cases} \tag{38b}$$

3.9 Summary notes

The mathematical approach presented above clear shows the following features. The five material parameters for the solutions of the boundary-value problems in the transversely isotropic solid are isolated and separated from the independent spatial coordinates (x, y, z) in the physical domain and/or (ξ, η, z) in the transform domain. They are presented in the four square matrices and the three material characteristic roots. The function of the two lateral coordinates (ξ, η) in the transform domain is consolidated into the function of the lateral radial distance $\rho (= \sqrt{\xi^2 + \eta^2})$.

The governing equations and general solutions are decoupled into the two systems of anti-symmetry and axial-symmetry about the vertical *z*-axis, which is consistent with the axial symmetry of the material property of the transversely isotropic solid about the *z*-axis. The general solutions are all presented in matrix form and the functions of the material matrices and the roots and the lateral radial distance ρ and the vertical coordinate z are clearly separated and identified.

The two independent variables ρ and z are always working together as a combined variable ρz. The body force term is also clear. There are only the following two types of integrations for the body force vector $\mathbf{g}(\xi, \eta, z)$.

$$\int_a^z e^{-\chi \rho (z-\varsigma)} \mathbf{g}(\xi, \eta, z) d\varsigma \tag{39a}$$

$$\int_a^z \rho (z-\varsigma) e^{-\chi \rho (z-\varsigma)} \mathbf{g}(\xi, \eta, z) d\varsigma \tag{39b}$$

where $\chi = \gamma_0, \gamma_1, \gamma_2$ or γ_3.

If the body force vector is concentrated on a horizontal plane, i.e., $\mathbf{g}(\xi,\eta,z)=\mathbf{g}(\xi,\eta)\delta(z-d)$, where δ is the Dirac delta function, the equations (39) become the following:

$$\int_a^z e^{-\chi\rho(z-\varsigma)}\mathbf{g}(\xi,\eta,z)d\varsigma = e^{-\chi\rho(z-d)}\mathbf{g}(\xi,\eta) \tag{40a}$$

$$\int_a^z \rho(z-\varsigma)e^{-\chi\rho(z-\varsigma)}\mathbf{g}(\xi,\eta,z)d\varsigma = \rho(z-d)e^{-\chi\rho(z-d)}\mathbf{g}(\xi,\eta) \tag{40b}$$

where $a<d<z$.

4 The Solution in Transform Domain

4.1 The boundary value problems

4.1.1 Material discretization

For simplicity and without loss of generality, this paper considered the specific type of boundary-value problems in an elastic solid of depth variable material properties and an infinite extent subject to a body force vector. The depth variations of material parameters are represented by a series of step functions (or a series of homogeneous and connected elastic layers). This material discretization technique can represent any variations in depth as long as the total number of the layers is large enough.

As a result, the *n*-layered elastic solid consisting of (*n*+2) dissimilar layers can be obtained. Each layer is homogeneous and has the five transversely isotropic elastic constants $(c_{1j},c_{2j},c_{3j},c_{4j},c_{5j})$, where $j=0,1,2,3,...,n,n+1$. The 0$^{\text{th}}$ layer occupies the region of upper halfspace. The $(n+1)^{\text{th}}$ layer occupies the region of lower halfspace. Between the upper and lower halfspaces, there are the *n* layers. The j$^{\text{th}}$ layer occupies a layer region of a finite thickness extent ($j=1,2,3,...,n$). In other words, (i) for $-\infty<z\le H_0^-$, it is the 0$^{\text{th}}$ homogeneous elastic halfspace; (ii) for $H_{j-1}^+\le z\le H_j^-$, it is the j^{th} homogeneous elastic layer with the layer thickness $h_j=H_j^+-H_{j-1}^-$; (iii) for $H_n^+\le z<+\infty$, it is the $(n+1)^{\text{th}}$ homogeneous elastic halfspace.

4.1.2 Interface conditions

Secondly, the interface connection conditions are needed to be prescribed for linking the layered solids together. For simplicity, they can be perfectly bonded together whilst other types of interface conditions can also be examined [76]. For this perfectly bonded interface connection, the displacement vector $\mathbf{u}(x,y,z)$ and the vertical stress vector $\mathbf{T}_z(x,y,z)$ are completely continuous at the horizontal interface between any two connected dissimilar elastic layers, *i.e.*,

$$\lim_{z\to H_j^-}\mathbf{u}(x,y,z)=\mathbf{u}(x,y,H_j)=\lim_{z\to H_j^+}\mathbf{u}(x,y,z) \tag{41a}$$

$$\lim_{z\to H_j^-}\mathbf{T}_z(x,y,z)=\mathbf{T}_z(x,y,H_j)=\lim_{z\to H_j^+}\mathbf{T}_z(x,y,z) \tag{41b}$$

4.1.3 Internal loading of body force vector

Thirdly, the distribution of the general body force vector $\mathbf{f}(x,y,z)$ is assumed to concentrate at an arbitrary horizontal plane $z=d$ in the layered elastic solids.

$$\mathbf{f}(x,y,z)=\mathbf{f}(x,y)\delta(z-d) \tag{42}$$

where $H_{k-1}^+\le d\le H_k^-$; δ is a Diract delta function, $1\le k\le n$. In particular, the situation for $-\infty<d\le H_0^-$ can be included by increasing a single layer of finite thickness $>H_0^- - d$ in the 0th elastic layer. Similarly, the situation for $H_n^+\le d<+\infty$ can be included by increasing a single layer of finite thickness $>d-H_n^+$ in the (n+1)th elastic layer.

Substituting the above body force condition into equations (17), the following results can be obtained for the internal loading variables.

$$\mathbf{g}(\xi,\eta,z)=\mathbf{g}(\xi,\eta)\delta(z-d) \tag{43a}$$

$$\mathbf{G}_v(\xi,\eta,z)=\mathbf{G}_v(\xi,\eta)\delta(z-d) \tag{43b}$$

$$\mathbf{G}_u(\xi,\eta,z)=\mathbf{G}_u(\xi,\eta)\delta(z-d) \tag{43c}$$

$$\mathbf{g}(\xi,\eta)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\mathbf{\Pi}^*\mathbf{f}(x,y)K^*dxdy \tag{43d}$$

4.1.4 The backward transfer matrix treatment

Using the general matrix solutions in the transform domain, many boundary value problems have been formulated and solved in n-layered elastic solids. Yue [11, 13, 14] has developed a backward transfer matrix method to obtain analytical solutions for elastic problems in the isotropic n-layered system. Details are respectively given in the following two sections for the solutions of $\mathbf{V}(z)$ and $\mathbf{U}(z)$ with transversely isotropic n-layered solids.

4.2 The solution of $\mathbf{V}(z)$

4.2.1 Solution of $\mathbf{V}(z)$ in terms of $\mathbf{V}(H_j^-)$ or $\mathbf{V}(H_j^+)$

The general matrix solutions in terms of the field variable matrix $\mathbf{V}(z)$ for each of the (n+2) layers can be expressed as follows.

(i) For the 0th layer of the upper halfspace $-\infty < z \le H_0^-$,

$$\mathbf{V}(z) = e^{-\gamma_{00}\rho(H_0 - z)}\mathbf{B}_0(\gamma_{00})\mathbf{V}(H_0^-) \tag{44a}$$

(ii) For the jth layer of finite thickness $H_{j-1}^+ \le z \le H_j^-$, $j = 1,2,...,k-1,k+1,...,n$ and $j \ne k$

$$\mathbf{V}(z) = e^{\gamma_{0j}\rho(z - H_{j-1})}\mathbf{A}_j^p(z - H_{j-1})\mathbf{V}(H_{j-1}^+) \tag{44b}$$

or

$$\mathbf{V}(z) = e^{\gamma_{0j}\rho(H_j - z)}\mathbf{A}_j^q(z - H_j)\mathbf{V}(H_j^-) \tag{44c}$$

(iii) For the kth layer of finite thickness $H_{k-1}^+ \le z \le H_k^-$, $k = 1,2,...,n-1,n$

$$\mathbf{V}(z) = \begin{cases} e^{\gamma_{0k}\rho(z-H_{k-1})}\mathbf{A}_k^p(z-H_{k-1})\mathbf{V}(H_{k-1}^+) & \text{for } H_{k-1}^+ \le z \le d^- \\ e^{\gamma_{0k}\rho(z-H_{k-1})}\mathbf{A}_k^p(z-H_{k-1})\mathbf{V}(H_{k-1}^+) - e^{\gamma_{0k}\rho(z-d)}\mathbf{A}_k^p(z-d)\mathbf{G}_v & \text{for } d^+ \le z \le H_k^- \end{cases} \tag{44d}$$

or

$$\mathbf{V}(z) = \begin{cases} e^{\gamma_{0k}\rho(H_{k-1}-z)}\mathbf{A}_k^q(z-H_k)\mathbf{V}(H_k^-) - e^{\gamma_{0k}\rho(d-z)}\mathbf{A}_k^q(z-d)\mathbf{G}_v & \text{for } H_{k-1}^+ \le z \le d^- \\ e^{\gamma_{0k}\rho(H_k-z)}\mathbf{A}_k^q(z-H_k)\mathbf{V}(H_k^-) & \text{for } d^+ \le z \le H_k^- \end{cases} \tag{44e}$$

(iv) For the (n+1)th layer of lower halfspace $H_n^+ \le z < +\infty$,

$$\mathbf{V}(z) = e^{-\gamma_{0(n+1)}\rho(z-H_n)}\mathbf{B}_{n+1}(-\gamma_{0(n+1)})\mathbf{V}(H_n^+) \tag{44f}$$

The basic solution matrices in equations (44b) to (44e) are defined as follows.

$$\mathbf{A}_j^p(s) = \mathbf{B}_j(\gamma_{0j}) + e^{-2\gamma_{0j}\rho s}\mathbf{B}_j(-\gamma_{0j}) \tag{45a}$$

$$\mathbf{A}_j^q(s) = e^{2\gamma_{0j}\rho s}\mathbf{B}_j(\gamma_{0j}) + \mathbf{B}_j(-\gamma_{0j}) \tag{45b}$$

where $\gamma_{0j} = \sqrt{c_{5j}/c_{4j}}$ and the constant matrix $\mathbf{B}_j(\chi)$ is given in (20c) for the jth elastic layer. The basic solution matrices do not have any functions of exponential growth, i.e., $e^{\gamma\rho|z|}$ with positive constant $\gamma(>0)$.

4.2.2 Solution of $\mathbf{V}(z)$ in terms of $\mathbf{V}(H_0)$ or $\mathbf{V}(H_n)$

Using the perfectly bonded interface condition (1), we have

$$\mathbf{V}_j(H_j^-) = \mathbf{V}_j(H_j) = \mathbf{V}_{j+1}(H_j) = \mathbf{V}_{j+1}(H_j^+) \tag{46}$$

Accordingly, using (44b), (44d) and (46), the matrix solution of $\mathbf{V}(z)$ for $H_0 \le z \le H_n$ can be uniformly expressed in terms of $\mathbf{V}(H_0)$ via the backward transfer matrix technique.

(i) For the *j*th layer of finite thickness $H_{j-1} \le z \le H_j$, $1 \le j \le k \le n$ and $z \le d^-$,

$$\mathbf{V}(z) = e^{\gamma_{0j}\rho(z-H_{j-1})+\gamma_{0(j-1)}\rho h_{j-1}+...+\gamma_{01}\rho h_1}\mathbf{A}_j^p(z-H_{j-1})\mathbf{A}_{j-1}^p(h_{j-1})...\mathbf{A}_1^p(h_1)\mathbf{V}(H_0) \tag{47a}$$

(ii) For the *j*th layer of finite thickness $H_{j-1} \le z \le H_{j-1}$, $n \ge j \ge k \ge 1$ and $z \ge d^+$,

$$\begin{aligned}\mathbf{V}(z) &= e^{\gamma_{0j}\rho(z-H_{j-1})+\gamma_{0(j-1)}\rho h_{j-1}+...+\gamma_{01}\rho h_1}\mathbf{A}_j^p(z-H_{j-1})\mathbf{A}_{j-1}^p(h_{j-1})...\mathbf{A}_1^p(h_1)\mathbf{V}(H_0)\\ &- e^{\gamma_{0j}\rho(z-H_{j-1})+\gamma_{0(j-1)}\rho h_{j-1}+...+\gamma_{0(k+1)}\rho h_{(k+1)}+\gamma_{0k}\rho(H_k-d)}\mathbf{A}_j^p(z-H_{j-1})\mathbf{A}_{j-1}^p(h_{j-1})...\mathbf{A}_{k+1}^p(h_{k+1})\mathbf{A}_k^p(H_k-d)\mathbf{G}_v\end{aligned} \tag{47b}$$

Similarly, using (44c), (44e) and (46) the matrix solution of $\mathbf{V}(z)$ for $H_0 \le z \le H_n$ can be uniformly expressed in terms of $\mathbf{V}(H_n)$ via the backward transfer matrix technique.

