Optimal retention for a stop-loss reinsurance under VaR risk measure with incomplete information

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Abstract

This paper considers the determination of optimal retention in a stop-loss reinsurance. Assume that we only have incomplete information on a risk X for an insurer, we calculate an upper bound for the value at risk (VaR) of the total loss of an insurer after stop-loss reinsurance arrangement. The adopted method is a distribution-free approximation which allows to construct the extremal random variables with respect to the stochastic dominance order and the stop-loss order. We develop an optimization criterion for deriving the optimal retention such that the obtained upper bound of the VaR attains the minimum. We formally establish the sufficient and necessary conditions for the existence of the nontrivial optimal stop-loss reinsurance. For illustration purpose, some numerical examples are included and compared with the results yielded in Theorem 2.1 of Cai and Tan (2007).

Keywords: Stop-loss reinsurance; expectation premium principle; optimal retention; value-at-risk; distribution-free approximation; stochastic orders

1. Introduction

The importance of managerial decisions related to optimal reinsurance has received considerable attention in actuarial literature. It usually involves

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formulating an optimization problem and obtaining its optimal solution under certain criterion. Recently, optimization criteria based on tail risk measures such as value at risk (VaR) and conditional value at risk (CVaR) have been used in many papers. Combined with different premium principles, the optimality results of optimal reinsurance are derived by minimizing VaR and CVaR of the insurer's total risk exposure. For instance, Cai and Tan (2007) determine explicitly the optimal retention level of a stop-loss reinsurance under the expectation premium principle. Tan et al. (2009) extend the study of Cai and Tan to other reinsurance premium principles associated with quota-share and stop-loss reinsurance. Motivated by Cai and Tan (2007), Cai et al. (2008) derive the optimal ceded loss functions among the class of increasing and convex ceded loss functions. Compared with Cai et al. (2008), Chi and Tan (2011) relax the constraints on the distribution of the aggregation loss and provide a more efficient and simpler proof. Moreover, Chi and Tan (2011) consider a feasible class with constraints on the ceded and retained loss function, i.e., both the ceded and retained loss functions are increasing. See also, Bernard and Tian (2009), Cheung (2010), Tan et al. (2011), and references therein.

In terms of optimal reinsurance models proposed in these papers, a common assumption is that the distribution function of the total loss is known and satisfies some desirable properties. Then the tail risk measures can be analyzed regularly for a certain confidence level, and the reinsurance premiums can be calculated according to the theory of premium principle. However, in practice, we may not have enough information to estimate the distribution of the total loss. For example, in catastrophe insurance, the loss data caused by the extreme event is scarce due to the low frequency of occurrence.

In the present paper, we assume that some incomplete information of the total loss is available, say its first two moments and support. More explicitly, let X be the total loss for an insurer, which belongs to the set $\mathcal{B} = \mathcal{B}(I; \mu, \sigma)$ of all nonnegative random variables with mean μ, standard deviation σ and support contained in the interval $I = [0, b]$, here $b = +\infty$ is allowed. Note that the partial knowledge is a reasonable assumption. This has been pointed out by several authors in actuarial and financial research, see e.g. Schepper and Heijnen (2007), Gerber and Smith (2008), De Schepper and Heijnen (2010), Wong and Zhang (2013), and references therein.

Following Cai and Tan (2007), the objective of this paper is to determine the optimal retention in a stop-loss reinsurance under expectation premium principle with the VaR optimization criterion. The difference is that only

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partial information of the total loss rather than its distribution function is known.

Actually, the optimization problem using VaR criterion for a stop-loss reinsurance involves two components in general: the evaluation of the VaR of the retained loss for the insurer and the calculation of pure reinsurance premium determined by certain principle, both require the knowledge of the distribution function of the risk X. When only incomplete information of X is available, the question arising here is whether we could find the optimal retention for a stop-loss reinsurance.

Inspired by distribution-free method, it is possible to derive stochastic bounds for certain risk in its moment space, which provides useful information in probabilistic modelling and has been widely adopted in actuarial literature. For example, Hürlimann (2001) calculates four plausible premium principles of risk X with known first two moments and bounded support. Given fixed few moments, Hürlimann (2002) yields the maximum value of VaR (CVaR) for risk X by construction extremal random variable with respect to (w.r.t.) the stochastic dominance order (stop-loss order). Assuming that the first two moments and support of X are known, Hürlimann (2005) uses the stop-loss ordered random variables to develop the analytical lower and upper bounds of X, and approximates pure premiums for excess of loss reinsurance with reinstatements. These papers have shown that the obtained approximations are accurate enough for practical purpose, especially when one agrees to calculate some risk measures, not based on the actual loss function, but based on stochastic bounds of the loss. For further reference, we refer readers to Hürlimann (2008a,b).

Consider a stop-loss reinsurance contract, the first part of this paper establishes an upper bound of the VaR of the total loss for the insurer. Furthermore, over the set $\mathcal{B}$, the VaR of the retained loss for the insurer is bounded by determining its maximum random variable w.r.t. the stochastic dominance order, and the reinsurance premium determined using expectation principle is bounded by constructing its maximum random variable w.r.t. the stop-loss order. The second part derives the optimal retention level as well as the sufficient and necessary conditions for the existence of the nontrivial optimal stop-loss reinsurance strategy by minimizing the obtained upper bound of the VaR.

