Shear wave velocity and stiffness of sand: the role of non-plastic fines

J. YANG* and X. LIU*

Current knowledge on the shear wave velocity ($V_s$) and associated stiffness ($G_0$) of sand is built mainly on the results of extensive laboratory studies on clean quartz sands. Often natural sands are not clean, but contain a certain amount of fines. The role of fines in altering the stiffness of sands is a matter of great concern, yet remains poorly understood. This paper presents an investigation into the problem through well-controlled laboratory experiments in conjunction with analysis and interpretation at the macro and micro scale. The laboratory experiments were conducted for a sequence of mixtures of clean quartz sand and crushed silica fines under saturated conditions, by the simultaneous use of the resonant column (RC) and bender element (BE) techniques. A broad range of states in terms of void ratio, confining stress and fines content was covered so as to obtain a comprehensive view on the effect of fines and the possible interplay with other factors. Both the RC and BE tests showed that $G_0$ tends to decrease continuously as the quantity of fines is increased and the reduction rates are similar; a similar stress dependence is also obtained for $G_0$ from both types of testing. Nevertheless, $G_0$ values obtained from BE tests are notably greater than those obtained from RC tests, and this effect of testing method is shown to be coupled with the sample reconstitution method. A new approach that allows unified characterisation of $G_0$ values for both clean sand and sand–fines mixtures is developed in a sound theoretical framework, thereby providing important insights into the various empirical correlations that involve $G_0$ (or $V_s$) in geotechnical engineering practice. A new micro-scale mechanism is also suggested for the observed effect of fines, which attributes the reduction of $G_0$ caused by fines to the decrease in the coordination number at an approximately constant void ratio.

KEYWORDS: dynamics; elasticity; laboratory tests; sands; stiffness

INTRODUCTION

The characterisation of shear wave velocity ($V_s$) and associated small-strain stiffness ($G_0$) for granular soils has been a subject of long-standing interest in soil mechanics and geotechnical engineering (Stokoe et al., 1999; Clayton, 2011). A sound knowledge has been developed over the last several decades, mainly through well-controlled laboratory experiments on clean, uniform, quartz sands (e.g. Hardin & Richart, 1963; Hardin & Drnevich, 1972; Iwasaki & Tatsuoka, 1977; Seed et al., 1986; Lo Presti et al., 1997; Kuwano et al., 2000; Youn et al., 2008; Wichtmann & Triantafyllidis, 2009; Gú et al., 2015; and the references therein). Among the various factors that may affect the stiffness property, void ratio and confining stress are recognised to be the most important ones, and several empirical equations accounting for the two factors are now commonly used in practice and in the development of constitutive models (Ishihara, 1996; Taibat & Dafalias, 2008). These empirical equations often take a general form as follows

$$G_0 = AF(e)\left(\frac{\sigma'}{p_a}\right)^n$$

where $\sigma'$ is the mean effective stress; $p_a$ is a reference stress, usually taken as the atmospheric pressure; $F(e)$ is a function of the void ratio $e$; and $A$ and $n$ are two best-fit parameters.

The exponent $n$ has received much discussion in the past (e.g. Goddard, 1990; Chang et al., 1991; McDowell & Bolton, 2001); it reflects the contact conditions at the grain scale and takes the value of 1/3 from the classical Hertz–Mindlin contact theory. The measured values, however, typically range between 0·35 and 0·6 for sands, and for simplicity the value of 0·5 is commonly adopted in empirical equations (Hicher, 1996; Ishihara, 1996).

Often natural sands are not clean, but contain a certain amount of fines (<63 μm). A number of experimental studies have shown that the presence of fines can alter the large-strain shear behaviour of clean sands under either monotonic or cyclic loading conditions (e.g. Lade & Yamamuro, 1997; Polito & Martin, 2001; Thevanayagam et al., 2002). A concrete example is given in Fig. 1, which shows that the liquefaction susceptibility of saturated Toyoura sand can be significantly enhanced by the addition of non-plastic silica fines (Yang & Wei, 2012). Concerns have been raised over such issues as what impact fines have on the shear wave velocity and the associated stiffness of sands, and whether the empirical equations developed from experiments on clean sands are applicable for sands with fines. In the current literature, however, available studies addressing these issues are limited compared with the enormous body of studies on clean sands.