(i) For the *j*th layer of finite thickness $H_{j-1} \le z \le H_j$, $n \ge j \ge k \ge 1$ and $z \ge d^+$,

$$\mathbf{V}(z) = e^{\gamma_{0j}\rho(H_j-z)+\gamma_{0(j+1)}\rho h_{j+1}+...+\gamma_{0n}\rho h_n}\mathbf{A}_j^q(z-H_j)\mathbf{A}_{j+1}^q(h_{j+1})...\mathbf{A}_n^q(h_n)\mathbf{V}(H_n) \tag{48a}$$

(ii) For the *j*th layer of finite thickness $H_{j-1} \le z \le H_j$, $1 \le j \le k \le n$ and $z \le d^-$,

$$\begin{aligned}\mathbf{V}(z) &= e^{\gamma_{0j}\rho(H_j-z)+\gamma_{0(j+1)}\rho h_{j+1}+...+\gamma_{0n}\rho h_n}\mathbf{A}_j^q(z-H_j)\mathbf{A}_{j+1}^q(-h_{j+1})...\mathbf{A}_n^q(-h_n)\mathbf{V}(H_n)\\ &- e^{\gamma_{0j}\rho(H_j-z)+\gamma_{0(j+1)}\rho h_{j+1}+...+\gamma_{0(k-1)}\rho h_{(k-1)}+\gamma_{0k}\rho(d-H_{k-1})}\mathbf{A}_j^q(z-H_j)\mathbf{A}_{j+1}^q(-h_{j+1})...\mathbf{A}_{k-1}^q(-h_{k-1})\mathbf{A}_k^q(H_{k-1}-d)\mathbf{G}_v\end{aligned} \tag{48b}$$

4.2.3 Solution of $\mathbf{V}(H_0)$ in terms of $\mathbf{G}_v$

Substituting $z = H_n$ into equation (47b), the following two algebraic boundary equations can be obtained for governing $\mathbf{V}(H_0)$ and $\mathbf{V}(H_n)$.

$$\begin{aligned}\mathbf{V}(H_n) &= e^{\gamma_{0n}\rho h_n + \gamma_{0(n-1)}\rho h_{n-1} + ... + \gamma_{01}\rho h_1} \mathbf{A}_n^p(h_n)\mathbf{A}_{n-1}^p(h_{n-1})...\mathbf{A}_1^p(h_1)\mathbf{V}(H_0) \\ &- e^{\gamma_{0n}\rho h_n + \gamma_{0(n-1)}\rho h_{n-1} + ... + \gamma_{0(k+1)}\rho h_{(k+1)} + \gamma_{0k}\rho(H_k - d)} \mathbf{A}_n^p(h_n)\mathbf{A}_{n-1}^p(h_{n-1})...\mathbf{A}_{k+1}^p(h_{k+1})\mathbf{A}_k^p(H_k - d)\mathbf{G}_v\end{aligned} \tag{49}$$

Since the 0th layer is an upper elastic halfspace, it has one independent algebraic boundary equation governing $\mathbf{V}(H_0)$ from (33). It can be expressed as follows.

$$\mathbf{q}_0 \mathbf{V}(H_0) = 0 \tag{50a}$$

where

$$\mathbf{q}_0 = \begin{bmatrix} 1 & \dfrac{-1}{c_{40}\gamma_{00}} \end{bmatrix} \tag{50b}$$

Thirdly, since the (n+1)th layer is a lower elastic halfspace, it has one independent algebraic boundary equation governing $\mathbf{V}(H_n)$ from (36). It can be expressed as follows.

$$\mathbf{p}_{n+1}\mathbf{V}(H_n) = 0 \tag{51a}$$

$$\mathbf{p}_{n+1} = \begin{bmatrix} 1 & \dfrac{1}{c_{4(n+1)}\gamma_{0(n+1)}} \end{bmatrix} \tag{51b}$$

Using (49) and (51a), the one algebraic boundary equation (51a) can be re-expressed as follows.

$$\begin{aligned}&e^{\gamma_{0n}\rho h_n + \gamma_{0(n-1)}\rho h_{n-1} + ... + \gamma_{01}\rho h_1} \mathbf{p}_{n+1}\mathbf{A}_n^p(h_n)\mathbf{A}_{n-1}^p(h_{n-1})...\mathbf{A}_1^p(h_1)\mathbf{V}(H_0) = \\ &e^{\gamma_{0n}\rho h_n + \gamma_{0(n-1)}\rho h_{n-1} + ... + \gamma_{0(k+1)}\rho h_{(k+1)} + \gamma_{0k}\rho(H_k - d)} \mathbf{p}_{n+1}\mathbf{A}_n^p(h_n)\mathbf{A}_{n-1}^p(h_{n-1})...\mathbf{A}_{k+1}^p(h_{k+1})\mathbf{A}_k^p(H_k - d)\mathbf{G}_v\end{aligned} \tag{52}$$

Using equations (50a) and (52), the following set of two linear algebraic boundary equations can be obtained for the unknown two variables $\mathbf{V}(H_0)$.

$$\mathbf{q}_0 \mathbf{V}(H_0) = 0 \tag{53a}$$

$$\begin{aligned}&\mathbf{p}_{n+1}\mathbf{A}_n^p(h_n)\mathbf{A}_{n-1}^p(h_{n-1})...\mathbf{A}_1^p(h_1)\mathbf{V}(H_0) = e^{-\gamma_{01}\rho h_1 - ... - \gamma_{0(k-2)}\rho h_{k-2} - \gamma_{0(k-1)}\rho h_{k-1} - \gamma_{0k}\rho(d - H_{k-1})} \\ &\mathbf{p}_{n+1}\mathbf{A}_n^p(h_n)\mathbf{A}_{n-1}^p(h_{n-1})...\mathbf{A}_{k+1}^p(h_{k+1})\mathbf{A}_k^p(H_k - d)\mathbf{G}_v\end{aligned} \tag{53b}$$

Or they can be re-expressed in the following matrix form,

$$
\begin{aligned}
&\begin{bmatrix} \mathbf{q}_0 \\ \mathbf{p}_{n+1}\mathbf{A}_n^p(h_n)\mathbf{A}_{n-1}^p(h_{n-1})...\mathbf{A}_1^p(h_1) \end{bmatrix}\mathbf{V}(H_0) = e^{-\gamma_{01}\rho h_1 - ... -\gamma_{0(k-2)}\rho h_{k-2} - \gamma_{0(k-1)}\rho h_{k-1} - \gamma_{0k}\rho(d-H_{k-1})} \\
&\begin{bmatrix} \mathbf{0} \\ \mathbf{p}_{n+1}\mathbf{A}_n^p(h_n)\mathbf{A}_{n-1}^p(h_{n-1})...\mathbf{A}_{k+1}^p(h_{k+1})\mathbf{A}_k^p(H_k-d) \end{bmatrix}\mathbf{G}_v
\end{aligned} \tag{54}
$$

From (54), the solution of $\mathbf{V}(H_0)$ is given exactly as follows.

$$
\mathbf{V}(H_0) = e^{-\gamma_{01}\rho h_1 - ... -\gamma_{0(k-2)}\rho h_{k-2} - \gamma_{0(k-1)}\rho h_{k-1} - \gamma_{0k}\rho(d-H_{k-1})}\mathbf{N}_{Ap}\mathbf{G}_v \tag{55a}
$$

where $\mathbf{N}_{Ap}$ is given below and its exact expression is in Appendix A.

$$
\mathbf{N}_{Ap} = \mathbf{M}_{Ap}^{-1}\begin{bmatrix} \mathbf{0} \\ \mathbf{p}_{n+1}\mathbf{A}_n^p(h_n)\mathbf{A}_{n-1}^p(h_{n-1})...\mathbf{A}_{k+1}^p(h_{k+1})\mathbf{A}_k^p(H_k-d) \end{bmatrix} \tag{55b}
$$

where $\mathbf{M}_{Ap}^{-1}$ is the inverse matrix of the 2×2 coefficient matrix $\mathbf{M}_{Ap}$ and can be analytically derived in exact form (Appendix A). $\mathbf{M}_{Ap}$ is defined as follows.

$$
\mathbf{M}_{Ap} = \begin{bmatrix} \mathbf{q}_0 \\ \mathbf{p}_{n+1}\mathbf{A}_n^p(h_n)\mathbf{A}_{n-1}^p(h_{n-1})...\mathbf{A}_1^p(h_1) \end{bmatrix} \tag{55c}
$$

4.2.4 Solution of $\mathbf{V}(H_n)$ in terms of $\mathbf{G}_v$

Similarly, substituting $z = H_0$ into equations (48b), another two algebraic boundary equations can be obtained for governing $\mathbf{V}(H_0)$ and $\mathbf{V}(H_n)$. They are as follows.

$$
\begin{aligned}
\mathbf{V}(H_0) &= e^{\gamma_{01}\rho h_1 + \gamma_{02}\rho h_2 + ... + \gamma_{0n}\rho h_n}\mathbf{A}_1^q(-h_1)\mathbf{A}_2^q(-h_2)...\mathbf{A}_n^q(-h_n)\mathbf{V}(H_n) \\
&- e^{\gamma_{01}\rho h_1 + \gamma_{02}\rho h_2 + ... + \gamma_{0(k-1)}\rho h_{(k-1)} + \gamma_{0k}\rho(d-H_{k-1})}\mathbf{A}_1^q(-h_1)\mathbf{A}_2^q(-h_2)...\mathbf{A}_{k-1}^q(-h_{k-1})\mathbf{A}_k^q(H_{k-1}-d)\mathbf{G}_v
\end{aligned} \tag{56}
$$

Using (56), (50a) can be re-expressed as follows.

$$
\begin{aligned}
&e^{\gamma_{01}\rho h_1 + \gamma_{02}\rho h_2 + ... + \gamma_{0n}\rho h_n}\mathbf{q}_0\mathbf{A}_1^q(-h_1)\mathbf{A}_2^q(-h_2)...\mathbf{A}_n^q(-h_n)\mathbf{V}(H_n) = \\
&e^{\gamma_{01}\rho h_1 + \gamma_{02}\rho h_2 + ... + \gamma_{0(k-1)}\rho h_{(k-1)} + \gamma_{0k}\rho(d-H_{k-1})}\mathbf{q}_0\mathbf{A}_1^q(-h_1)\mathbf{A}_2^q(-h_2)...\mathbf{A}_{k-1}^q(-h_{k-1})\mathbf{A}_k^q(H_{k-1}-d)\mathbf{G}_v
\end{aligned} \tag{57}
$$

Using (51a) and (57), the following set of two linear algebraic equations can be obtained for the two unknown variables $\mathbf{V}(H_n)$ at the interface $z = H_n$.

$$
\mathbf{p}_{n+1}\mathbf{V}(H_n) = 0 \tag{58a}
$$

$$\mathbf{q}_0\mathbf{A}_1^q(-h_1)\mathbf{A}_2^q(-h_2)...\mathbf{A}_n^q(-h_n)\mathbf{V}(H_n) = e^{-\gamma_{0n}\rho h_n - \gamma_{0(n-1)}\rho h_{(n-1)} - ... - \gamma_{0(k+1)}\rho h_{(k+1)} - \gamma_{0k}\rho(H_k - d)}$$
$$\mathbf{q}_0\mathbf{A}_1^q(-h_1)\mathbf{A}_2^q(-h_2)...\mathbf{A}_{k-1}^q(-h_{k-1})\mathbf{A}_k^q(H_{k-1}-d)\mathbf{G}_v \tag{58b}$$

Or they can be re-expressed in the following matrix form,

$$\begin{bmatrix} \mathbf{p}_{n+1} \\ \mathbf{q}_0\mathbf{A}_1^q(-h_1)\mathbf{A}_2^q(-h_2)...\mathbf{A}_n^q(-h_n) \end{bmatrix}\mathbf{V}(H_n) = e^{-\gamma_{0n}\rho h_n - \gamma_{0(n-1)}\rho h_{(n-1)} - ... - \gamma_{0(k+1)}\rho h_{(k+1)} - \gamma_{0k}\rho(H_k - d)}$$
$$\begin{bmatrix} \mathbf{0} \\ \mathbf{q}_0\mathbf{A}_1^q(-h_1)\mathbf{A}_2^q(-h_2)...\mathbf{A}_{k-1}^q(-h_{k-1})\mathbf{A}_k^q(H_{k-1}-d) \end{bmatrix}\mathbf{G}_v \tag{59}$$

From (59), the solution of $\mathbf{V}(H_n)$ is given exactly as follows.

$$\mathbf{V}(H_n) = e^{-\gamma_{0n}\rho h_n - \gamma_{0(n-1)}\rho h_{(n-1)} - ... - \gamma_{0(k+1)}\rho h_{(k+1)} - \gamma_{0k}\rho(H_k - d)}\mathbf{N}_{Aq}\mathbf{G}_v \tag{60a}$$

where $\mathbf{N}_{Aq}$ is given below and its exact expression is in Appendix B.

$$\mathbf{N}_{Aq} = \mathbf{M}_{Aq}^{-1}\begin{bmatrix} \mathbf{0} \\ \mathbf{q}_0\mathbf{A}_1^q(-h_1)\mathbf{A}_2^q(-h_2)...\mathbf{A}_{k-1}^q(-h_{k-1})\mathbf{A}_k^q(H_{k-1}-d) \end{bmatrix} \tag{60b}$$

where $\mathbf{M}_{Aq}^{-1}$ is the inverse matrix of the 2×2 coefficient matrix $\mathbf{M}_{Aq}$ and can be analytically derived in exact form (Appendix B). $\mathbf{M}_{Aq}$ is defined as follows.

$$\mathbf{M}_{Aq} = \begin{bmatrix} \mathbf{p}_{n+1} \\ \mathbf{q}_0\mathbf{A}_1^q(-h_1)\mathbf{A}_2^q(-h_2)...\mathbf{A}_n^q(-h_n) \end{bmatrix} \tag{60c}$$

4.2.5 Solution of $\mathbf{V}(z)$ *in terms of* $\mathbf{G}_v$

Finally, substituting the solution of $\mathbf{V}(H_0)$ in the general matrix solutions expressed in terms of $\mathbf{V}(H_0)$, the solution of $\mathbf{V}(z)$ can be obtained for $-\infty < z \le d^-$. Similarly, substituting the solution of $\mathbf{V}(H_n)$ in the general matrix solutions expressed in terms of $\mathbf{V}(H_n)$, the solution of $\mathbf{V}(z)$ can be obtained for $d^+ \le z < +\infty$. Consequently, the solution of $\mathbf{V}(z)$ is expressed as follows.

$$\mathbf{V}(z) = \mathbf{\Psi}_V(\rho, z)\mathbf{G}_v \tag{61a}$$

where $-\infty < z < +\infty$, $0 \le \rho < +\infty$; $\mathbf{\Psi}_{V1}(\rho, z)$ is a square matrix of 2×2 elements and can be exactly expressed as follows.