The rest of the paper is structured as follows. Section 2 introduces the VaR based optimal stop-loss reinsurance model. Section 3 provides the distribution-free approximations and establishes an upper bound of the VaR

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of the total loss for the insurer. Section 4 derives the optimal retention and discusses the sufficient and necessary conditions for the existence of a non-trivial optimal stop-loss reinsurance strategy. The final Section 5 illustrates the results by numerical examples and compares with the results yielded in Theorem 2.1 of Cai and Tan (2007).

2. VaR based optimal reinsurance model

In this section, we establish the framework of the VaR risk measure based optimal stop-loss reinsurance model, which have been described in detail in Cai and Tan (2007).

Let the total loss for an insurer be X, where $X \in \mathcal{B}$. We define X_I and X_R, respectively, as the retained loss and the ceded loss random variables under stop-loss reinsurance arrangement. Then X_I and X_R are related to X as follows:

$$X_I = \left\{ \begin{array}{ll} X, & X \leq d \\ d, & X > d \end{array} \right. = X \wedge d \tag{1}$$

and

$$X_R = \left\{ \begin{array}{ll} 0, & X \leq d \\ X - d, & X > d \end{array} \right. = (X - d)_+, \tag{2}$$

where $0 \leq d \leq b$ is known as the retention, $x \wedge y := \min\{x, y\}$, and $(x)_+ := \max\{x, 0\}$.

With the stop-loss reinsurance contract, the insurer caps the risk exposure at the retention, and transfers to the reinsurer by the part that exceeds the retention. Note that $d = b$ denotes the special case where the insurer retains all loss, and $d = 0$ means that the insurer transfers all loss to the reinsurer. Consequently, the former case implies no reinsurance, and the latter case leads to full reinsurance.

In exchange of undertaking the risk, the insurer should pay a reinsurance premium to the reinsurer. Here, we assume that the reinsurance premium is determined by expectation principle and expressed as $\delta(d) = (1 + \rho)\pi_X(d)$, where $\rho > 0$ is the safety loading factor and

$$\pi_X(d) = E(X_R) = E(X - d)_+ \tag{3}$$

is the stop-loss premium. In what follows, we denote $\rho^* = (1 + \rho)$ for simplicity.

Suppose that the total risk exposure of the insurer in the presence of reinsurance is T. The above analysis indicates that T can be expressed as the sum of two components: the retained loss and the incurred reinsurance premium; that is,

$$T = X_I + \delta(d). \tag{4}$$

To determine the optimal retention of stop-loss reinsurance by minimizing the proposed risk measure associated with T, we now formally introduce the definition of VaR.

Definition 1. *The VaR of a random variable X at a confidence level $1 - \alpha$ where $0 < \alpha < 1$ is defined as*

$$VaR_\alpha(X) = \inf\{x : \Pr(X > x) \leq \alpha\}. \tag{5}$$

It is equivalent to the $100(1 - \alpha)$-th percentile of X. Hence,

$$VaR_\alpha(X) \leq x \Leftrightarrow \bar{F}_X(x) \leq \alpha, \tag{6}$$

where $\bar{F}_X(x) = 1 - F_X(x)$. In addition, if g is an increasing continuous function, then

$$VaR_\alpha(g(X)) = g(VaR_\alpha(X)). \tag{7}$$

Other properties of the VaR considered in this paper are its useful links with stochastic order, which will be presented in next section.

Analogously, we can define VaR in terms of the insurer's retained loss X_I and the insurer's total loss T, i.e., $VaR_\alpha(d, X_I) = \inf\{x : \Pr(X_I > x) \leq \alpha\}$ and $VaR_\alpha(d, T) = \inf\{x : \Pr(T > x) \leq \alpha\}$. Here, we introduce an argument d to the VaR notations to emphasize that these risk measures are functions of the retention d. From (4) and (7), we have

$$VaR_\alpha(d, T) = VaR_\alpha(d, X_I) + \delta(d). \tag{8}$$

Building upon these, the optimal retentions by minimizing the corresponding VaR can be summarized as:

$$VaR_\alpha(d^*, T) = \min_{0 \leq d \leq b} \{VaR_\alpha(d, X_I) + \delta(d)\}. \tag{9}$$

In Cai and Tan (2007), the authors establish necessary and sufficient conditions for the existence of the optimal retention for formula (9), where the

distribution function of the risk X plays a role in the resulting optimal solution. As previously mentioned, with only partial information of X, neither $VaR_\alpha(d, X_I)$ nor $\delta(d)$ can be derived analytically. Therefore, it is hopeless to determine the optimal retention d^* in formula (9) for this case.

However, notice that in (9), we obtain that

$$VaR_\alpha(d, X_I) = VaR_\alpha(X) \wedge d, \quad \delta(d) = \rho^* \pi_X(d)$$

are two functionals of X, where the first equation holds due to (7). These two functionals preserve, respectively, the stochastic dominance order and the stop-loss order. Consequently, these orders exploiting results can be used to bound the functionals of X by determining their extremal values over the set $\mathcal{B}$, which will be explicitly introduced in next section.

3. Distribution-free approximations

In this section, we first introduce some notations of stochastic orders which will be used later on.