Recent notable work on the effect of fines includes that by Wichtmann et al. (2015), who conducted a structured resonant column testing programme on a quartz sand mixed with a non-plastic quartz powder of varying quantities (0–20% by mass). The study showed that the small-strain stiffness ($G_0$) decreased with increasing fines content (FC) up to about 10%, but a further increase of FC to 20% did not cause measurable changes in $G_0$. This result is not in full agreement with that of Salgado et al. (2000), which was derived from laboratory experiments on Ottawa sand.

Manuscript received 13 September 2015; revised manuscript accepted 13 January 2016. Published online ahead of print 17 February 2016.

Discussion on this paper closes on 1 November 2016, for further details see p. ii.

* Department of Civil Engineering, The University of Hong Kong, Hong Kong.

500
mixed with non-plastic silica fines (FC = 0–20%) by using piezoceramic bender elements in a triaxial device. Those experiments showed that the value of $G_0$ continuously decreased with an increase of FC up to 20% and became as low as 40% of the clean sand at the highest FC; a dramatic variation of the stress exponent $n$ was also measured, ranging from 0.435 at FC = 0 to 0.809 at FC = 20%. Given that both Ottawa sand and silica fines are hard-grained materials and given the range of confining pressure applied, the value of 0.809 appears to be unusually high compared with the reported values in the literature. Moreover, a dramatic variation in the stress exponent does not appear to be in agreement with the observations of Iwasaki & Tatsuoka (1977) and Chien & Oh (2002) that the stress exponent is insensitive to the presence of fines.

The experimental data from previous studies provide a useful reference for understanding the effect of fines. Nevertheless, the diverse observations indicate that the problem remains highly complex and not yet fully understood. Previous studies have often involved different materials (in terms of grain shape, size distribution and mineralogy) and different testing methods, making it difficult to evaluate the discrepancies through direct comparison. For example, the bender element (BE) tests of Salgado et al. (2000) were performed on saturated specimens prepared by slurry deposition, whereas the resonant column (RC) tests of Wichtmann et al. (2015) were conducted on dry specimens prepared by air pluviation. Several studies (e.g. Nakagawa et al., 1997) have shown that under otherwise similar conditions, $G_0$ values measured on dry specimens are not, as usually assumed, exactly the same as those of saturated specimens. In particular, the presence of fines raises a concern about the effect of grain segregation in the deposition process and a concern about the effect of grain size ratio.

Unlike the widely recognised technique of RC testing, BE testing is not yet standardised worldwide, partly because of the variability of results (e.g. Jovicic et al., 1996; Lee & Santamarina, 2005; Yamashita et al., 2009). The variability is mainly associated with the determination of the travel time of shear waves. As demonstrated by Yamashita et al. (2009) and Yang & Gu (2013), even for uniform clean sand and glass beads, significantly different travel times and consequently different $V_s$ and $G_0$ values may be derived when signals are not properly interpreted. It may therefore be speculated that the observed discrepancies on the effect of fines might be caused by the uncertainty in signal interpretation or attributable to the effect of testing methods. However, the literature is lacking solid data showing how the presence of fines affects the shear wave signals in sand specimens and whether considerable uncertainty tends to emerge when fines are present. Systematic data sets are needed that allow a meaningful comparison of BE and RC measurements on sand–fines mixtures and at the same time can serve as a useful reference in the validation and calibration of numerical simulations and theoretical developments in this important realm.

With the aim of addressing the above concerns, a specifically designed experimental programme has been carried out using an apparatus that incorporates both RC and BE functions. The apparatus allows RC and BE testing to be performed on an identical specimen, thus affording a more reliable and convincing comparison. All specimens were tested under the saturated rather than the dry condition, because the former is more relevant to practical situations. To obtain a comprehensive view on the effect of fines and the possible interplay with other factors, the experimental programme covered a wide range of conditions in terms of void ratio, confining stress and fines content. This paper presents the main results along with a detailed discussion and interpretation from the macro-scale and micro-scale perspectives. A new approach is put forward that allows unified characterisation of $G_0$ values for both clean sand and sand–fines mixtures in a theoretical framework, and a micro-scale mechanism is also suggested to explain the observed effect of fines.