(i) For the 0th layer of the upper halfspace $-\infty < z \le H_0 \le d^-$,

$$\mathbf{\Psi}_V(\rho, z) = e^{-\gamma_{00}\rho(H_0 - z) - \gamma_{01}\rho h_1 - \ldots - \gamma_{0(k-2)}\rho h_{k-2} - \gamma_{0(k-1)}\rho h_{k-1} - \gamma_{0k}\rho(d - H_{k-1})}\mathbf{B}_0(\gamma_{00})\mathbf{N}_{Ap} \tag{61b}$$

(ii) For the jth layer of finite thickness $H_{j-1} \le z \le H_j$ and $z \le d^-$, $j = 1,2,\ldots,k-1,k(\le n)$

$$\mathbf{\Psi}_V(\rho, z) = e^{-\gamma_{0j}\rho(H_j - z) - \gamma_{0(j+1)}\rho h_{j+1} - \ldots - \gamma_{0(k-1)}\rho h_{k-1} - \gamma_{0k}\rho(d - H_{k-1})}\mathbf{A}_j^p(z - H_{j-1})\mathbf{A}_{j-1}^p(h_{j-1})\ldots\mathbf{A}_1^p(h_1)\mathbf{N}_{Ap} \tag{61c}$$

(iii) For the jth layer of finite thickness $H_{j-1} \le z \le H_j$ and $z \ge d^+$, $j = k, k+1,\ldots,n$

$$\mathbf{\Psi}_V(\rho, z) = e^{-\gamma_{0j}\rho(z - H_{j-1}) - \gamma_{0(j-1)}\rho h_{(j-1)} - \ldots - \gamma_{0(k+1)}\rho h_{(k+1)} - \gamma_{0k}\rho(H_k - d)}\mathbf{A}_j^q(z - H_j)\mathbf{A}_{j+1}^q(h_{j+1})\ldots\mathbf{A}_n^q(h_n)\mathbf{N}_{Aq} \tag{61d}$$

(iv) For the (n+1)th layer of lower halfspace $d^+ \le H_n^+ \le z < +\infty$,

$$\mathbf{\Psi}_V(\rho, z) = e^{-\gamma_{0(n+1)}\rho(z - H_n) - \gamma_{0n}\rho h_n - \gamma_{0(n-1)}\rho h_{(n-1)} - \ldots - \gamma_{0(k+1)}\rho h_{(k+1)} - \gamma_{0k}\rho(H_k - d)}\mathbf{B}_{n+1}(-\gamma_{0(n+1)})\mathbf{N}_{Aq} \tag{61e}$$

4.3 The solution of $\mathbf{U}(z)$

4.3.1 Solution of $\mathbf{U}(z)$ in terms of $\mathbf{U}(H_j^-)$ or $\mathbf{U}(H_j^+)$

Similarly, the general matrix solutions in terms of the field variable matrix $\mathbf{U}(z)$ for each of the (n+2) layers can be expressed as follows.

(i) For the 0th layer of the upper halfspace $-\infty < z \le H_0^-$,

$$\mathbf{U}(z) = \begin{cases} \left[e^{-\gamma_{10}\rho(H_0 - z)}\mathbf{C}_0(\gamma_{10}) - e^{-\gamma_{20}\rho(H_0 - z)}\mathbf{C}_0(\gamma_{20})\right]\mathbf{U}(H_0^-) & \text{for } \Delta_0 \ne 0 \\ e^{-\gamma_{30}\rho(H_0 - z)}\left[\mathbf{D}_0(\gamma_{30}) + \gamma_{30}\rho(z - H_0)\mathbf{E}_0(\gamma_{30})\right]\mathbf{U}(H_0^-) & \text{for } \Delta_0 = 0 \end{cases} \tag{62a}$$

(ii) For the jth layer of finite thickness $H_{j-1}^+ \le z \le H_j^-$, $j = 1,2,\ldots,k-1,k+1,\ldots,n$ and $j \ne k$

$$\mathbf{U}(z) = e^{\gamma_{\alpha j}\rho(z - H_{j-1})}\mathbf{Q}_j^p(z - H_{j-1})\mathbf{U}(H_{j-1}^+) \tag{62b}$$

or

$$\mathbf{U}(z) = e^{\gamma_{\alpha j}\rho(H_{j-1} - z)}\mathbf{Q}_j^q(z - H_j)\mathbf{U}(H_j^-) \tag{62c}$$

(iii) For the kth layer of finite thickness $H_{k-1}^+ \le z \le H_k^-$, $k = 1,2,\ldots,n-1,n$

$$\mathbf{U}(z)=\begin{cases} e^{\gamma_{ak}\rho(z-H_{k-1})}\mathbf{Q}_k^p(z-H_{k-1})\mathbf{U}(H_{k-1}^+) & \text{for } H_{k-1}^+\le z\le d^- \\ e^{\gamma_{ak}\rho(z-H_{k-1})}\mathbf{Q}_k^p(z-H_{k-1})\mathbf{U}(H_{k-1}^+)-e^{\gamma_{ak}\rho(z-d)}\mathbf{Q}_k^p(z-d)\mathbf{G}_u & \text{for } d^+\le z\le H_k^- \end{cases} \quad (62d)$$

or

$$\mathbf{U}(z)=\begin{cases} e^{\gamma_{ak}\rho(H_{k-1}-z)}\mathbf{Q}_k^q(z-H_k)\mathbf{U}(H_k^-)-e^{\gamma_{ak}\rho(d-z)}\mathbf{Q}_k^q(z-d)\mathbf{G}_u & \text{for } H_{k-1}^+\le z\le d^- \\ e^{\gamma_{ak}\rho(H_k-z)}\mathbf{Q}_k^q(z-H_k)\mathbf{U}(H_k^-) & \text{for } d^+\le z\le H_k^- \end{cases} \quad (62e)$$

(iv) For the (n+1)th layer of lower halfspace $H_n^+\le z<+\infty$,

$$\mathbf{U}(z)=\begin{cases} \left[e^{-\gamma_{1(n+1)}\rho(z-H_n)}\mathbf{C}_{n+1}(-\gamma_{1(n+1)})-e^{-\gamma_{2(n+1)}\rho(z-H_n)}\mathbf{C}_{n+1}(-\gamma_{2(n+1)})\right]\mathbf{U}(H_n^+) & \text{for } \Delta\ne 0 \\ e^{-\gamma_{3(n+1)}\rho(z-H_n)}\left[\mathbf{D}_{n+1}(-\gamma_{3(n+1)})+\gamma_{3(n+1)}\rho(z-H_n)\mathbf{E}_{n+1}(-\gamma_{3(n+1)})\right]\mathbf{U}(H_n^+) & \text{for } \Delta=0 \end{cases} \quad (62f)$$

The basic solution matrices in equations (62a) to (62e) are defined as follows.

$$\mathbf{Q}_j^p(s)=\begin{cases} \mathbf{C}_j(\gamma_{1j})+e^{-2\gamma_{1j}\rho s}\mathbf{C}_j(-\gamma_{1j})-e^{-(\gamma_{1j}-\gamma_{2j})\rho s}\mathbf{C}_j(\gamma_{2j})-e^{-(\gamma_{1j}+\gamma_{2j})\rho s}\mathbf{C}_j(-\gamma_{2j}) & \text{for } \Delta\ne 0 \\ \mathbf{D}_j(\gamma_{3j})+\gamma_{3j}\rho s\mathbf{E}_j(\gamma_{3j})+e^{-2\gamma_{3j}\rho s}\left(\mathbf{D}_{0j}(-\gamma_{3j})+\gamma_{3j}\rho s\mathbf{E}_j(-\gamma_{3j})\right) & \text{for } \Delta=0 \end{cases} \quad (63a)$$

$$\mathbf{Q}_j^q(s)=\begin{cases} e^{2\gamma_{1j}\rho s}\mathbf{C}_j(\gamma_{1j})+\mathbf{C}_j(-\gamma_{1j})-e^{(\gamma_{1j}+\gamma_{2j})\rho s}\mathbf{C}_j(\gamma_{2j})-e^{(\gamma_{1j}-\gamma_{2j})\rho s}\mathbf{C}_j(-\gamma_{2j}) & \text{for } \Delta\ne 0 \\ e^{2\gamma_{3j}\rho s}\left(\mathbf{D}_j(\gamma_{3j})+\gamma_{3j}\rho s\mathbf{E}_j(\gamma_{3j})\right)+\mathbf{D}_j(-\gamma_{3j})+\gamma_{3j}\rho s\mathbf{E}_j(-\gamma_{3j}) & \text{for } \Delta=0 \end{cases} \quad (63b)$$

where the three constant matrices $\mathbf{C}_j(\chi)$, $\mathbf{D}_j(\chi)$ and $\mathbf{E}_j(\chi)$, and the roots $\gamma_{1j},\gamma_{2j},\gamma_{3j}$ are obtained by replacing the five elastic constants in equations (23-24) with the five elastic constants of the jth layer. The basic solution matrices do not have any functions of exponential growth, i.e., $e^{\gamma\rho|s|}$ with positive constant $\gamma(>0)$.

4.3.2 Solution of $\mathbf{U}(z)$ *in terms of* $\mathbf{U}(H_0)$ *or* $\mathbf{U}(H_n)$

Using the perfectly bonded interface condition (1), we have

$$\mathbf{U}_j(H_j^-)=\mathbf{U}_j(H_j)=\mathbf{U}_{j+1}(H_j)=\mathbf{U}_{j+1}(H_j^+) \quad (64)$$

Using (62b), (62d) and (64), $\mathbf{U}(z)$ can be uniformly expressed $\mathbf{U}(H_0)$ at $z=H_0$ via the backward transfer matrix technique.

(i) For the jth layer of finite thickness $H_{j-1}\le z\le H_j$, $1\le j\le k\le n$ and $z\le d^-$,

$$\mathbf{U}(z)=e^{\gamma_{aj}\rho(z-H_{j-1})+\gamma_{a(j-1)}\rho h_{j-1}+...+\gamma_{a1}\rho h_1}\mathbf{Q}_j^p(z-H_{j-1})\mathbf{Q}_{j-1}^p(h_{j-1})...\mathbf{Q}_1^p(h_1)\mathbf{U}(H_0) \quad (65a)$$

(ii) For the jth layer of finite thickness $H_{j-1}\le z\le H_{j-1}$, $n\ge j\ge k\ge 1$ and $z\ge d^+$,

$$\mathbf{U}(z)=e^{\gamma_{aj}\rho(z-H_{j-1})+\gamma_{a(j-1)}\rho h_{j-1}+...+\gamma_{a1}\rho h_1}\mathbf{Q}_j^p(z-H_{j-1})\mathbf{Q}_{j-1}^p(h_{j-1})...\mathbf{Q}_1^p(h_1)\mathbf{U}(H_0)$$
$$-e^{\gamma_{aj}\rho(z-H_{j-1})+\gamma_{a(j-1)}\rho h_{j-1}+...+\gamma_{a(k+1)}\rho h_{(k+1)}+\gamma_{ak}\rho(H_k-d)}\mathbf{Q}_j^p(z-H_{j-1})\mathbf{Q}_{j-1}^p(h_{j-1})...\mathbf{Q}_{k+1}^p(h_{k+1})\mathbf{Q}_k^p(H_k-d)\mathbf{G}_u \tag{65b}$$

Similarly, using (62c), (62e) and (64), $\mathbf{U}(z)$ can be uniformly expressed in terms of $\mathbf{U}(H_n)$ via the backward transfer matrix technique.

(i) For the *j*th layer of finite thickness $H_{j-1}\le z\le H_j$, $n\ge j\ge k\ge 1$ and $z\ge d^+$,

$$\mathbf{U}(z)=e^{\gamma_{0j}\rho(H_j-z)+\gamma_{0(j+1)}\rho h_{j+1}+...+\gamma_{0n}\rho h_n}\mathbf{Q}_j^q(z-H_j)\mathbf{Q}_{j+1}^q(h_{j+1})...\mathbf{Q}_n^q(h_n)\mathbf{U}(H_n) \tag{66a}$$

(ii) For the *j*th layer of finite thickness $H_{j-1}\le z\le H_j$, $1\le j\le k\le n$ and $z\le d^-$,

$$\mathbf{U}(z)=e^{\gamma_{aj}\rho(H_j-z)+\gamma_{a(j+1)}\rho h_{j+1}+...+\gamma_{an}\rho h_n}\mathbf{Q}_j^q(z-H_j)\mathbf{Q}_{j+1}^q(-h_{j+1})...\mathbf{Q}_n^q(-h_n)\mathbf{U}(H_n)$$
$$-e^{\gamma_{aj}\rho(H_j-z)+\gamma_{a(j+1)}\rho h_{j+1}+...+\gamma_{a(k-1)}\rho h_{(k-1)}+\gamma_{ak}\rho(d-H_{k-1})}\mathbf{Q}_j^q(z-H_j)\mathbf{Q}_{j+1}^q(-h_{j+1})...\mathbf{Q}_{k-1}^q(-h_{k-1})\mathbf{Q}_k^q(H_{k-1}-d)\mathbf{G}_u \tag{66b}$$

4.3.3 Solution of $\mathbf{U}(H_0)$ *in terms of* $\mathbf{G}_u$

Substituting $z=H_n$ into equations (65b), the following four algebraic boundary equations can be obtained for governing $\mathbf{U}(H_0)$ and $\mathbf{U}(H_n)$.

$$\begin{aligned}\mathbf{U}(H_n)&=e^{\gamma_{an}\rho h_n+\gamma_{a(n-1)}\rho h_{n-1}+...+\gamma_{a1}\rho h_1}\mathbf{Q}_n^p(h_n)\mathbf{Q}_{n-1}^p(h_{n-1})...\mathbf{Q}_1^p(h_1)\mathbf{U}(H_0)\\&-e^{\gamma_{an}\rho h_n+\gamma_{a(n-1)}\rho h_{n-1}+...+\gamma_{a(k+1)}\rho h_{(k+1)}+\gamma_{ak}\rho(H_k-d)}\mathbf{Q}_n^p(h_n)\mathbf{Q}_{n-1}^p(h_{n-1})...\mathbf{Q}_{k+1}^p(h_{k+1})\mathbf{Q}_k^p(H_k-d)\mathbf{G}_u\end{aligned} \tag{67}$$

Since the 0th layer is an upper elastic halfspace, it has two independent algebraic boundary equations governing $\mathbf{U}(H_0)$ from (34). It can be expressed as follows.

$$\mathbf{P}_{q0}\mathbf{U}(H_0)=\mathbf{0} \tag{68a}$$

where

$$\mathbf{P}_{q0}=\begin{bmatrix}2c_{a0} & 1-\dfrac{c_{20}}{\sqrt{c_{10}c_{30}}} & 0 & -\dfrac{1}{c_{40}}-\dfrac{1}{\sqrt{c_{10}c_{30}}}\\ \sqrt{\dfrac{c_{10}}{c_{30}}}-\dfrac{c_{20}}{c_{30}} & 2c_{a0} & -\dfrac{1}{c_{30}}-\dfrac{\sqrt{c_{10}c_{30}}}{c_{30}c_{40}} & 0\end{bmatrix} \tag{68b}$$

Thirdly, since the (n+1)th layer is a lower elastic halfspace, it has two independent algebraic boundary equations governing $\mathbf{U}(H_n)$ from (37). It can be expressed as follows.