Definition 2. *Let X_1 and X_2 be two random variables. Then X_1 is said to be smaller than X_2 in stochastic dominance order, denoted by $X_1 \leq_{st} X_2$, if the inequality $\bar{F}_1(x) \leq \bar{F}_2(x)$ holds for all $x \in \mathbb{R}$, where $\bar{F}_i$ is the survival function of X_i, for $i = 1, 2$.*

Definition 3. *Let X_1 and X_2 be two random variables. Then X_1 is said to be smaller than X_2 in stop-loss order, written as $X_1 \leq_{sl} X_2$, if $\pi_1(x) \leq \pi_2(x)$, for all $x \in \mathbb{R}$, where $\pi_i(x) = E(X_i - x)_+$ is the stop-loss transform of X_i, for $i = 1, 2$.*

In actuarial science, stochastic orders have been widely discussed to compare the riskiness of different random situations. Standard reference is Denuit et al. (2006). As a sub-stream of this research, the optimality criterion by minimizing the retained risk w.r.t. certain stochastic order has general application in optimal reinsurance theory. See, for instance, Denuit and Vermandele (1998), Denuit and Vermandele (1999), Cai and Wei (2012).

Theoretically, given a partial order between random variables and some class of random variables, it is possible to construct extremal random variables w.r.t. this partial order. We now formally construct these extremal random variables w.r.t., respectively, the stochastic dominance order and the stop-loss order for $X \in \mathcal{B}$.

Let $F_*(x)$ and $F^*(x)$ be the Chebyshev-Markov extremal distributions over the space $\mathcal{B}$, which are solutions of the extremal moment problems

$$\bar{F}_*(x) := \min_{X \in \mathcal{B}}\{\bar{F}_X(x)\}, \ \ \bar{F}^*(x) := \max_{X \in \mathcal{B}}\{\bar{F}_X(x)\}, \tag{10}$$

where $\bar{F}_*(x) = 1 - F_*(x)$ and $\bar{F}^*(x) = 1 - F^*(x)$.

Random variables with distributions $F_*(x)$ and $F^*(x)$ are denoted by X_* and X^*, and are extremal w.r.t. the stochastic dominance order, that is

$$X_* \leq_{st} X \leq_{st} X^*, \text{ for all } X \in \mathcal{B}.$$

Similarly, the minimal and maximal stop-loss transforms over the space $\mathcal{B}$ are defined as

$$\pi_l(d) := \min_{X \in \mathcal{B}}\{\pi_X(d)\}, \ \ \pi_u(d) := \max_{X \in \mathcal{B}}\{\pi_X(d)\}. \tag{11}$$

From (3), a one-to-one correspondence between a distribution function and its stop-loss transform shows that $\bar{F}_X(x) = -\frac{d}{dx}\pi_X(x)$. Then, we define minimal and maximal stop-loss ordered random variables X_l and X_u by specifying their distribution functions

$$F_l(x) = 1 + \frac{d}{dx}\pi_l(x), \ \ F_u(x) = 1 + \frac{d}{dx}\pi_u(x). \tag{12}$$

These are extremal in the sense that

$$X_l \leq_{sl} X \leq_{sl} X_u, \text{ for all } X \in \mathcal{B}.$$

Therefore, the following relationships

$$\min_{X \in \mathcal{B}}\{VaR_\alpha(X)\} = VaR_\alpha(X_*), \ \ \max_{X \in \mathcal{B}}\{VaR_\alpha(X)\} = VaR_\alpha(X^*) \tag{13}$$

and

$$\min_{X \in \mathcal{B}}\{E(X - d)_+\} = E(X_l - d)_+, \ \ \max_{X \in \mathcal{B}}\{E(X - d)_+\} = E(X_u - d)_+ \tag{14}$$

hold with these notations.

Recall formula (9), the maximum value of $VaR_\alpha(d, X_I)$ and $\delta(d)$ over the set $\mathcal{B}$ can be derived analytically as follows.

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Proposition 1. *The maximum of $VaR_\alpha(d, X_I)$ for $X \in \mathcal{B}$ is*

Case 1: $\alpha \leq \frac{\sigma^2}{\sigma^2+(b-\mu)^2}$, $\qquad \max\limits_{X\in\mathcal{B}}\{VaR_\alpha(d, X_I)\} = d.$

Case 2: $\frac{\sigma^2}{\sigma^2+(b-\mu)^2} \leq \alpha \leq \frac{\mu^2}{\sigma^2+\mu^2}$, $\max\limits_{X\in\mathcal{B}}\{VaR_\alpha(d, X_I)\} = \left(\mu + \sqrt{\frac{1-\alpha}{\alpha}}\sigma\right) \wedge d.$

Case 3: $\alpha \geq \frac{\mu^2}{\sigma^2+\mu^2}$, $\qquad \max\limits_{X\in\mathcal{B}}\{VaR_\alpha(d, X_I)\} = \left(\mu + \frac{(1-\alpha)b\mu-\sigma^2}{\alpha b-\mu}\right) \wedge d.$