**TEST MATERIALS AND METHODS**

In the laboratory experiments Toyoura sand was used as the base sand and crushed silica fines were used as the additive. Using artificially created mixtures allows good control of grain characteristics and facilitates experimental repeatability so that any more complex effects or uncertainties are eliminated. Table 1 gives the basic physical properties of the two materials, and Fig. 2 shows their particle size distribution curves together with microscopy images. Toyoura sand is a uniform quartz sand with sub-angular to sub-rounded grains, whereas the crushed silica fines are composed of non-plastic angular grains. To produce a sequence of sand–fines mixtures, the quantity of crushed silica fines was varied from 0 to 30% by mass. The threshold...
with the free-bottom-fixed and top-free configuration which, compared
air-filled cell pressure up to 1 MPa. The resonant column is
specimen 50 mm in diameter and 100 mm high, with an
position system, as shown in Fig. 3. It can accommodate a soil
features and a robust signal conditioning and data acqui-
were sand-dominated.

The apparatus used in the study has both RC and BE
functions (Yang & Gu, 2013): for the former a set of aluminium bars of different dimensions was used to establish a
calibration curve for the frequency-dependent mass polar
moment of inertia of the drive head, whereas for the latter the
calibration was conducted to determine the system delay,
including the response time of the bender elements and the
travel time in the cables, and to check the phase relationship
between the input and output signals.

All specimens were prepared by the moist tamping method
(Ishihara, 1996) in conjunction with the under-compaction
technique (Ladd, 1978). This method was chosen because it
can produce a very wide range of soil densities and has the
advantage of preventing segregation and enhancing uniformity. As previous studies focused mainly on medium-
dense and dense samples that would exhibit strain-hardening
rather than contractive, liquefaction behaviour, the testing
programme has purposely included a number of specimens in
the loose state. All specimens were saturated in two stages:
initially by flushing the specimen with carbon dioxide and
de-aired water, and then by applying back pressure.
Specimens with a Skempton B-value greater than 0.95 were
considered saturated. After saturation, each specimen was
subjected to an isotropic confining stress in stages, typically
at 50, 100, 200, 400 and 500 kPa. When the specimen was
brought to a specific confining pressure level, it was
consolidated for about 15 min so that the reading of the
internal linear variable differential transducer (LVDT)
became stable and the volume change was measured; then
the BE test was performed under a range of excitation
frequencies. Following the BE test, the RC test was
performed on the same specimen for the purpose of
comparison of the stiffness measurements. The strain level
involved in all tests was in the order of $10^{-5}$ or below.
A summary of the testing series is given in Table 2.

RESULTS AND ANALYSIS

Measurements from BE tests

In each BE test a set of sinusoid signals at various
frequencies (from 1 to 40 kHz) was used as the excitation,
and the received signals were examined in a whole view to
better identify the travel time of the shear wave. The signal
corresponding to the excitation frequency of 10 kHz was
found to consistently yield a clear arrival of the shear wave in
both clean sand and mixed soil specimens. This is in good
agreement with the observation of Yang & Gu (2013) on
samples of uniform glass beads tested in the same apparatus.
This result also agrees with the observation of Brignoli et al.
(1996) from their pulse tests on uniform Ticino sand that the
most interpretable waveforms typically occurred in the range
of 3–10 kHz for specimens of 100–140 mm high. As an
example, Fig. 4(a) shows a set of received signals in a clean
sand specimen under a range of confining stresses, from as
low as 50 kPa to as high as 500 kPa; for the purpose of
comparison, Fig. 4(b) shows the received signals in a mixed
specimen with 10% fines at a similar void ratio.