$$\mathbf{P}_{p(n+1)}\mathbf{U}(H_n)=0 \tag{69a}$$

$$\mathbf{P}_{p(n+1)}=\begin{bmatrix} 2c_{a(n+1)} & \dfrac{c_{2(n+1)}}{\sqrt{c_{1(n+1)}c_{3(n+1)}}}-1 & 0 & \dfrac{1}{c_{4(n+1)}}+\dfrac{1}{\sqrt{c_{1(n+1)}c_{3(n+1)}}} \\ \dfrac{c_{2(n+1)}}{c_{3(n+1)}}-\sqrt{\dfrac{c_{1(n+1)}}{c_{3(n+1)}}} & 2c_{a(n+1)} & \dfrac{1}{c_{3(n+1)}}+\dfrac{\sqrt{c_{1(n+1)}c_{3(n+1)}}}{c_{3(n+1)}c_{4(n+1)}} & 0 \end{bmatrix} \tag{69b}$$

Using (67), (69a) can be re-expressed as follows.

$$\begin{aligned} &e^{\gamma_{an}\rho h_n+\gamma_{a(n-1)}\rho h_{n-1}+...+\gamma_{a1}\rho h_1}\mathbf{P}_{p(n+1)}\mathbf{Q}_n^p(h_n)\mathbf{Q}_{n-1}^p(h_{n-1})...\mathbf{Q}_1^p(h_1)\mathbf{U}(H_0)= \\ &e^{\gamma_{an}\rho h_n+\gamma_{a(n-1)}\rho h_{n-1}+...+\gamma_{a(k+1)}\rho h_{(k+1)}+\gamma_{ak}\rho(H_k-d)}\mathbf{P}_{p(n+1)}\mathbf{Q}_n^p(h_n)\mathbf{Q}_{n-1}^p(h_{n-1})...\mathbf{Q}_{k+1}^p(h_{k+1})\mathbf{Q}_k^p(H_k-d)\mathbf{G}_u \end{aligned} \tag{70}$$

Using (68a) and (70), the following set of four linear algebraic equations governing $\mathbf{U}(H_0)$ with four known variables can be obtained.

$$\mathbf{P}_{q0}\mathbf{U}(H_0)=\mathbf{0} \tag{71a}$$

$$\begin{aligned} &\mathbf{P}_{p(n+1)}\mathbf{Q}_n^p(h_n)\mathbf{Q}_{n-1}^p(h_{n-1})...\mathbf{Q}_1^p(h_1)\mathbf{U}(H_0)=e^{-\gamma_{a1}\rho h_1-...-\gamma_{a(k-2)}\rho h_{k-2}-\gamma_{a(k-1)}\rho h_{k-1}-\gamma_{ak}\rho(d-H_{k-1})} \\ &\mathbf{P}_{p(n+1)}\mathbf{Q}_n^p(h_n)\mathbf{Q}_{n-1}^p(h_{n-1})...\mathbf{Q}_{k+1}^p(h_{k+1})\mathbf{Q}_k^p(H_k-d)\mathbf{G}_u \end{aligned} \tag{71b}$$

Or they can be re-expressed in the following matrix form,

$$\begin{aligned} &\begin{bmatrix} \mathbf{P}_{q0} \\ \mathbf{P}_{p(n+1)}\mathbf{Q}_n^p(h_n)\mathbf{Q}_{n-1}^p(h_{n-1})...\mathbf{Q}_1^p(h_1) \end{bmatrix}\mathbf{U}(H_0)=e^{-\gamma_{a1}\rho h_1-...-\gamma_{a(k-2)}\rho h_{k-2}-\gamma_{a(k-1)}\rho h_{k-1}-\gamma_{ak}\rho(d-H_{k-1})} \\ &\begin{bmatrix} 0 \\ \mathbf{P}_{p(n+1)}\mathbf{Q}_n^p(h_n)\mathbf{Q}_{n-1}^p(h_{n-1})...\mathbf{Q}_{k+1}^p(h_{k+1})\mathbf{Q}_k^p(H_k-d) \end{bmatrix}\mathbf{G}_u \end{aligned} \tag{72}$$

From (25), the solution of $\mathbf{U}(H_0)$ is given exactly as follows.

$$\mathbf{U}(H_0)=e^{-\gamma_{a1}\rho h_1-...-\gamma_{a(k-2)}\rho h_{k-2}-\gamma_{a(k-1)}\rho h_{k-1}-\gamma_{ak}\rho(d-H_{k-1})}\mathbf{N}_{Qp}\mathbf{G}_u \tag{73a}$$

where $\mathbf{N}_{Qp}$ is given below and its exact expression is in Appendix C.

$$\mathbf{N}_{Qp}=\mathbf{M}_{Qp}^{-1}\begin{bmatrix} \mathbf{0} \\ \mathbf{P}_{p(n+1)}\mathbf{Q}_n^p(h_n)\mathbf{Q}_{n-1}^p(h_{n-1})...\mathbf{Q}_{k+1}^p(h_{k+1})\mathbf{Q}_k^p(H_k-d) \end{bmatrix} \tag{73b}$$

where $\mathbf{M}_{Qp}^{-1}$ is the inverse matrix of the 4×4 coefficient matrix $\mathbf{M}_{Qp}$ and can be analytically derived in exact form (Appendix C). $\mathbf{M}_{Qp}$ is defined as follows.

$$\mathbf{M}_{Qp} = \begin{bmatrix} \mathbf{P}_{q0} \\ \mathbf{P}_{p(n+1)}\mathbf{Q}_n^p(h_n)\mathbf{Q}_{n-1}^p(h_{n-1})...\mathbf{Q}_1^p(h_1) \end{bmatrix} \tag{73c}$$

4.3.4 Solution of $\mathbf{U}(H_n)$ *in terms of* $\mathbf{G}_u$

Similarly, substituting $z = H_0$ into equations (66b), another four algebraic boundary equations can be obtained for governing $\mathbf{U}(H_0)$ and $\mathbf{U}(H_n)$. They are as follows.

$$\begin{aligned} \mathbf{U}(H_0) &= e^{\gamma_{a1}\rho h_1 + \gamma_{a2}\rho h_2 + ... + \gamma_{an}\rho h_n}\mathbf{Q}_1^q(-h_1)\mathbf{Q}_2^q(-h_2)...\mathbf{Q}_n^q(-h_n)\mathbf{U}(H_n) \\ &- e^{\gamma_{a1}\rho h_1 + \gamma_{a2}\rho h_2 + ... + \gamma_{a(k-1)}\rho h_{(k-1)} + \gamma_{ak}\rho(d - H_{k-1})}\mathbf{Q}_1^q(-h_1)\mathbf{Q}_2^q(-h_2)...\mathbf{Q}_{k-1}^q(-h_{k-1})\mathbf{Q}_k^q(H_{k-1} - d)\mathbf{G}_u \end{aligned} \tag{74}$$

Using (74), (68a) can be re-expressed as follows.

$$\begin{aligned} & e^{\gamma_{a1}\rho h_1 + \gamma_{a2}\rho h_2 + ... + \gamma_{an}\rho h_n}\mathbf{P}_{q0}\mathbf{Q}_1^q(-h_1)\mathbf{Q}_2^q(-h_2)...\mathbf{Q}_n^q(-h_n)\mathbf{U}(H_n) = \\ & e^{\gamma_{a1}\rho h_1 + \gamma_{a2}\rho h_2 + ... + \gamma_{a(k-1)}\rho h_{(k-1)} + \gamma_{ak}\rho(d - H_{k-1})}\mathbf{P}_{q0}\mathbf{Q}_1^q(-h_1)\mathbf{Q}_2^q(-h_2)...\mathbf{Q}_{k-1}^q(-h_{k-1})\mathbf{Q}_k^q(H_{k-1} - d)\mathbf{G}_u \end{aligned} \tag{75}$$

Using (69a) and (75), the following set of four linear algebraic equations governing $\mathbf{U}(H_n)$ with four unknown variables can be obtained

$$\mathbf{P}_{p(n+1)}\mathbf{U}(H_n) = 0 \tag{76a}$$

$$\begin{aligned} & \mathbf{P}_{q0}\mathbf{Q}_1^q(-h_1)\mathbf{Q}_2^q(-h_2)...\mathbf{Q}_n^q(-h_n)\mathbf{U}(H_n) = e^{-\gamma_{an}\rho h_n - \gamma_{a(n-1)}\rho h_{n-1} - ... - \gamma_{a(k+1)}\rho h_{(k+1)} - \gamma_{ak}\rho(H_k - d)} \\ & \mathbf{P}_{q0}\mathbf{Q}_1^q(-h_1)\mathbf{Q}_2^q(-h_2)...\mathbf{Q}_{k-1}^q(-h_{k-1})\mathbf{Q}_k^q(H_{k-1} - d)\mathbf{G}_u \end{aligned} \tag{76b}$$

Or they can be re-expressed in the following matrix form,

$$\begin{aligned} & \begin{bmatrix} \mathbf{P}_{p(n+1)} \\ \mathbf{P}_{q0}\mathbf{Q}_1^q(-h_1)\mathbf{Q}_2^q(-h_2)...\mathbf{Q}_n^q(-h_n) \end{bmatrix}\mathbf{U}(H_n) = e^{-\gamma_{an}\rho h_n - \gamma_{a(n-1)}\rho h_{n-1} - ... - \gamma_{a(k+1)}\rho h_{(k+1)} - \gamma_{ak}\rho(H_k - d)} \\ & \begin{bmatrix} \mathbf{0} \\ \mathbf{P}_{q0}\mathbf{Q}_1^q(-h_1)\mathbf{Q}_2^q(-h_2)...\mathbf{Q}_{k-1}^q(-h_{k-1})\mathbf{Q}_k^q(H_{k-1} - d) \end{bmatrix}\mathbf{G}_u \end{aligned} \tag{77}$$

From (77), the solution of $\mathbf{U}(H_n)$ is given exactly as follows.

$$\mathbf{U}(H_n) = e^{-\gamma_{an}\rho h_n - \gamma_{a(n-1)}\rho h_{n-1} - ... - \gamma_{a(k+1)}\rho h_{(k+1)} - \gamma_{ak}\rho(H_k - d)}\mathbf{N}_{Qq}\mathbf{G}_u \tag{78a}$$

where $\mathbf{N}_{Qq}$ is given below and its exact expression is in Appendix D.

$$\mathbf{N}_{Qq} = \mathbf{M}_{Qq1}^{-1} \begin{bmatrix} \mathbf{0} \\ \mathbf{P}_{q0}\mathbf{Q}_1^q(-h_1)\mathbf{Q}_2^q(-h_2)\ldots\mathbf{Q}_{k-1}^q(-h_{k-1})\mathbf{Q}_k^q(H_{k-1}-d) \end{bmatrix} \tag{78b}$$

where $\mathbf{M}_{Qq}^{-1}$ is the inverse matrix of the 4×4 coefficient matrix $\mathbf{M}_{Qq}$ and can be analytically derived in exact form (Appendix D). $\mathbf{M}_{Qq}$ is defined as follows.

$$\mathbf{M}_{Qq} = \begin{bmatrix} \mathbf{P}_{p(n+1)} \\ \mathbf{P}_{q0}\mathbf{Q}_1^q(-h_1)\mathbf{Q}_2^q(-h_2)\ldots\mathbf{Q}_n^q(-h_n) \end{bmatrix} \tag{78c}$$

4.2.5 Solution of $\mathbf{U}(z)$ *in terms of* $\mathbf{G}_u$

Substituting the solution of $\mathbf{U}(H_0)$ in (62a) and (65a), the solution of $\mathbf{U}(z)$ can be obtained for $-\infty < z \le d^-$. Similarly, substituting the solution of $\mathbf{U}(H_n)$ in (62f) and (66a), the solution of $\mathbf{U}(z)$ can be obtained for $d^+ \le z < +\infty$. Consequently, the solution of $\mathbf{U}(z)$ is expressed as follows.

$$\mathbf{U}(z) = \mathbf{\Psi}_U(\rho, z)\mathbf{G}_u \tag{79a}$$

where $-\infty < z < +\infty$, $0 \le \rho < +\infty$; $\mathbf{\Psi}_U(\rho, z)$ is a square matrix of 4×4 elements and can be exactly expressed as follows.

(i) For the 0th layer of the upper halfspace $-\infty < z \le H_0$,

$$\mathbf{\Psi}_{U1}(\rho, z) = \begin{cases} e^{-\gamma_{a1}\rho h_1 - \ldots - \gamma_{a(k-2)}\rho h_{k-2} - \gamma_{a(k-1)}\rho h_{k-1} - \gamma_{ak}\rho(d-H_{k-1})} \\ \left[e^{-\gamma_{10}\rho(H_0-z)}\mathbf{C}_0(\gamma_{10}) - e^{-\gamma_{20}\rho(H_0-z)}\mathbf{C}_0(\gamma_{20})\right]\mathbf{N}_{Qp} & \text{for } \Delta_0 \ne 0 \\ e^{-\gamma_{30}\rho(H_0-z) - \gamma_{a1}\rho h_1 - \ldots - \gamma_{a(k-2)}\rho h_{k-2} - \gamma_{a(k-1)}\rho h_{k-1} - \gamma_{ak}\rho(d-H_{k-1})} \\ \left[\mathbf{D}_0(\gamma_{30}) + \gamma_{30}\rho(z-H_0)\mathbf{E}_0(\gamma_{30})\right]\mathbf{N}_{Qp} & \text{for } \Delta_0 = 0 \end{cases} \tag{79b}$$

(ii) For the *j*th layer of finite thickness $H_{j-1} \le z \le H_j$ and $z \le d^-$, $j = 1,2,\ldots,k-1,k(\le n)$

$$\mathbf{\Psi}_{U1}(\rho, z) = e^{-\gamma_{aj}\rho(H_j - z) - \gamma_{a(j+1)}\rho h_{j+1} - \ldots - \gamma_{a(k-1)}\rho h_{k-1} - \gamma_{ak}\rho(d-H_{k-1})}\mathbf{Q}_j^p(z-H_{j-1})\mathbf{Q}_{j-1}^p(h_{j-1})\ldots\mathbf{Q}_1^p(h_1)\mathbf{N}_{Qp} \tag{79c}$$