Proof. The distribution function of the normalized random variable $\frac{X^*-\mu}{\sigma}$ has been summarized in Hürlimann (2002) (TABLE III.1). Then, after some transformations and algebraic operations, the distribution functions of X^* can be described in tabular form:

condition	$F^*(x)$
$0 \leq x \leq \mu - \frac{\sigma^2}{b-\mu}$	0
$\mu - \frac{\sigma^2}{b-\mu} \leq x < \mu + \frac{\sigma^2}{\mu}$	$\frac{\sigma^2+(b-\mu)(x-\mu)}{bx}$
$\mu + \frac{\sigma^2}{\mu} \leq x < b$	$\frac{(x-\mu)^2}{\sigma^2+(x-\mu)^2}$
$x = b$	1

Inserting this into the formula

$$\max_{X\in\mathcal{B}}\{VaR_\alpha(d, X_I)\} = \max_{X\in\mathcal{B}}\{VaR_\alpha(X \wedge d)\}$$
$$= \max_{X\in\mathcal{B}}\{VaR_\alpha(X)\} \wedge d$$
$$= VaR_\alpha(X^*) \wedge d$$

yields the desired results. $\qquad\qquad\square$

Proposition 2. *For $X \in \mathcal{B}$, the maximum stop-loss transform of X equals*

Case 1: $0 \leq d \leq \frac{\sigma^2+\mu^2}{2\mu}$, $\qquad \max\limits_{X\in\mathcal{B}}\{\pi_X(d)\} = \mu\left(1 - \frac{\mu d}{\sigma^2+\mu^2}\right).$

Case 2: $\frac{\sigma^2+\mu^2}{2\mu} \leq d \leq \frac{b+\mu}{2} - \frac{\sigma^2}{2(b-\mu)}$, $\max\limits_{X\in\mathcal{B}}\{\pi_X(d)\} = \frac{\sqrt{\sigma^2+(d-\mu)^2}}{2} - \frac{d-\mu}{2}.$

Case 3: $\frac{b+\mu}{2} - \frac{\sigma^2}{2(b-\mu)} \leq d \leq b$, $\qquad \max\limits_{X\in\mathcal{B}}\{\pi_X(d)\} = \frac{\sigma^2(b-d)}{\sigma^2+(b-\mu)^2}.$

Proof. From Hürlimann (2001) (Table 1), one obtains the survival function of the normalized random variable $\frac{X_u-\mu}{\sigma}$. Similarly, to obtain the maximum of $\pi_u(d)$, we first calculate the distribution of X_u, which is determined by the following table:

condition	$F_u(x)$
$0 < x \leq \frac{\sigma^2 + \mu^2}{2\mu}$	$\frac{\mu^2}{\sigma^2 + \mu^2}$
$\frac{\sigma^2 + \mu^2}{2\mu} \leq x \leq \frac{1}{2}\left(b + \mu - \frac{\sigma^2}{b-\mu}\right)$	$\frac{1}{2}\left(1 - \frac{x-\mu}{\sqrt{\sigma^2 + (x-\mu)^2}}\right)$
$\frac{1}{2}\left(b + \mu - \frac{\sigma^2}{b-\mu}\right) \leq x < b$	$\frac{\sigma^2}{\sigma^2 + (b-\mu)^2}$
$x = b$	1

By the definition of stop-loss transform, it follows

$$\max_{X \in \mathcal{B}}\{\pi_X(d)\} = \pi_u(d) = E(X_u - d)_+$$

$$= \int_d^b (x - d)dF_u(x).$$

The desired results are concluded after some detailed calculations. $\square$

Remark 1. *Based on Hürlimann (2008a,b), two comments need be stated here. First, the minimum of $VaR_\alpha(d, X_I)$ and $\pi_X(d)$ can be derived since the distribution function of X_* and X_l are all analytic. Second, one can carry on a similar analysis for the space of random variables with information extended to moments known up to $n = 3, 4$, except that the mathematical operations are more complex.*

Combining these two propositions, an upper bound for $VaR_\alpha(d, T)$, denoted by $VaR_\alpha^*(d, T) := \max_{X \in \mathcal{B}}\{VaR_\alpha(d, X_I)\} + \max_{X \in \mathcal{B}}\{\delta(d)\}$, is obtained after substituting the stochastic ordered bounds for X to calculate the corresponding risk measures. This motivates us to minimize $VaR_\alpha^*(d, T)$ for seeking the optimal level of retention. Mathematically, it is equivalent to an optimization problem

$$
\begin{aligned}
d^* &= \arg\min_{0 \leq d \leq b}\{VaR_\alpha^*(d, T)\} \\
&= \arg\min_{0 \leq d \leq b}\{\max_{X \in \mathcal{B}}\{VaR_\alpha(d, X_I)\} + \max_{X \in \mathcal{B}}\{\delta(d)\}\} \\
&= \arg\min_{0 \leq d \leq b}\{VaR_\alpha(X^*) \wedge d + \rho^*\pi_u(d)\},
\end{aligned} \tag{15}
$$

where the objective function

$$OBF(d) := VaR_\alpha(X^*) \wedge d + \rho^* \pi_u(d)$$

is analytic according to the conclusions of Proposition 1 and Proposition 2.

Remark 2. *The optimization (15) produces a conservative solution because the objective function is provided by using the maximum of $VaR_\alpha(d, X_I)$ and $\pi_X(d)$ over the set $\mathcal{B}$. Therefore, an optimization using the minimal values as well as the average of these approximations may be a proper alternative.*

4. Optimal retention

In order not to complicate the formulae, we use a short notation for the following intervals:

- $I_1 = \left[0, \frac{\sigma^2 + \mu^2}{2\mu}\right]$,

- $I_2 = \left[\frac{\sigma^2 + \mu^2}{2\mu}, \frac{b+\mu}{2} - \frac{\sigma^2}{2(b-\mu)}\right]$,

- $I_3 = \left[\frac{b+\mu}{2} - \frac{\sigma^2}{2(b-\mu)}, b\right]$.