For either the clean sand specimen or the mixed soil
specimen, the arrival of the shear wave can be clearly
identified (marked by a downward solid triangle in each
waveform). As the confining stress increased, the travel time
of the shear wave decreased accordingly. A comparison of
Figs 4(a) and 4(b) indicates that the waveforms in the mixed
soil specimen are similar to their counterparts in the clean sand specimen, suggesting that the presence of fines would not introduce notable uncertainty in signal interpretation. Nevertheless, the presence of fines was found to increase the shear wave travel time. This effect can be observed more clearly in Fig. 5, where received signals in four specimens with different quantities of fines (FC = 0–30%) are compared. Note that all four specimens were carefully controlled to achieve a similar state (e = 0.86–0.87, \(\sigma'\) = 100 kPa) so as to afford a meaningful comparison.

A more comprehensive view of \(G_0\) values measured under various conditions is given in Fig. 6, where \(G_0\) values are shown as a function of void ratio for samples at different quantities of fines and at different confining pressures. It is clear from the plots that \(G_0\) is dependent on void ratio, confining stress and the percentage of fines. Under otherwise similar conditions, \(G_0\) increases with decreasing void ratio and with increasing confining stress, but it decreases with increasing fines content. A notable feature of the results in Fig. 6 is that the void ratio dependence of the sand–fines mixtures appears to be similar to that of clean sand, and this dependence seems to be insensitive to changes in

<table>
<thead>
<tr>
<th>Material</th>
<th>State 1 (e, (\sigma'))</th>
<th>State 2 (e, (\sigma'))</th>
<th>State 3 (e, (\sigma'))</th>
<th>State 4 (e, (\sigma'))</th>
<th>State 5 (e, (\sigma'))</th>
</tr>
</thead>
<tbody>
<tr>
<td>Clean sand FC = 0%</td>
<td>(0.903,50)</td>
<td>(0.899,100)</td>
<td>(0.893,200)</td>
<td>(0.793,400)</td>
<td>(0.881,500)</td>
</tr>
<tr>
<td></td>
<td>(0.805,50)</td>
<td>(0.802,100)</td>
<td>(0.798,200)</td>
<td>(0.842,400)</td>
<td>(0.791,500)</td>
</tr>
<tr>
<td></td>
<td>(0.887,50)</td>
<td>(0.883,100)</td>
<td>(0.878,200)</td>
<td>(0.867,500)</td>
<td>(0.839,500)</td>
</tr>
<tr>
<td></td>
<td>(0.859,50)</td>
<td>(0.855,100)</td>
<td>(0.850,200)</td>
<td>(0.912,500)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(0.937,50)</td>
<td>(0.933,100)</td>
<td>(0.926,200)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>FC = 5%</td>
<td>(0.934,50)</td>
<td>(0.929,100)</td>
<td>(0.921,200)</td>
<td>(0.905,500)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(0.802,50)</td>
<td>(0.800,100)</td>
<td>(0.796,200)</td>
<td>(0.802,500)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(0.820,50)</td>
<td>(0.817,100)</td>
<td>(0.812,200)</td>
<td>(0.802,500)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(0.805,50)</td>
<td>(0.801,100)</td>
<td>(0.804,200)</td>
<td>(0.802,500)</td>
<td></td>
</tr>
<tr>
<td>FC = 10%</td>
<td>(0.874,50)</td>
<td>(0.870,100)</td>
<td>(0.864,200)</td>
<td>(0.855,400)</td>
<td>(0.852,500)</td>
</tr>
<tr>
<td></td>
<td>(0.815,50)</td>
<td>(0.811,100)</td>
<td>(0.806,200)</td>
<td>(0.794,400)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(0.809,50)</td>
<td>(0.805,100)</td>
<td>(0.799,200)</td>
<td>(0.789,400)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(0.874,50)</td>
<td>(0.870,100)</td>
<td>(0.863,200)</td>
<td>(0.853,400)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(0.934,50)</td>
<td>(0.927,100)</td>
<td>(0.916,200)</td>
<td>(0.896,500)</td>
<td></td>
</tr>
<tr>
<td>FC = 20%</td>
<td>(0.810,50)</td>
<td>(0.806,100)</td>
<td>(0.800,200)</td>
<td>(0.791,400)</td>
<td>(0.787,500)</td>
</tr>
<tr>
<td></td>
<td>(0.813,50)</td>
<td>(0.809,100)</td>
<td>(0.803,200)</td>
<td>(0.794,400)</td>
<td>(0.790,500)</td>
</tr>
<tr>
<td></td>
<td>(0.881,50)</td>
<td>(0.874,100)</td>
<td>(0.862,200)</td>
<td>(0.841,400)</td>
<td>(0.832,500)</td>
</tr>
<tr>
<td></td>
<td>(0.936,50)</td>
<td>(0.926,100)</td>
<td>(0.910,200)</td>
<td>(0.881,500)</td>
<td></td>
</tr>
<tr>
<td>FC = 30%</td>
<td>(0.810,50)</td>
<td>(0.805,100)</td>
<td>(0.799,200)</td>
<td>(0.788,400)</td>
<td>(0.783,500)</td>
</tr>
<tr>
<td></td>
<td>(0.875,50)</td>
<td>(0.867,100)</td>
<td>(0.855,200)</td>
<td>(0.836,400)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(0.865,50)</td>
<td>(0.858,100)</td>
<td>(0.848,200)</td>
<td>(0.831,400)</td>
<td>(0.824,500)</td>
</tr>
<tr>
<td></td>
<td>(0.930,50)</td>
<td>(0.917,100)</td>
<td>(0.898,200)</td>
<td>(0.898,200)</td>
<td></td>
</tr>
</tbody>
</table>