(iii) For the *j*th layer of finite thickness $H_{j-1} \le z \le H_j$ and $z \ge d^+$, $j = k, k+1, \ldots, n$

$$\mathbf{\Psi}_{U1}(\rho, z) = e^{-\gamma_{aj}\rho(z-H_{j-1}) - \gamma_{a(j-1)}\rho h_{(j-1)} - \ldots - \gamma_{a(k+1)}\rho h_{(k+1)} - \gamma_{ak}\rho(H_k - d)}\mathbf{Q}_j^q(z-H_j)\mathbf{Q}_{j+1}^q(h_{j+1})\ldots\mathbf{Q}_n^q(h_n)\mathbf{N}_{Qq} \tag{79d}$$

(iv) For the (*n*+1)th layer of lower halfspace $H_n \le z < +\infty$,

$$\mathbf{\Psi}_{U1}(\rho,z)=\begin{cases} e^{-\gamma_{an}\rho h_n-\gamma_{a(n-1)}\rho h_{n-1}-\ldots-\gamma_{a(k+1)}\rho h_{(k+1)}-\gamma_{ak}\rho(H_k-d)} \\ \left[e^{-\gamma_{1(n+1)}\rho(z-H_n)}\mathbf{C}_{n+1}(-\gamma_{1(n+1)})-e^{-\gamma_{2(n+1)}\rho(z-H_n)}\mathbf{C}_{n+1}(-\gamma_{2(n+1)})\right]\mathbf{N}_{Qq} & \text{for } \Delta_{n+1}\neq 0 \\ e^{-\gamma_{3(n+1)}\rho(z-H_n)-\gamma_{an}\rho h_n-\gamma_{a(n-1)}\rho h_{n-1}-\ldots-\gamma_{a(k+1)}\rho h_{(k+1)}-\gamma_{ak}\rho(H_k-d)} \\ \left[\mathbf{D}_{n+1}(-\gamma_{3(n+1)})+\gamma_{3(n+1)}\rho(z-H_n)\mathbf{E}_{n+1}(-\gamma_{3(n+1)})\right]\mathbf{N}_{Qq} & \text{for } \Delta_{n+1}=0 \end{cases} \tag{79e}$$

4.4 The solution of $\mathbf{w}(z)$ and $\mathbf{Y}_z(z)$ in terms of $\mathbf{g}(\xi,\eta)$

The solution of $\mathbf{V}(z)$ and $\mathbf{U}(z)$ is given in (61a) and (79a) in terms of the two loading matrices $\mathbf{G}_v(\xi,\eta)$ and $\mathbf{G}_u(\xi,\eta)$. $\mathbf{V}(z)$ and $\mathbf{U}(z)$ can be re-expressed as follows.

$$\mathbf{V}(z)=\begin{pmatrix} w_2 \\ \tau_2 \end{pmatrix}=\begin{pmatrix} \Phi_{bb}(\rho,z) & \Phi_{22}(\rho,z) \\ \Psi_{bb}(\rho,z) & \Psi_{22}(\rho,z) \end{pmatrix}\begin{pmatrix} 0 \\ g_2 \end{pmatrix} \tag{80a}$$

$$\mathbf{U}(z)=\begin{pmatrix} w_1 \\ w_3 \\ \tau_3 \\ \tau_1 \end{pmatrix}=\begin{pmatrix} \Phi_{ac}(\rho,z) & \Phi_{aa}(\rho,z) & \Phi_{13}(\rho,z) & \Phi_{11}(\rho,z) \\ \Phi_{cc}(\rho,z) & \Phi_{ca}(\rho,z) & \Phi_{33}(\rho,z) & \Phi_{31}(\rho,z) \\ \Psi_{cc}(\rho,z) & \Psi_{ca}(\rho,z) & \Psi_{33}(\rho,z) & \Psi_{31}(\rho,z) \\ \Psi_{ac}(\rho,z) & \Psi_{aa}(\rho,z) & \Psi_{13}(\rho,z) & \Psi_{11}(\rho,z) \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ g_3 \\ g_1 \end{pmatrix} \tag{80b}$$

As a result, the solution of $\mathbf{w}(z)$ and $\mathbf{Y}_z(z)$ can be expressed as follows in terms of the body force loading vector $\mathbf{g}(\xi,\eta)$

$$\mathbf{w}(\xi,\eta,z)=\mathbf{\Phi}(\rho,z)\mathbf{g}(\xi,\eta) \tag{81a}$$

$$\mathbf{Y}_z(\xi,\eta,z)=\mathbf{\Psi}(\rho,z)\mathbf{g}(\xi,\eta) \tag{81b}$$

where

$$\mathbf{\Phi}(\rho,z)=\begin{pmatrix} \Phi_{11}(\rho,z) & 0 & \Phi_{13}(\rho,z) \\ 0 & \Phi_{22}(\rho,z) & 0 \\ \Phi_{31}(\rho,z) & 0 & \Phi_{33}(\rho,z) \end{pmatrix} \tag{81c}$$

$$\boldsymbol{\psi}(\rho,z)=\begin{pmatrix} \Psi_{11}(\rho,z) & 0 & \Psi_{13}(\rho,z) \\ 0 & \Psi_{22}(\rho,z) & 0 \\ \Psi_{31}(\rho,z) & 0 & \Psi_{33}(\rho,z) \end{pmatrix} \tag{81d}$$

4.5 Summary Notes

The solutions of $\mathbf{V}(z)$ and $\mathbf{U}(z)$ (or $\mathbf{w}(z)$ and $\mathbf{Y}_z(z)$) are exactly derived, formulated and expressed in matrix forms. The many layers are treated with matrix production. It just increases one step of production of the associated two solution matrices for adding or increasing one layer solid with different elastic constants. The backward transfer matrix technique eliminates the existence of functions of the exponential growth in the solution in the transform domain and maintains the advantages of the conventional forward transfer matrix method.

The two inverse matrices have just the dimensions of 2×2 and 4×4 for $\mathbf{V}(z)$ and $\mathbf{U}(z)$, respectively and can be derived analytically. The solution matrices $\mathbf{\Psi}_V(\rho,z)$ and $\mathbf{\Psi}_U(\rho,z)$ (or $\mathbf{\Phi}(\rho,z)$ and $\mathbf{\Psi}(\rho,z)$) are functions of ρz, ρh_j, and the elastic constants $(c_{1j},c_{2j},c_{3j},c_{4j},c_{5j})$. They are independent to the actual forms of the internal loading vector $\mathbf{G}_v$ and $\mathbf{G}_u$ (or $\mathbf{g}(\xi,\eta)$), which makes them applicable to many actual distributions of the internal loadings.

5 The Solution in Physical Domain

5.1 General

In this section, the method for deriving and formulating solutions in physical domain is presented. The solution representations in (14) and (17) are used for this purpose. In addition to solution in Cartesian coordinate system, the solution in cylindrical coordinate system can also be derived and formulated and expressed directly and systematically.

5.2 The solution in Cartesian coordinate system

5.2.1 Solution in inverse double Fourier transform integrals

Using (14), (17) and (81), the solution of the field variable vectors $\mathbf{u}(x,y,z)$, $\mathbf{T}_z(x,y,z)$, and $\mathbf{\Gamma}_p(x,y,z)$ in the layered solid $(-\infty < x,y,z < +\infty)$ due to the internal loading concentrated on a horizontal plane, i.e., $\mathbf{f}(x,y,z)=\mathbf{f}(x,y)\delta(z-d)$, can be expressed as follows in the Cartesian coordinate system.

$$\mathbf{u}(x,y,z)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\frac{1}{\rho}\mathbf{\Pi\Phi}(\rho,z)\mathbf{\Pi}^*\tilde{\mathbf{f}}(\xi,\eta)Kd\xi d\eta \tag{82a}$$

$$\mathbf{T}_z(x,y,z)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\mathbf{\Pi\Psi}(\rho,z)\mathbf{\Pi}^*\tilde{\mathbf{f}}(\xi,\eta)Kd\xi d\eta \tag{82b}$$

$$\mathbf{\Gamma}_p(x,y,z)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\mathbf{\Pi}_p\mathbf{\Phi}(\rho,z)\mathbf{\Pi}^*\tilde{\mathbf{f}}(\xi,\eta)Kd\xi d\eta \tag{82c}$$

where $(-\infty < x, y, z < +\infty)$ and the body force vector $\tilde{\mathbf{f}}(\xi,\eta)$ in the transform domain is expressed as follows,

$$\tilde{\mathbf{f}}(\xi,\eta) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\mathbf{f}(x,y)K^{*}dxdy \tag{82d}$$

5.2.2 Solution for concentrated point body force vector

The fundamental solutions due to the point type of body force vector $\mathbf{f}$ concentrated at the original point (0,0,0) can be expressed as follows.

$$\mathbf{f}(x,y) = \delta(x)\delta(y)\mathbf{f}_c \tag{83a}$$

So, we have

$$\tilde{\mathbf{f}}(\xi,\eta) = \frac{\mathbf{f}_c}{2\pi} \tag{83b}$$

Consequently, the solution of $\mathbf{u}(x,y,z)$, $\mathbf{T}_z(x,y,z)$, and $\mathbf{\Gamma}_p(x,y,z)$ can be expressed as follows

$$\mathbf{u}(x,y,z) = \mathbf{G}_u(x,y,z)\mathbf{f}_c \tag{84a}$$

$$\mathbf{T}_z(x,y,z) = \mathbf{G}_z(x,y,z)\mathbf{f}_c \tag{84b}$$

$$\mathbf{\Gamma}_p(x,y,z) = \mathbf{G}_p(x,y,z)\mathbf{f}_c \tag{84c}$$

where the Green's functions are

$$2\pi\mathbf{G}_u(x,y,z) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\frac{1}{\rho}\mathbf{\Pi}\mathbf{\Phi}(\rho,z)\mathbf{\Pi}^{*}Kd\xi d\eta \tag{85a}$$

$$2\pi\mathbf{G}_z(x,y,z) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\mathbf{\Pi}\mathbf{\Psi}(\rho,z)\mathbf{\Pi}^{*}Kd\xi d\eta \tag{85b}$$

$$2\pi\mathbf{G}_p(x,y,z) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\mathbf{\Pi}_p\mathbf{\Phi}(\rho,z)\mathbf{\Pi}^{*}Kd\xi d\eta \tag{85c}$$

The relationships of the independent variables between the Cartesian and cylindrical coordinates in the physical domain can be defined as follows,

$$
\begin{aligned}
x &= r\cos\theta \\
y &= r\sin\theta \\
z &= z \\
r &= \sqrt{x^2+y^2}
\end{aligned}
\tag{86a}
$$

Similarly, the relationships of the independent variables between the Cartesian and cylindrical coordinates in the transform domain can be defined as follows,

$$
\begin{aligned}
\xi &= \rho\sin\varphi \\
\eta &= \rho\cos\varphi \\
z &= z \\
\rho &= \sqrt{\xi^2+\eta^2}
\end{aligned}
\tag{86b}
$$

The identity Bessel functions of order m can be expressed as follow.

$$
J_m = J_m(\rho r) = \frac{1}{2\pi}\int_0^{2\pi} e^{\pm i(\rho r\sin\theta - m\theta)}d\theta, \qquad m = 0,\pm1,\pm2,\pm3,... \tag{87a}
$$

$$
J_{-m} = (-1)^m J_m \qquad m = 0,\pm1,\pm2,\pm3,... \tag{87b}
$$

Consequently, the Green's functions in (85) can be simplified as the following Hankel transform integrals with the semi-infinite interval from 0 to $+\infty$.

$$
2\pi\mathbf{G}_u(x,y,z) = \int_0^{+\infty}\begin{pmatrix} \Phi_1 & 0 & 0 \\ 0 & \Phi_1 & 0 \\ 0 & 0 & \Phi_{33} \end{pmatrix} J_0 d\rho - \int_0^{+\infty}\begin{pmatrix} \frac{x^2-y^2}{r^2}\Phi_2 J_2 & \frac{2xy}{r^2}\Phi_2 J_2 & \frac{x}{r}\Phi_{13}J_1 \\ \frac{2xy}{r^2}\Phi_2 J_2 & \frac{x^2-y^2}{r^2}\Phi_2 J_2 & \frac{y}{r}\Phi_{13}J_1 \\ -\frac{x}{r}\Phi_{31}J_1 & -\frac{y}{r}\Phi_{31}J_1 & 0 \end{pmatrix} d\rho \tag{88a}
$$

$$
2\pi\mathbf{G}_z(x,y,z) = \int_0^{+\infty}\begin{pmatrix} \Psi_1 & 0 & 0 \\ 0 & \Psi_1 & 0 \\ 0 & 0 & \Psi_{33} \end{pmatrix} J_0 \rho d\rho - \int_0^{+\infty}\begin{pmatrix} \frac{x^2-y^2}{r^2}\Psi_2 J_2 & \frac{2xy}{r^2}\Psi_2 J_2 & \frac{x}{r}\Psi_{13}J_1 \\ \frac{2xy}{r^2}\Psi_2 J_2 & \frac{y^2-x^2}{r^2}\Psi_2 J_2 & \frac{y}{r}\Psi_{13}J_1 \\ -\frac{x}{r}\Psi_{31}J_1 & -\frac{y}{r}\Psi_{31}J_1 & 0 \end{pmatrix} \rho d\rho \tag{88b}
$$

$$
\begin{aligned}
2\pi\mathbf{G}_p(x,y,z) = &-\frac{1}{2}\int_0^{+\infty}\begin{pmatrix} \frac{x}{r}(2\Phi_1+2\Phi_2)J_1 & \frac{y}{r}\Phi_2 J_1 & \Phi_{13}J_0 \\ \frac{x}{r}\Phi_1 J_1 & \frac{x}{r}\Phi_1 J_1 & 0 \\ \frac{x}{r}\Phi_2 J_1 & \frac{y}{r}(2\Phi_1+2\Phi_2)J_1 & \Phi_{13}J_0 \end{pmatrix}\rho d\rho \\
&+\frac{1}{2}\int_0^{+\infty}\begin{pmatrix} \left(\frac{4x^3}{r^3}-\frac{3x}{r}\right)\Phi_2 J_3 & -\left(\frac{4y^3}{r^3}-\frac{3y}{r}\right)\Phi_2 J_3 & \frac{x^2-y^2}{r^2}\Phi_{13}J_2 \\ -\left(\frac{4y^3}{r^3}-\frac{3y}{r}\right)\Phi_2 J_3 & -\left(\frac{4x^3}{r^3}-\frac{3x}{r}\right)\Phi_2 J_3 & \frac{2xy}{r^2}\Phi_{13}J_1 \\ -\left(\frac{4x^3}{r^3}-\frac{3x}{r}\right)\Phi_2 J_3 & \left(\frac{4y^3}{r^3}-\frac{3y}{r}\right)\Phi_2 J_3 & \frac{y^2-x^2}{r^2}\Phi_{13}J_2 \end{pmatrix}\rho d\rho
\end{aligned}
\tag{88c}
$$

where $\Phi_1 = \frac{1}{2}(\Phi_{11} + \Phi_{22})$, $\Phi_2 = \frac{1}{2}(\Phi_{11} - \Phi_{22})$, $\Psi_1 = \frac{1}{2}(\Psi_{11} + \Psi_{22})$, $\Psi_1 = \frac{1}{2}(\Psi_{11} - \Psi_{22})$. There are only sixteen Hankel transform integrals in (88) and are defined as follows.

$$G_{u0}(r,z) = \int_0^{+\infty} \Phi(\rho,z) J_L(\rho r) d\rho \tag{89a}$$

$$G_{p1}(r,z) = \int_0^{+\infty} \Phi(\rho,z) J_L(\rho r) \rho d\rho \tag{89b}$$

$$G_{z1}(r,z) = \int_0^{+\infty} \Psi(\rho,z) J_L(\rho r) \rho d\rho \tag{89c}$$

where $\Phi(\rho,z) = \Phi_1, \Phi_2, \Phi_{13}, \Phi_{31}, \Phi_{33}$; $\Psi(\rho,z) = \Psi_1, \Psi_2, \Psi_{13}, \Psi_{31}, \Psi_{33}$;L = 1, 2, 3, or 4.