Moreover, as the measure of reinsurance premium, ρ^* will play a critical role in the solution of the VaR optimization (15). Thus, we also give two notations for convenience:

$$\Delta_1 = \frac{\sigma^2 + \mu^2}{\mu^2} \quad \text{and} \quad \Delta_2 = \frac{\sigma^2 + (b-\mu)^2}{\sigma^2},$$

where $\Delta_1 \leq \Delta_2$ holds by moment inequalities.

Our purpose in this section is to determine the optimal retention d^* in the interval $I_1 \cup I_2 \cup I_3$ for the optimization (15). The key results are verified in the following theorem.

Theorem 1. *Consider the VaR optimization (15).*
(a) For $\alpha \leq \frac{\sigma^2}{\sigma^2 + (b-\mu)^2}$, the optimal retention d^* and the minimum value of VaR are determined as follows:

condition	d^*	$OBF(d^*)$
$1 < \rho^* < \Delta_1$	0	$\rho^* \mu$
$\rho^* = \Delta_1$	any number in I_1	$\mu + \frac{\sigma}{\mu}\sigma$
$\Delta_1 < \rho^* < \Delta_2$	$\mu + \frac{\sigma(\rho^*-2)}{2\sqrt{\rho^*-1}}$	$\mu + \sqrt{\rho^*-1}\sigma$
$\rho^* = \Delta_2$	any number in I_3	b
$\rho^* > \Delta_2$	b	b

(b) For $\frac{\sigma^2}{\sigma^2+(b-\mu)^2} < \alpha \leq \frac{\mu^2}{\sigma^2+\mu^2}$, the optimal retention d^* and the corresponding minimum value of VaR are given by:

condition	d^*	$OBF(d^*)$
$1 < \rho^* < \Delta_1$	0	$\rho^*\mu$
$\rho^* = \Delta_1$	any number in I_1	$\mu + \frac{\sigma}{\mu}\sigma$
$\Delta_1 < \rho^* < \frac{1}{\alpha}$	$\mu + \frac{\sigma(\rho^*-2)}{2\sqrt{\rho^*-1}}$	$\mu + \sqrt{\rho^*-1}\sigma$
$\rho^* \geq \frac{1}{\alpha}$	b	$\mu + \sqrt{\frac{1-\alpha}{\alpha}}\sigma$

(c) For $\alpha > \frac{\mu^2}{\sigma^2+\mu^2}$, the optimal retention d^* and the minimum value of VaR are described as:

condition	d^*	$OBF(d^*)$
$1 < \rho^* < \Delta_1$	u^*	$\min\left\{\rho^*\mu, \ \mu + \frac{(1-\alpha)b\mu-\sigma^2}{\alpha b-\mu}\right\}$
$\rho^* \geq \Delta_1$	b	$\mu + \frac{(1-\alpha)b\mu-\sigma^2}{\alpha b-\mu}$

where $u^* = 0$, for $\rho^*\mu < \mu + \frac{(1-\alpha)b\mu-\sigma^2}{\alpha b-\mu}$; $u^* = b$, otherwise.

Proof. Case 1: If $\alpha \leq \frac{\sigma^2}{\sigma^2+(b-\mu)^2}$, then

$$OBF(d) = \begin{cases} d + \rho^*\mu\left(1 - \frac{\mu d}{\sigma^2+\mu^2}\right), & d \in I_1, \\ d + \rho^*\left(\frac{\sqrt{\sigma^2+(d-\mu)^2}}{2} - \frac{d-\mu}{2}\right), & d \in I_2, \\ d + \rho^*\frac{\sigma^2(b-d)}{\sigma^2+(b-\mu)^2}, & d \in I_3. \end{cases} \quad (16)$$

Observe that from (16), $OBF(d)$ is continuous of d on the interval $I_1 \cup I_2 \cup I_3$. Taking its first two derivatives, the following five situations can be identified.

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(1) If $1 < \rho^* < \Delta_1$, then $OBF(d)$ is strictly increasing on $I_1 \cup I_2 \cup I_3$. It follows that the optimal retention attains at $d^* = 0$, and the minimum value of $OBF(d)$ equals $\rho^*\mu$.

(2) If $\rho^* = \Delta_1$, then $OBF(d)$ is strictly increasing on $I_2 \cup I_3$, and takes a constant on $d \in I_1$. Therefore, the optimal retention d^* can be any number on I_1 with $OBF(d^*) = \mu + \frac{\sigma}{\mu}\sigma$.

(3) If $\Delta_1 < \rho^* < \Delta_2$, then $OBF(d)$ is decreasing on I_1 and increasing on I_3. On the interval I_2, the first order condition of $OBF(d)$ shows that it has at least one turning point. Furthermore, the second order condition presents that $OBF(d)$ is convex on I_2. These imply that $OBF(d)$ has one and only one turning point on I_2. Therefore, not only the local minimum value of $OBF(d)$ on the interval I_2, but also its global minimum value on the interval $I_1 \cup I_2 \cup I_3$ reaches at the unique turning point, i.e., $d^* = \mu + \frac{\sigma(\rho^*-2)}{2\sqrt{\rho^*-1}}$ and $OBF(d^*) = \mu + \sqrt{\rho^* - 1}\sigma$.