5.2.3 Solution in double convolution integrals

Using convolution integral theorem and the Green's functions (88), the solution for $\mathbf{f}(x,y,z) = \mathbf{f}(x,y)\delta(z-d)$ in (82) can be further expressed in the following two-dimensional convolution integrals.

$$\mathbf{u}(x,y,z) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \mathbf{G}_u(x-s, y-t, z)\mathbf{f}(s,t) ds dt \tag{90a}$$

$$\mathbf{T}_z(x,y,z) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \mathbf{G}_z(x-s, y-t, z)\mathbf{f}(s,t) ds dt \tag{90b}$$

$$\mathbf{\Gamma}_p(x,y,z) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \mathbf{G}_p(x-s, y-t, z)\mathbf{f}(s,t) ds dt \tag{90c}$$

where $-\infty < x, y, z < +\infty$.

For a general body force loading $\mathbf{f}(x,y,z)$, the solution can be derived from the following three-dimensional convolution integrals.

$$\mathbf{u}(x,y,z,f) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \mathbf{G}_u(x-s, y-t, z-\varsigma)\mathbf{f}(s,t,\varsigma) ds dt d\varsigma \tag{91a}$$

$$\mathbf{T}_z(x,y,z) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \mathbf{G}_z(x-s, y-t, z-\varsigma)\mathbf{f}(s,t,\varsigma) ds dt d\varsigma \tag{91b}$$

$$\boldsymbol{\Gamma}_p(x,y,z)=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\mathbf{G}_p(x-s,y-t,z-\varsigma)\mathbf{f}(s,t,\varsigma)dsdtd\varsigma \tag{91c}$$

where $-\infty < x,y,z < +\infty$.

5.3 The solution in cylindrical coordinate system

5.3.1 General matrix solution representation

The five field variable vectors can be defined as follows in the cylindrical coordinate system $(0 \le r < +\infty; 0 \le \theta < 2\pi; a \le z \le b)$.

$$\mathbf{u}=\begin{pmatrix}u_r\\u_\theta\\u_z\end{pmatrix},\ \mathbf{T}_z=\begin{pmatrix}\sigma_{rz}\\\sigma_{\theta z}\\\sigma_{zz}\end{pmatrix},\mathbf{T}_p=\begin{pmatrix}\sigma_{rr}\\\sigma_{r\theta}\\\sigma_{\theta\theta}\end{pmatrix},\boldsymbol{\Gamma}_p=\begin{pmatrix}\varepsilon_{rr}\\\varepsilon_{r\theta}\\\varepsilon_{\theta\theta}\end{pmatrix},\boldsymbol{\Gamma}_z=\begin{pmatrix}\varepsilon_{rz}\\\varepsilon_{\theta z}\\\varepsilon_{zz}\end{pmatrix} \tag{92}$$

The above five sets of vectors for the fifteen field variables in the physical domain can be also represented by two unknown vectors $\mathbf{w}(\rho,\varphi,z)$ and $\mathbf{Y}_z(\rho,\varphi,z)$ in the transform domain in the cylindrical coordinate system [11, 12]. The solution representations can be given as follows.

$$\mathbf{u}(r,\theta,z)=\frac{1}{2\pi}\int_0^{+\infty}\int_0^{2\pi}\boldsymbol{\Pi}_c\mathbf{w}(\rho,\varphi,z)K_c d\varphi d\rho \tag{93a}$$

$$\mathbf{T}_z(r,\theta,z)=\frac{1}{2\pi}\int_0^{+\infty}\int_0^{2\pi}\boldsymbol{\Pi}_c\mathbf{Y}_z(\rho,\varphi,z)K_c\rho d\varphi d\rho \tag{93b}$$

$$\boldsymbol{\Gamma}_p(r,\theta,z)=\frac{1}{2\pi}\int_0^{+\infty}\int_0^{2\pi}\boldsymbol{\Pi}_{cp}\mathbf{w}(\rho,\varphi,z)K_c\rho d\varphi d\rho \tag{93c}$$

$$\mathbf{T}_p(r,\theta,z)=\mathbf{C}_{pz}\mathbf{T}_z(r,\theta,z)+\mathbf{C}_{pp}\boldsymbol{\Gamma}_p(r,\theta,z) \tag{93d}$$

$$\boldsymbol{\Gamma}_z(r,\theta,z)=\mathbf{C}_{zz}\mathbf{T}_z(r,\theta,z)+\mathbf{C}_{zp}\boldsymbol{\Gamma}_p(r,\theta,z) \tag{93e}$$

where $K_c = e^{i\rho r\sin(\theta+\varphi)}$; $i=\sqrt{-1}$; The four elastic parameter matrices $\mathbf{C}_{pz},\mathbf{C}_{pp}$, $\mathbf{C}_{zz}$ and $\mathbf{C}_{zp}$ are given in (14h) to (14j). The coordinate coefficient matrices $\boldsymbol{\Pi}_c$ and $\boldsymbol{\Pi}_{cp}$ are defined by

$$\boldsymbol{\Pi}_c=\begin{pmatrix}i\sin(\theta+\varphi) & i\cos(\theta+\varphi) & 0\\ i\cos(\theta+\varphi) & -i\sin(\theta+\varphi) & 0\\ 0 & 0 & 1\end{pmatrix} \tag{94a}$$

$$\mathbf{\Pi}_{cp} = -\frac{1}{2}\begin{pmatrix} 1-\cos 2(\theta+\varphi) & \sin 2(\theta+\varphi) & 0 \\ \sin 2(\theta+\varphi) & \cos 2(\theta+\varphi) & 0 \\ 1+\cos 2(\theta+\varphi) & -\sin 2(\theta+\varphi) & 0 \end{pmatrix} \tag{94b}$$

The two unknown field variable vectors $\mathbf{w}(\rho,\varphi,z)$ and $\mathbf{Y}_z(\rho,\varphi,z)$ are defined as follows.

$$\mathbf{w} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}, \ \mathbf{Y}_z = \begin{pmatrix} \tau_1 \\ \tau_2 \\ \tau_3 \end{pmatrix}. \tag{95}$$

They can be re-expressed by $\mathbf{u}(r,\theta,z)$ and $\mathbf{T}_z(r,\theta,z)$ in the cylindrical coordinate system as follows.

$$\mathbf{w}(\rho,\varphi,z) = \frac{\rho}{2\pi}\int_0^{+\infty}\int_0^{2\pi}\mathbf{\Pi}_c^*\mathbf{u}(r,\theta,z)K_c^* r\,d\theta\,dr \tag{96a}$$

$$\mathbf{Y}_z(\rho,\varphi,z) = \frac{1}{2\pi}\int_0^{+\infty}\int_0^{2\pi}\mathbf{\Pi}_c^*\mathbf{T}_z(r,\theta,z)K_c^* r\,d\theta\,dr \tag{96b}$$

where $K_c^* = e^{-i\rho r\sin(\theta+\varphi)}$; the coordinate coefficient matrices $\mathbf{\Pi}_c^*$ is defined by

$$\mathbf{\Pi}_c^* = \begin{pmatrix} -i\sin(\theta+\varphi) & -i\cos(\theta+\varphi) & 0 \\ -i\cos(\theta+\varphi) & i\sin(\theta+\varphi) & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{96c}$$

Similarly, the body force vector $\mathbf{f}(r,\theta,z)$ and its counterpart $\mathbf{g}(\rho,\varphi,z)$ in the cylindrical coordinate system can be expressed as follows.

$$\mathbf{f}(r,\theta,z) = \frac{1}{2\pi}\int_0^{+\infty}\int_0^{2\pi}\mathbf{\Pi}_c\mathbf{g}(\rho,\varphi,z)K_c\rho\,d\varphi\,d\rho \tag{97a}$$

$$\mathbf{g}(\rho,\varphi,z) = \frac{1}{2\pi}\int_0^{+\infty}\int_0^{2\pi}\mathbf{\Pi}_c^*\mathbf{f}(r,\theta,z)K_c^* r\,d\theta\,dr \tag{97b}$$

where

$$\mathbf{f} = \begin{pmatrix} f_r \\ f_\theta \\ f_z \end{pmatrix}; \quad \mathbf{g} = \begin{pmatrix} g_1 \\ g_2 \\ g_3 \end{pmatrix}. \tag{97c}$$

5.3.2 Solution in inverse double Fourier transform integrals

It has been shown that the partial differential equations (2) to (6) can be reduced to the two sets of two and four first-order linear ordinary different equations (18) and (19). Similar to the formulation process in the Cartesian coordinate system, the solution of $\mathbf{w}(z)$ and $\mathbf{Y}_z(z)$ due to the internal loading of $\mathbf{f}(x,y,z)=\mathbf{f}(x,y)\delta(z-d)$ can also be expressed as follows.

$$\mathbf{w}(z)=\mathbf{\Phi}(\rho,z)\mathbf{g}(\rho,\varphi) \tag{98a}$$

$$\mathbf{Y}_z(z)=\mathbf{\Psi}(\rho,z)\mathbf{g}(\rho,\varphi) \tag{98b}$$

$$\mathbf{g}(\rho,\varphi)=\frac{1}{2\pi}\int_0^{+\infty}\int_0^{2\pi}\mathbf{\Pi}_c^*\mathbf{f}(r,\theta)K_c^*rd\theta dr \tag{98c}$$

where $\mathbf{\Phi}(\rho,z)$ and $\mathbf{\Psi}(\rho,z)$ are given in (81c) and (81d) respectively.

The solution of $\mathbf{u}(r,\theta,z)$, $\mathbf{T}_z(r,\theta,z)$ and $\mathbf{\Gamma}_p(r,\theta,z)$ in (93) due to the internal loading of $\mathbf{f}(r,\theta,z)=\mathbf{f}(r,\theta)\delta(z-d)$ can be expressed as follows in the cylindrical coordinate systems.

$$\mathbf{u}(r,\theta,z)=\frac{1}{2\pi}\int_0^{+\infty}\int_0^{2\pi}\mathbf{\Pi}_c\mathbf{\Phi}(\rho,z)\mathbf{g}(\rho,\varphi)K_c d\varphi d\rho \tag{99a}$$

$$\mathbf{T}_z(r,\theta,z)=\frac{1}{2\pi}\int_0^{+\infty}\int_0^{2\pi}\mathbf{\Pi}_c\mathbf{\Psi}(\rho,z)\mathbf{g}(\rho,\varphi)K_c\rho d\varphi d\rho \tag{99b}$$

$$\mathbf{\Gamma}_p(r,\theta,z)=\frac{1}{2\pi}\int_0^{+\infty}\int_0^{2\pi}\mathbf{\Pi}_{cp}\mathbf{\Phi}(\rho,z)\mathbf{g}(\rho,\varphi)K_c\rho d\varphi d\rho \tag{99c}$$

where $0\le r<+\infty$; $0\le\theta<2\pi$; $-\infty<z<+\infty$.

5.3.3 Solution in Fourier series and Hankel transform integrals

The solution in (99) can be further expressed in terms of Fourier series and Hankel transforms as follows.

$$\mathbf{u}(r,\theta,z)=\sum_{m=-\infty}^{m=+\infty}\int_0^{+\infty}\mathbf{\Pi}_m(\rho r)\mathbf{\Phi}(\rho,z)\mathbf{g}_m(\rho)d\rho e^{im\theta} \tag{100a}$$

$$\mathbf{T}_z(r,\theta,z)=\sum_{m=-\infty}^{m=+\infty}\int_0^{+\infty}\mathbf{\Pi}_m(\rho r)\mathbf{\Psi}(\rho,z)\mathbf{g}_m(\rho)\rho d\rho e^{im\theta} \tag{100b}$$

$$\boldsymbol{\Gamma}_p(r,\theta,z) = \sum_{m=-\infty}^{m=+\infty} \int_0^{+\infty} \boldsymbol{\Pi}_{pm}(\rho r)\boldsymbol{\Phi}(\rho,z)\mathbf{g}_m(\rho)d\rho e^{im\theta} \tag{100c}$$

where

$$\boldsymbol{\Pi}_{cm}(\rho r) = \frac{1}{2}\begin{pmatrix} J_{m-1} - J_{m+1} & i(J_{m-1} + J_{m+1}) & 0 \\ i(J_{m-1} + J_{m+1}) & -(J_{m-1} - J_{m+1}) & 0 \\ 0 & 0 & 2J_m \end{pmatrix} \tag{101a}$$

$$\boldsymbol{\Pi}_{cpm}(\rho r) = -\frac{1}{2}\begin{pmatrix} J_m & 0 & 0 \\ 0 & 0 & 0 \\ J_m & 0 & 0 \end{pmatrix} + \frac{1}{4}\begin{pmatrix} J_{m-2} + J_{m+2} & i(J_{m-2} - J_{m+2}) & 0 \\ i(J_{m-2} - J_{m+2}) & -(J_{m-2} + J_{m+2}) & 0 \\ -(J_{m-2} + J_{m+2}) & -i(J_{m-2} - J_{m+2}) & 0 \end{pmatrix} \tag{101b}$$

$$J_m = J_m(\rho r) \quad m = 0, \pm 1, \pm 2, \pm 3, \ldots \text{ and } i = \sqrt{-1} \tag{101c}$$

The body force vector **g** and **f** can be further expressed in terms of Fourier series and Hankel transforms as follows.

$$\mathbf{f}(r,\theta) = \sum_{m=-\infty}^{m=+\infty} \mathbf{f}_m(r)e^{im\theta} \tag{102a}$$

$$\mathbf{f}_m(r) = \frac{1}{2\pi}\int_0^{2\pi} \mathbf{f}(r,\theta)e^{-im\theta}d\theta \tag{102b}$$

$$\mathbf{g}(\rho,\varphi) = \sum_{m=-\infty}^{m=+\infty} \mathbf{g}_m(\rho)e^{-im\varphi} \tag{102c}$$

$$\mathbf{g}_m(\rho) = \frac{1}{2\pi}\int_0^{2\pi} \mathbf{g}(\rho,\varphi)e^{im\varphi}d\varphi \tag{102d}$$

where $\mathbf{g}_m(\rho)$ and $\mathbf{f}_m(r)$ have the following relationship.

$$\mathbf{g}_m(\rho) = \int_0^{+\infty} \boldsymbol{\Pi}^*_{cm}(\rho r)\mathbf{f}_m(r)rdr \tag{103a}$$

where

$$\mathbf{\Pi}_{cm}^{*}(\rho r)=\frac{1}{2\pi}\int_{0}^{2\pi}\mathbf{\Pi}^{*}(\rho r)e^{-i\rho r\sin(\theta+\varphi)+im(\varphi+\theta)}d\varphi$$

$$=\frac{1}{2}\begin{pmatrix} J_{m-1}-J_{m+1} & -i(J_{m-1}+J_{m+1}) & 0 \\ -i(J_{m-1}+J_{m+1}) & -(J_{m-1}-J_{m+1}) & 0 \\ 0 & 0 & 2J_{m} \end{pmatrix} \tag{103b}$$

5.3.4 Solution in Fourier series and Green's functions

The solution in (100) can be further expressed in terms of Fourier series and Green's functions as follows.

$$\mathbf{u}(r,\theta,z)=\sum_{m=-\infty}^{m=+\infty}\int_{0}^{+\infty}\mathbf{G}_{um}(r,r_0,z)\mathbf{f}_m(r_0)r_0dr_0e^{im\theta} \tag{104a}$$

$$\mathbf{T}_z(r,\theta,z)=\sum_{m=-\infty}^{m=+\infty}\int_{0}^{+\infty}\mathbf{G}_{zm}(r,r_0,z)\mathbf{f}_m(r_0)r_0dr_0e^{im\theta} \tag{104b}$$

$$\mathbf{\Gamma}_p(r,\theta,z)=\sum_{m=-\infty}^{m=+\infty}\int_{0}^{+\infty}\mathbf{G}_{pm}(r,r_0,z)\mathbf{f}_m(r_0)r_0dr_0e^{im\theta} \tag{104c}$$

where the Green's functions are defined as follows.

$$\mathbf{G}_{um}(r,r_0,z)=\int_{0}^{+\infty}\mathbf{\Pi}_{cm}(\rho r)\mathbf{\Phi}(\rho,z)\mathbf{\Pi}_{cm}^{*}(\rho r_0)d\rho \tag{105a}$$

$$\mathbf{G}_{zm}(r,r_0,z)=\int_{0}^{+\infty}\mathbf{\Pi}_{cm}(\rho r)\mathbf{\Psi}(\rho,z)\mathbf{\Pi}_{cm}^{*}(\rho r_0)\rho d\rho \tag{105b}$$

$$\mathbf{G}_{pm}(r,r_0,z)=\int_{0}^{+\infty}\mathbf{\Pi}_{cpm}(\rho r)\mathbf{\Phi}(\rho,z)\mathbf{\Pi}_{cm}^{*}(\rho r_0)\rho d\rho \tag{105c}$$

5.3.5 Solution of body force vector concentrated at a circular ring

The equations governing the body force vector uniformly concentrated on the circular ring can be expressed as follows.

$$\mathbf{f}(r,\theta)=\frac{\delta(r-r_0)}{2\pi r}\mathbf{f}_c \tag{106a}$$

$$\mathbf{f}_m(r,\theta)=\frac{1}{2\pi}\int_0^{2\pi}\mathbf{f}(r,\theta)e^{-im\theta}d\theta=\frac{1}{2\pi}\int_0^{2\pi}\frac{\delta(r-r_0)}{2\pi r}e^{-im\theta}d\theta\mathbf{f}_c$$
$$=\begin{cases}\dfrac{\delta(r-r_0)}{2\pi r}\mathbf{f}_c & m=0\\ 0 & m=\pm1,\pm2,\pm3...\end{cases}\tag{106b}$$

$$\mathbf{g}_m(\rho)=\int_0^{+\infty}\mathbf{\Pi}^*_{cm}(\rho r)\mathbf{f}_m(r)rdr=\begin{cases}\displaystyle\int_0^{+\infty}\mathbf{\Pi}^*_{c0}(\rho r)\frac{\delta(r-r_0)}{2\pi r}rdr\mathbf{f}_c & m=0\\ 0 & m=\pm1,\pm2,...\end{cases}$$
$$=\begin{cases}\dfrac{1}{2\pi}\mathbf{\Pi}^*_{c0}(\rho r_0)\mathbf{f}_c & m=0\\ 0 & m=\pm1,\pm2,...\end{cases}\tag{106c}$$

The corresponding fundamental singular solutions due to the body force vector uniformly concentrated on the circular ring can then be expressed as follows.

$$\mathbf{u}(r,\theta,z)=\mathbf{u}(r,r_0,z)=\frac{1}{2\pi}\mathbf{G}_{u0}(r,r_0,z)\mathbf{f}_c\tag{107a}$$

$$\mathbf{T}_z(r,\theta,z)=\mathbf{T}_z(r,r_0,z)=\frac{1}{2\pi}\mathbf{G}_{z0}(r,r_0,z)\mathbf{f}_c\tag{107b}$$

$$\mathbf{\Gamma}_p(r,\theta,z)=\mathbf{\Gamma}_p(r,r_0,z)=\frac{1}{2\pi}\mathbf{G}_{p0}(r,r_0,z)\mathbf{f}_c\tag{107c}$$

where the Green's functions are fined as follows.

$$\mathbf{G}_{u0}(r,r_0,z)=\frac{1}{2\pi}\int_0^{+\infty}\mathbf{\Pi}_{c0}(\rho r)\mathbf{\Phi}(\rho,z)\mathbf{\Pi}^*_{c0}(\rho r_0)d\rho\tag{108a}$$

$$\mathbf{G}_{z0}(r,r_0,z)=\frac{1}{2\pi}\int_0^{+\infty}\mathbf{\Pi}_{c0}(\rho r)\mathbf{\Psi}(\rho,z)\mathbf{\Pi}^*_{c0}(\rho r_0)\rho d\rho\tag{108b}$$

$$\mathbf{G}_{p0}(r,r_0,z)=\frac{1}{2\pi}\int_0^{+\infty}\mathbf{\Pi}_{cp0}(\rho r)\mathbf{\Phi}(\rho,z)\mathbf{\Pi}^*_{c0}(\rho r_0)\rho d\rho\tag{108c}$$

where

$$\mathbf{\Pi}_{c0}(\rho r)=\begin{pmatrix}-J_1 & 0 & 0\\ 0 & J_1 & 0\\ 0 & 0 & J_0\end{pmatrix}=\mathbf{\Pi}^*_{c0}(\rho r)\tag{109a}$$

$$\mathbf{\Pi}_{cp0}(\rho r) = -\frac{1}{2}\begin{pmatrix} J_0 - J_2 & 0 & 0 \\ 0 & J_2 & 0 \\ J_0 + J_2 & 0 & 0 \end{pmatrix} \tag{109b}$$

5.4 Summary notes

The solution has been systematically derived and formulated with the matrix approach in both Cartesian and cylindrical coordinate systems. The basic solution matrices $\mathbf{\Phi}(\rho, z)$ and $\mathbf{\Psi}(\rho, z)$ are related only to the material properties of the *n*-layered solids. They are suitable and applicable to various internal body force loadings in both Cartesian and cylindrical coordinate systems. The other three stresses $\mathbf{T}_p$ and three strains $\mathbf{\Gamma}_z$ can be obtained using the Hooke's law (6) and the solution of $\mathbf{T}_z$ and $\mathbf{\Gamma}_p$ in (14d-14e) in Cartesian coordinate system and in (93d-93e) in cylindrical coordinate system.

Most importantly, the solutions given in above equations are in the forms of improper integrals of infinite intervals either over the entire horizontal plane or from 0 to $+\infty$. The improper integrals have many depending parameters including $5\times(n+2)$ elastic constants $(c_{ij}, i = 1,.2,3,4,5; j = 0,1,..., n, n+1)$, n layer thicknesses $(h_j, j = 1,2,..., n)$, the three independent variables (x, y, z) and the applied loading vectors. The following three questions have to be answered. Do they converge? What are their singularities? Do they satisfy the governing partial differential equations and the boundary and interface conditions? These questions are analytically and rigorously examined and verified in the companion paper [24].

6 Summary and Conclusions

It is evident that the mathematical theory of linear elasticity is one of the most classical field theories in mechanics and physics. Many well-known scientists and mathematicians made contributions to its development and establishment over 200 years from 1638 to 1838. They include Galileo Galilei (1564 – 1642), Robert Hooke (1635 – 1703), Isaac Newton (1642 – 1726), Edme Mariotte (1620 – 1684), Jacob Bernoulli (1655 – 1705), Daniel Bernoulli FRS (1700 – 1782), Leonhard Euler (1707 – 1783), Charles-Augustin de Coulomb (1736 – 1806), Thomas Young (1773 – 1829), and Siméon Denis Poisson (1781 – 1840), Claude-Louis Navier (1785 – 1836), Augustin-Louis Cauchy (1789 – 1857), George Green (1793 – 1841), and Gabriel Léon Jean Baptiste Lamé (1795 – 1870).

Furthermore, the boundary-value problems of classical elasticity in *n*-layered and graded solids are also one of the classical problems. For more than 150 years from 1838 to present, many well-known scientists and mathematicians have made tremendous efforts to mathematically and analytically derive and formulate solutions in closed-forms for these boundary-value problems. They include Barré de Saint-Venant (1797 – 1886), Gustav Robert

Kirchhoff (1824 – 1887), George Biddell Airy (1801 – 1892), William Thomson (or Lord Kelvin) (1824 – 1907), Joseph Valentin Boussinesq (1842 – 1929), and Raymond David Mindlin (1906 – 1987). However, exact solutions and/or fundamental singular solutions in closed form are still very limited in literature.

From 1984 to 1995, the author investigated the boundary value problems of the classical elasticity in *n*-layered solids of either isotropy or transversely isotropy. He derived and formulated the solutions in equations (82) to (109) exactly and analytically for an arbitrary number *n* of elastic layers with different material properties. He used the classical mathematical tools and presented the mathematical derivation and formulation and the solutions in matrix forms. Using the symmetry and anti-symmetry of the *n*-layered solids of transverse isotropy, the author broke down the solutions and separated them into several blocks (or matrices) including material matrices, layering matrices, independent variable coordinate matrices, and applied loading terms. Consequently, many solutions in *n*-layered solids are obtained systematically and automatically. The solutions are also systematically expressed in both Cartesian and cylindrical coordinate systems. Therefore, the researchers at Research Centre Jülich and Massachusetts Institute of Technology have shortly named these mathematical formulations and solutions as Yue's approach, Yue's treatment, Yue's method and Yue's solution.

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Appendix A Exact Expression of $\mathbf{V}(H_0)$ in terms of $\mathbf{G}_v$

The equation (54) can be re-expressed as follow.

$$\begin{bmatrix} M_{Ap11} & M_{Ap12} \\ M_{Ap21} & M_{Ap22} \end{bmatrix} \mathbf{V}(H_0) = e^{-\gamma_{01}\rho h_1 - \ldots - \gamma_{0(k-2)}\rho h_{k-2} - \gamma_{0(k-1)}\rho h_{k-1} - \gamma_{0k}\rho(d-H_{k-1})} \begin{bmatrix} 0 & 0 \\ C_{Ap1} & C_{Ap2} \end{bmatrix} \mathbf{G}_v \qquad \text{(a1)}$$

From (a1), the solution of $\mathbf{V}(H_0)$ is given exactly as follows.

$$\mathbf{V}(H_0) = e^{-\gamma_{01}\rho h_1 - \ldots - \gamma_{0(k-2)}\rho h_{k-2} - \gamma_{0(k-1)}\rho h_{k-1} - \gamma_{0k}\rho(d-H_{k-1})} \mathbf{N}_{Ap} \mathbf{G}_v \qquad \text{(a2)}$$

where $\mathbf{N}_{Ap}$ can be re-expressed as follows.

$$\mathbf{N}_{Ap} = \frac{1}{|\mathbf{M}_{Ap}|} \begin{bmatrix} |\mathbf{M}_{Ap11}| & |\mathbf{M}_{Ap12}| \\ |\mathbf{M}_{Ap21}| & |\mathbf{M}_{Ap22}| \end{bmatrix} \qquad \text{(a3)}$$

The five determinants $|\mathbf{M}_{Ap}|$, $|\mathbf{M}_{Ap11}|$, $|\mathbf{M}_{Ap12}|$, $|\mathbf{M}_{Ap21}|$ and $|\mathbf{M}_{Ap22}|$ in (a3) can be expressed as follows.

$$|\mathbf{M}_{Ap}| = \begin{vmatrix} M_{Ap11} & M_{Ap12} \\ M_{Ap21} & M_{Ap22} \end{vmatrix} \qquad \text{(a4)}$$

$$|\mathbf{M}_{Ap11}| = \begin{vmatrix} 0 & M_{Ap12} \\ C_{Ap1} & M_{Ap22} \end{vmatrix}, \quad |\mathbf{M}_{Ap12}| = \begin{vmatrix} 0 & M_{Ap12} \\ C_{Ap2} & M_{Ap22} \end{vmatrix} \qquad \text{(a5)}$$

$$|\mathbf{M}_{Ap21}| = \begin{vmatrix} M_{Ap11} & 0 \\ M_{Ap21} & C_{Ap1} \end{vmatrix}, \quad |\mathbf{M}_{Ap22}| = \begin{vmatrix} M_{Ap11} & 0 \\ M_{Ap21} & C_{Ap2} \end{vmatrix}. \qquad \text{(a6)}$$

The determinants of the five 2×2 square matrices in (a4)-(a6) can be obtained using the following formula.