(4) If $\rho^* = \Delta_2$, then $OBF(d)$ is strictly decreasing on $I_1 \cup I_2$, and takes a constant on I_3. It follows immediately that d^* can be an arbitrary number on I_3 and $OBF(d^*) = b$.

(5) If $\rho^* > \Delta_2$, then $OBF(d)$ is strictly decreasing on $I_1 \cup I_2 \cup I_3$. It implies that $d^* = b$ and $OBF(d^*) = b$.

Case 2: If $\frac{\sigma^2}{\sigma^2+(b-\mu)^2} < \alpha \le \frac{\mu^2}{\sigma^2+\mu^2}$, then

$$
OBF(d) = \begin{cases} \left(\mu + \sqrt{\frac{1-\alpha}{\alpha}}\sigma\right) \wedge d + \rho^*\mu\left(1 - \frac{\mu d}{\sigma^2+\mu^2}\right), & d \in I_1, \\ \left(\mu + \sqrt{\frac{1-\alpha}{\alpha}}\sigma\right) \wedge d + \rho^*\left(\frac{\sqrt{\sigma^2+(d-\mu)^2}}{2} - \frac{d-\mu}{2}\right), & d \in I_2, \quad (17) \\ \left(\mu + \sqrt{\frac{1-\alpha}{\alpha}}\sigma\right) \wedge d + \rho^*\frac{\sigma^2(b-d)}{\sigma^2+(b-\mu)^2}, & d \in I_3, \end{cases}
$$

where $\mu + \sqrt{\frac{1-\alpha}{\alpha}}\sigma \in \left[\mu + \frac{\sigma}{\mu}\sigma, b\right] \subset I_2 \cup I_3$. After an observation, we can divide the range of d into two parts, i.e.,

$$
\left[0, \mu + \sqrt{\frac{1-\alpha}{\alpha}}\sigma\right] \text{ and } \left(\mu + \sqrt{\frac{1-\alpha}{\alpha}}\sigma, b\right].
$$

For $d \in \left(\mu + \sqrt{\frac{1-\alpha}{\alpha}}\sigma, b\right]$, it follows $\mu + \sqrt{\frac{1-\alpha}{\alpha}}\sigma \wedge d = \mu + \sqrt{\frac{1-\alpha}{\alpha}}\sigma$. Inserting this into formula (17), we obtain that $OBF(d)$ is strictly decreasing on $\left[\mu + \sqrt{\frac{1-\alpha}{\alpha}}\sigma, b\right]$, which indicates the optimal retention $d_1^* = b$ and $OBF(d_1^*) = \mu + \sqrt{\frac{1-\alpha}{\alpha}}\sigma$.

For $d \in \left[0, \mu + \sqrt{\frac{1-\alpha}{\alpha}}\sigma\right]$, it follows $\mu + \sqrt{\frac{1-\alpha}{\alpha}}\sigma \wedge d = d$. Through some substitutions, (17) reduces to (15) with support $I_1 \cup I_4$, where $I_4 := \left[\frac{\sigma^2+\mu^2}{2\mu}, \mu + \sqrt{\frac{1-\alpha}{\alpha}}\sigma\right] \subset I_2 \cup I_3$. Based on the analysis in Case 1, the optimal retention on $I_1 \cup I_4$ can be derived by a similar reasoning. Here, we denote the optimal results as d_2^* and $OBF(d_2^*)$ in this situation.

A comparison between $OBF(d_1^*)$ and $OBF(d_2^*)$ yields the final version of d^* and $OBF(d^*)$ as follows.

(1) If $1 < \rho^* < \Delta_1$, then $d^* = d_2^* = 0$ and $OBF(d^*) = OBF(d_2^*) = \rho^*\mu$.

(2) If $\rho^* = \Delta_1$, then $d^* = d_2^*$, where d^* can take any number on I_1 and $OBF(d^*) = \mu + \frac{\sigma}{\mu}\sigma$.

(3) If $\Delta_1 < \rho^* < \Delta_2$, then the optimality depends on the relationship between $\mu + \frac{\sigma(\rho^*-2)}{2\sqrt{\rho^*-1}}$ and $\mu + \sqrt{\frac{1-\alpha}{\alpha}}\sigma$:

(i) if $\frac{\rho^*-2}{2\sqrt{\rho^*-1}} \geq \sqrt{\frac{1-\alpha}{\alpha}}$, from formula (17), then $OBF(d)$ is strictly decreasing on $I_1 \cup I_2 \cup I_3$. Thus, $d^* = d_1^* = b$ and $OBF(d^*) = \mu + \sqrt{\frac{1-\alpha}{\alpha}}\sigma$;