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}. \qquad \text{(a7)}$$

Appendix B Exact Expression of $\mathbf{V}(H_n)$ in terms of $\mathbf{G}_v$

The equation (59) can be re-expressed as follow.

$$\begin{bmatrix} M_{Aq11} & M_{Aq12} \\ M_{Aq21} & M_{Aq22} \end{bmatrix}\mathbf{V}(H_n) = e^{-\gamma_{0n}\rho h_n - \gamma_{0(n-1)}\rho h_{(n-1)} - \ldots - \gamma_{0(k+1)}\rho h_{(k+1)} - \gamma_{0k}\rho(H_k - d)}\begin{bmatrix} 0 & 0 \\ C_{Aq1} & C_{Aq2} \end{bmatrix}\mathbf{G}_v \qquad \text{(b1)}$$

From (b1), the solution of $\mathbf{V}(H_n)$ is given exactly as follows.

$$\mathbf{V}(H_n) = e^{-\gamma_{0n}\rho h_n - \gamma_{0(n-1)}\rho h_{(n-1)} - \ldots - \gamma_{0(k+1)}\rho h_{(k+1)} - \gamma_{0k}\rho(H_k - d)}\mathbf{N}_{Aq}\mathbf{G}_v \qquad \text{(b2)}$$

where $\mathbf{N}_{Aq}$ can be re-expressed as follows.

$$\mathbf{N}_{Aq} = \frac{1}{|\mathbf{M}_{Aq}|}\begin{bmatrix} |\mathbf{M}_{Aq11}| & |\mathbf{M}_{Aq12}| \\ |\mathbf{M}_{Aq21}| & |\mathbf{M}_{Aq22}| \end{bmatrix} \qquad \text{(b3)}$$

The five determinants $|\mathbf{M}_{Aq}|$, $|\mathbf{M}_{Aq11}|$, $|\mathbf{M}_{Aq12}|$, $|\mathbf{M}_{Aq21}|$ and $|\mathbf{M}_{Aq22}|$ in (b3) can be given below and can be expressed exactly using the formula (a7).

$$|\mathbf{M}_{Aq}| = \begin{vmatrix} M_{Aq11} & M_{Aq12} \\ M_{Aq21} & M_{Aq22} \end{vmatrix} \qquad \text{(b4)}$$

$$|\mathbf{M}_{Aq11}| = \begin{vmatrix} 0 & M_{Aq12} \\ C_{Aq1} & M_{Aq22} \end{vmatrix}, \quad |\mathbf{M}_{Aq12}| = \begin{vmatrix} 0 & M_{Aq12} \\ C_{Aq2} & M_{Aq22} \end{vmatrix} \qquad \text{(b5)}$$

$$|\mathbf{M}_{Aq21}| = \begin{vmatrix} M_{Aq11} & 0 \\ M_{Aq21} & C_{Aq1} \end{vmatrix}, \quad |\mathbf{M}_{Aq22}| = \begin{vmatrix} M_{Aq11} & 0 \\ M_{Aq21} & C_{Aq2} \end{vmatrix} \qquad \text{(b6)}$$

Appendix C Exact Expression of $\mathrm{U}(H_0)$ in terms of $\mathbf{G}_u$

The equation (72) can be re-expressed as follow.

$$
\begin{bmatrix} M_{Qp11} & M_{Qp12} & M_{Qp13} & M_{Qp14} \\ M_{Qp21} & M_{Qp22} & M_{Qp23} & M_{Qp24} \\ M_{Qp31} & M_{Qp32} & M_{Qp33} & M_{Qp34} \\ M_{Qp41} & M_{Qp42} & M_{Qp43} & M_{Qp44} \end{bmatrix} \mathbf{U}(H_0) = e^{-\gamma_{a1}\rho h_1 - \ldots - \gamma_{a(k-2)}\rho h_{k-2} - \gamma_{a(k-1)}\rho h_{k-1} - \gamma_{ak}\rho(d-H_{k-1})}
\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ C_{Qp31} & C_{Qp32} & C_{Qp33} & C_{Qp34} \\ C_{Qp41} & C_{Qp42} & C_{Qp43} & C_{Qp44} \end{bmatrix} \mathbf{G}_u \tag{c1}
$$

From (c1), the solution of $\mathbf{U}(H_0)$ is given exactly as follows.

$$
\mathbf{U}(H_0) = e^{-\gamma_{a1}\rho h_1 - \ldots - \gamma_{a(k-2)}\rho h_{k-2} - \gamma_{a(k-1)}\rho h_{k-1} - \gamma_{ak}\rho(d-H_{k-1})} \mathbf{N}_{Qp} \mathbf{G}_u \tag{c2}
$$

where

$$
\mathbf{N}_{Qp} = \frac{1}{|\mathbf{M}_{Qp}|} \begin{bmatrix} |\mathbf{M}_{Qp11}| & |\mathbf{M}_{Qp12}| & |\mathbf{M}_{Qp13}| & |\mathbf{M}_{Qp14}| \\ |\mathbf{M}_{Qp21}| & |\mathbf{M}_{Qp22}| & |\mathbf{M}_{Qp23}| & |\mathbf{M}_{Qp24}| \\ |\mathbf{M}_{Qp31}| & |\mathbf{M}_{Qp32}| & |\mathbf{M}_{Qp33}| & |\mathbf{M}_{Qp34}| \\ |\mathbf{M}_{Qp41}| & |\mathbf{M}_{Qp42}| & |\mathbf{M}_{Qp34}| & |\mathbf{M}_{Qp44}| \end{bmatrix} \tag{c3}
$$

The seventeen determinants in (c3) are expressed as follows.

$$
|\mathbf{M}_{Qp}| = \begin{vmatrix} M_{Qp11} & M_{Qp12} & M_{Qp13} & M_{Qp14} \\ M_{Qp21} & M_{Qp22} & M_{Qp23} & M_{Qp24} \\ M_{Qp31} & M_{Qp32} & M_{Qp33} & M_{Qp34} \\ M_{Qp41} & M_{Qp42} & M_{Qp43} & M_{Qp44} \end{vmatrix} \tag{c4}
$$

$$
|\mathbf{M}_{Qp1j}| = \begin{vmatrix} 0 & M_{Qp12} & M_{Qp13} & M_{Qp14} \\ 0 & M_{Qp22} & M_{Qp23} & M_{Qp24} \\ C_{Qp3j} & M_{Qp32} & M_{Qp33} & M_{Qp34} \\ C_{Qp4j} & M_{Qp42} & M_{Qp43} & M_{Qp44} \end{vmatrix}; |\mathbf{M}_{Qp2j}| = \begin{vmatrix} M_{Qp11} & 0 & M_{Qp13} & M_{Qp14} \\ M_{Qp21} & 0 & M_{Qp23} & M_{Qp24} \\ M_{Qp31} & C_{Qp3j} & M_{Qp33} & M_{Qp34} \\ M_{Qp41} & C_{Qp4j} & M_{Qp43} & M_{Qp44} \end{vmatrix} \tag{c5}
$$

$$
|\mathbf{M}_{Qp3j}| = \begin{vmatrix} M_{Qp11} & M_{Qp12} & 0 & M_{Qp14} \\ M_{Qp21} & M_{Qp22} & 0 & M_{Qp24} \\ M_{Qp31} & M_{Qp32} & C_{Qp3j} & M_{Qp34} \\ M_{Qp41} & M_{Qp42} & C_{Qp4j} & M_{Qp44} \end{vmatrix}; |\mathbf{M}_{Qp4j}| = \begin{vmatrix} M_{Qp11} & M_{Qp12} & M_{Qp13} & 0 \\ M_{Qp21} & M_{Qp22} & M_{Qp23} & 0 \\ M_{Qp31} & M_{Qp32} & M_{Qp33} & C_{Qp3j} \\ M_{Qp41} & M_{Qp42} & M_{Qp43} & C_{Qp4j} \end{vmatrix} \tag{c6}
$$

where $j = 1,2,3,4$.

The determinants of the seventeen 4×4 square matrices in (c4)-(c6) can be obtained exactly using the following formulae.

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \cdot \begin{vmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{vmatrix} - \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} \cdot \begin{vmatrix} a_{32} & a_{34} \\ a_{42} & a_{44} \end{vmatrix}$$
$$+ \begin{vmatrix} a_{11} & a_{14} \\ a_{21} & a_{24} \end{vmatrix} \cdot \begin{vmatrix} a_{32} & a_{33} \\ a_{42} & a_{43} \end{vmatrix} + \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} \cdot \begin{vmatrix} a_{31} & a_{34} \\ a_{41} & a_{44} \end{vmatrix} \tag{c7}$$
$$- \begin{vmatrix} a_{12} & a_{14} \\ a_{22} & a_{24} \end{vmatrix} \cdot \begin{vmatrix} a_{31} & a_{33} \\ a_{41} & a_{43} \end{vmatrix} + \begin{vmatrix} a_{13} & a_{14} \\ a_{23} & a_{24} \end{vmatrix} \cdot \begin{vmatrix} a_{31} & a_{32} \\ a_{41} & a_{42} \end{vmatrix}$$

where the determinant of a 2×2 square matrix is given in (a7).

Appendix D Exact Expression of $\mathbf{U}(H_n)$ in terms of $\mathbf{G}_u$

The equation (77) can be re-expressed as follow.

$$\begin{bmatrix} M_{Qq11} & M_{Qq12} & M_{Qq13} & M_{Qq14} \\ M_{Qq21} & M_{Qq22} & M_{Qq23} & M_{Qq24} \\ M_{Qq31} & M_{Qq32} & M_{Qq33} & M_{Qq34} \\ M_{Qq41} & M_{Qq42} & M_{Qq43} & M_{Qq44} \end{bmatrix} \mathbf{U}(H_n) = e^{-\gamma_{an}\rho h_n - \gamma_{a(n-1)}\rho h_{n-1} - \ldots - \gamma_{a(k+1)}\rho h_{(k+1)} - \gamma_{ak}\rho(H_k - d)}$$
$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ C_{Qq31} & C_{Qq32} & C_{Qq33} & C_{Qq34} \\ C_{Qq41} & C_{Qq42} & C_{Qq43} & C_{Qq44} \end{bmatrix} \mathbf{G}_u \tag{d1}$$

From (d1), the solution of $\mathbf{U}(H_n)$ is given exactly as follows.

$$\mathbf{U}(H_n) = e^{-\gamma_{an}\rho h_n - \gamma_{a(n-1)}\rho h_{n-1} - \ldots - \gamma_{a(k+1)}\rho h_{(k+1)} - \gamma_{ak}\rho(H_k - d)} \mathbf{N}_{Qq} \mathbf{G}_u \tag{d2}$$

where

$$\mathbf{N}_{Qq} = \frac{1}{|\mathbf{M}_{Qq}|} \begin{bmatrix} |\mathbf{M}_{Qq11}| & |\mathbf{M}_{Qq12}| & |\mathbf{M}_{Qq13}| & |\mathbf{M}_{Qq14}| \\ |\mathbf{M}_{Qq21}| & |\mathbf{M}_{Qq22}| & |\mathbf{M}_{Qq23}| & |\mathbf{M}_{Qq24}| \\ |\mathbf{M}_{Qq31}| & |\mathbf{M}_{Qq32}| & |\mathbf{M}_{Qq33}| & |\mathbf{M}_{Qq34}| \\ |\mathbf{M}_{Qq41}| & |\mathbf{M}_{Qq42}| & |\mathbf{M}_{Qq34}| & |\mathbf{M}_{Qq44}| \end{bmatrix} \tag{d3}$$

The seventeen determinants in (d3) are given below and can be exactly expressed using the formula (c7).

$$\left|\mathbf{M}_{Qq}\right| = \begin{vmatrix} M_{Qq11} & M_{Qq12} & M_{Qq13} & M_{Qq14} \\ M_{Qq21} & M_{Qq22} & M_{Qq23} & M_{Qq24} \\ M_{Qq31} & M_{Qq32} & M_{Qq33} & M_{Qq34} \\ M_{Qq41} & M_{Qq42} & M_{Qq43} & M_{Qq44} \end{vmatrix} \tag{d4}$$

$$\left|\mathbf{M}_{Qq1j}\right| = \begin{vmatrix} 0 & M_{Qq12} & M_{Qq13} & M_{Qq14} \\ 0 & M_{Qq22} & M_{Qq23} & M_{Qq24} \\ C_{Qq3j} & M_{Qq32} & M_{Qq33} & M_{Qq34} \\ C_{Qq4j} & M_{Qq42} & M_{Qq43} & M_{Qq44} \end{vmatrix}; \left|\mathbf{M}_{Qq2j}\right| = \begin{vmatrix} M_{Qq11} & 0 & M_{Qq13} & M_{Qq14} \\ M_{Qq21} & 0 & M_{Qq23} & M_{Qq24} \\ M_{Qq31} & C_{Qq3j} & M_{Qq33} & M_{Qq34} \\ M_{Qq41} & C_{Qq4j} & M_{Qq43} & M_{Qq44} \end{vmatrix} \tag{d5}$$

$$\left|\mathbf{M}_{Qq3j}\right| = \begin{vmatrix} M_{Qq11} & M_{Qq12} & 0 & M_{Qq14} \\ M_{Qq21} & M_{Qq22} & 0 & M_{Qq24} \\ M_{Qq31} & M_{Qq32} & C_{Qq3j} & M_{Qq34} \\ M_{Qq41} & M_{Qq42} & C_{Qq4j} & M_{Qq44} \end{vmatrix}; \left|\mathbf{M}_{Qq4j}\right| = \begin{vmatrix} M_{Qq11} & M_{Qq12} & M_{Qq13} & 0 \\ M_{Qq21} & M_{Qq22} & M_{Qq23} & 0 \\ M_{Qq31} & M_{Qq32} & M_{Qq33} & C_{Qq3j} \\ M_{Qq41} & M_{Qq42} & M_{Qq43} & C_{Qq4j} \end{vmatrix} \tag{d6}$$

where $j = 1,2,3,4$.