(ii) if $\frac{\rho^*-2}{2\sqrt{\rho^*-1}} < \sqrt{\frac{1-\alpha}{\alpha}}$, then

$$OBF(d^*) = \begin{cases} \mu + \sqrt{\frac{1-\alpha}{\alpha}}\sigma, & if \ \rho^* > \frac{1}{\alpha}, \\ \mu + \sqrt{\rho^* - 1}, & otherwise, \end{cases}$$

with corresponding d^* as follows

$$d^* = \begin{cases} b, & if \ \rho^* > \frac{1}{\alpha}, \\ \mu + \frac{\sigma(\rho^*-2)}{2\sqrt{\rho^*-1}}, & otherwise. \end{cases}$$

Note that $\frac{\rho^*-2}{2\sqrt{\rho^*-1}} \geq \sqrt{\frac{1-\alpha}{\alpha}}$ implies $\rho^* > \frac{1}{\alpha}$. Therefore, we can summarize that:

if $\Delta_1 < \rho^* < \frac{1}{\alpha}$, then $d^* = \mu + \frac{\sigma(\rho^*-2)}{2\sqrt{\rho^*-1}}$ and $OBF(d^*) = \mu + \sqrt{\rho^* - 1}\sigma$;

if $\frac{1}{\alpha} \leq \rho^* < \Delta_2$, then $d^* = b$ and $OBF(d^*) = \mu + \sqrt{\frac{1-\alpha}{\alpha}}\sigma$.

(4) If $\rho^* \geq \Delta_2$, then $d^* = d_1^* = b$ and $OBF(d^*) = \mu + \sqrt{\frac{1-\alpha}{\alpha}}\sigma$.

Case 3: For $\alpha > \frac{\mu^2}{\sigma^2+\mu^2}$, the proof is quite similar to those of Case 1 and Case 2, which is omitted here. $\square$

The results from Theorem 1 tell us that, with the known incomplete information of risk X, the optimal retention depends only on the reinsurer's

loading factor and the first two moments and support of the total loss.

Letting $b \to +\infty$ in Theorem 1, we can also derive the optimal results when the support of the total loss X is infinite. Here, the details are ignored.

Following Tan et al. (2009), in terms of the solution to optimization (15), the optimal stop-loss reinsurance can be classified as either trivial or nontrivial. By trivial optimal stop-loss reinsurance, we mean that it is optimal to have either no reinsurance or full reinsurance, i.e., either $d^* = b$ or $d^* = 0$. On the other hand, the optimal stop-loss reinsurance is nontrivial if the optimal retention d^* lies in the open interval $(0, b)$. Then based on Theorem 1, the following sufficient and necessary conditions are established for the existence of nontrivial optimal stop-loss reinsurance.

Remark 3. *(a) For $\alpha \leq \frac{\sigma^2}{\sigma^2 + (b-\mu)^2}$, the optimal stop-loss reinsurance is nontrivial if and only if $\Delta_1 < \rho^* < \Delta_2$ holds.*

(b) For $\frac{\sigma^2}{\sigma^2 + (b-\mu)^2} < \alpha \leq \frac{\mu^2}{\sigma^2 + \mu^2}$, the optimal stop-loss reinsurance is nontrivial if and only if $\Delta_1 < \rho^* < \frac{1}{\alpha}$.

(c) For $\alpha > \frac{\mu^2}{\sigma^2 + \mu^2}$, the optimal stop-loss reinsurance is always trivial.

5. Numerical illustration

In this section, we offer three examples to illustrate the results in Theorem 1. Here, we denote $\Omega := (\mu, \sigma, b)$ as the known information of the total loss X for simplicity. For comparison purpose, we also assume that X has known distribution function, satisfying certain condition Ω. Then the VaR-optimization introduced in formula (9) can be solved according to Theorem 2.1 of Cai and Tan (2007), where the optimal retention and the corresponding minimum VaR are denoted by d^{**} and $OBF(d^{**})$, respectively. Let $IM := \left| \frac{OBF(d^*) - OBF(d^{**})}{OBF(d^{**})} \right| \times 100\%$ be the implicit margin between the $OBF(d^*)$ and $OBF(d^{**})$, which can be found in the last column of the following tables.

Example 1. *Assuming that $\alpha = 0.05$, Table 1 provides the optimal retention d^* and the corresponding $OBF(d^*)$ by varying Ω for $\rho^* = 2.1$ and $\rho^* = 2.5$, respectively. Under the known information Ω, we further suppose X is truncated exponential distributed with survival function*

$$S_X(x) = \begin{cases} \frac{e^{-0.001x} - e^{-0.001b}}{1 - e^{-0.001b}}, & 0 \leq x \leq b, \\ 0, & x > b. \end{cases}$$

Table 1: Optimal solutions comparison for truncated exponential distribution with fixed $\alpha = 0.05$

ρ^*	Ω	d^*	$OBF(d^*)$	d^{**}	$OBF(d^{**})$	IM
2.1	(1000, 1000, 100000)	1047.67	2048.81	741.94	2158.36	5.07
	(1000, 1000, 50000)	1047.67	2048.81	741.94	2158.36	5.07
	(999.54, 997.73, 10000)	1047.11	2045.97	741.89	2157.4	5.16
	(995.85, 984.3, 7500)	1042.77	2018.2	741.33	2149.51	5.64
	(966.08, 910.64, 5000)	1009.5	1921.17	734.55	2085.63	7.89
2.5	(1000, 1000, 100000)	1204.12	2224.74	916.29	2443.44	8.95
	(1000, 1000, 50000)	1204.12	2224.74	916.29	2443.44	8.95
	(999.54, 997.73, 10000)	1203.21	2221.5	916.22	2442.3	9.04
	(995.85, 984.3, 7500)	1196.77	2201.37	915.46	2432.97	9.52
	(966.08, 910.64, 5000)	1151.96	2081.37	906.23	2357.44	11.71

According to Theorem 2.1 of Cai and Tan (2007), d^{**} and $OBF(d^{**})$ are calculated and presented in Table 1.

Table 1 shows that as the reinsurance premium (ρ^*) increases, the optimal retentions d^* and d^{**} increase. As a result, the insurer's total risk exposure becomes more dangerous since both $OBF(d^*)$ and $OBF(d^{**})$ increase. On the other hand, for fixed parameters, d^* is always larger than d^{**}, while the relationship between $OBF(d^*)$ and $OBF(d^{**})$ is just opposite. These confirm with the fact that our optimization criterion products a conservative result. Table 1 shows that a large value ρ^* contains a relatively large implicit margin.

Example 2. *For fixed $\alpha = 0.05$, we obtain d^* and $OBF(d^*)$ for different value of Ω in Table 2, where $\rho^* = 2.3$ and $\rho^* = 2.5$ guarantee the optimal stop-loss reinsurance is non-trivial. Furthermore, let X be truncated Pareto distributed with available information Ω, whose survival function satisfies*

$$S_X(x) = \begin{cases} \frac{S_0(x) - S_0(b)}{1 - S_0(b)}, & 0 \leq x \leq b, \\ 0, & x > b, \end{cases}$$

Table 2: Optimal solutions comparison for truncated Pareto distribution with fixed $\alpha = 0.05$

ρ^*	Ω	d^*	$OBF(d^*)$	d^{**}	$OBF(d^{**})$	IM
2.3	(1000, 1118.03, 100000)	1147.08	2274.75	781.72	2318.44	1.88
	(1000, 1118.02, 50000)	1147.08	2274.75	781.72	2318.44	1.88
	(993.68, 1085.01, 10000)	1136.42	2230.78	781	2303.76	3.17
	(980.53, 1039.45, 7500)	1117.28	2165.68	778.76	2273.1	4.73
	(932.21, 920.41, 5000)	1053.3	1981.65	766.52	2159.74	8.25
2.5	(1000, 1118.03, 100000)	1228.22	2369.31	863.62	2462.93	3.8
	(1000, 1118.02, 50000)	1228.21	2369.43	863.62	2462.92	3.8
	(993.68, 1085.01, 10000)	1215.12	2322.54	862.78	2447	5.09
	(980.53, 1039.45, 7500)	1192.7	2253.59	860.18	2413.73	6.63
	(932.21, 920.41, 5000)	1120.09	2059.49	845.96	2290.88	10.1

where $S_0(x) = \left(\frac{9000}{x+9000} \right)^{10}$. Taking a similar calculation procedure with Example 1, we obtain the optimal solutions d^{**} and $OBF(d^{**})$ in Table 2.

Example 3. Given $\rho^* = 2.4$ and $\rho^* = 2.5$, d^* and $OBF(d^*)$ are concluded in Table 3 by varying Ω for $\alpha = 0.05$. As a further comparison, for given information Ω, it is supposed that X has a truncated Burr distribution with survival function

$$S_X(x) = \begin{cases} \frac{S_1(x)-S_1(b)}{1-S_1(b)}, & 0 \le x \le b, \\ 0, & x > b, \end{cases}$$

where $S_1(x) = \left(\frac{10000^{0.95}}{x^{0.95}+10000^{0.95}} \right)^{11}$. After some computation, we have optimal solutions d^{**} and $OBF(d^{**})$ in Table 3.

Observations from Table 2 and Table 3 lead to similar results as that from Table 1.

Table 3: Optimal solutions comparison for truncated Burr distribution with fixed $\alpha = 0.05$

ρ^*	Ω	d^*	$OBF(d^*)$	d^{**}	$OBF(d^{**})$	IM
2.4	(909.16, 1064.79, 100000)	1089.14	2169.04	726.62	2196.84	1.26
	(909.16, 1064.78, 50000)	1089.14	2169.03	726.63	2196.84	1.27
	(903.68, 1034.45, 10000)	1078.54	2127.66	726	2183.57	2.56
	(892.41, 992.93, 7500)	1060.24	2067.26	724.11	2156.15	4.12
	(851.08, 884.37, 5000)	1000.57	1897.48	713.79	2055.06	7.67
2.5	(909.16, 1064.79, 100000)	1126.51	2213.26	763.84	2264.17	2.25
	(909.16, 1064.78, 50000)	1126.51	2213.25	763.84	2264.17	2.25
	(903.68, 1034.45, 10000)	1114.8	2170.62	763.17	2250.35	3.54
	(892.41, 992.93, 7500)	1095.09	2108.49	761.13	2221.82	5.1
	(851.08, 884.37, 5000)	1031.6	1934.21	750.02	2116.64	8.61

For fixed $\rho^* = 2.5$, compared with results from these three tables for certain row, we find that the values of IM in Table 3 are always the smallest, and the largest values are in Table 1. These are mainly because the truncated Burr distribution used in Table 3 has the heaviest right tail, and the right tail of the truncated exponential distribution presented in Table 1 is the lightest.

Acknowledgements

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