Fig. 5. Shear wave signals in saturated sand specimens with different percentages of fines

Fig. 4. Shear wave signals at various effective confining stresses: (a) clean sand specimen; (b) sand with 10% fines
confining stress. The state dependence of $G_0$ is of particular interest and will be discussed in more detail in sections that follow.

**Comparison of BE and RC measurements**

It is of interest to examine whether a similar effect of fines on $V_s$ and $G_0$ can be obtained from RC tests. Fig. 7 shows an example of the frequency response of four specimens with different quantities of fines. All specimens were brought to a similar state ($e = 0.85–0.86$, $\sigma' = 200$ kPa) so that any observed difference can be attributed mainly to the effect of fines. Clearly, even a small amount of fines (FC = 5%) is able to cause a notable shift of the resonant frequency to the low frequency end, thus leading to a reduction of $V_s$ and $G_0$ in accordance with the relations as follows

$$f_n = \frac{1}{2\pi} \frac{I_0}{I} \frac{L}{V_s}$$

(2)

where $f_n$ is the resonant frequency; $I$ is the mass polar moment of inertia of the specimen and $I_0$ is the mass polar moment of inertia of the added mass; and $L$ is the height of the specimen.

To facilitate comparison with BE measurements, $G_0$ values measured from RC tests are also presented as a function of void ratio for samples at various percentages of fines and confining stresses, as given in Fig. 8. By comparing the results in Figs 6 and 8, it is possible to conclude that both the BE and RC tests tend to yield a similar effect of fines and also that the void ratio dependence of $G_0$ obtained from both types of testing appears to be similar.

On the other hand, for a given specimen at a given state, the $G_0$ value measured by BE testing is notably greater than that measured by RC testing. For example, for a clean sand specimen at a confining pressure of 100 kPa and a void ratio of 0.802, the $G_0$ value obtained from BE testing is 118.6 MPa, which is approximately 20% larger than the RC measurement under the same state; meanwhile, for a mixed soil specimen (FC = 20%) at a confining pressure of 400 kPa and a void ratio of 0.791, the $G_0$ value measured by RC testing is 140.9 MPa, which is approximately 24% less than the BE measurement. This observation is interesting, and it warrants a further comparison of $G_0$ values obtained by all BE and RC tests, as given in Fig. 9, where RC test data are plotted against their BE counterparts and the diagonal line represents the equality line. Clearly, for either clean sand or sand–fines mixtures, $G_0$ values determined by BE tests are...
consistently larger than those from RC tests by approximately 20%. The possible reasons for this difference may include: (a) the strain level involved in BE tests is relatively lower than that involved in RC tests, and (b) the RC test measures the overall stiffness of a specimen, whereas the BE test measures the central part of the specimen between the transmitter and the receiver, which is likely to be stiffer than the whole specimen owing to the boundary effect.

The observed difference between BE and RC measurements is bigger than that obtained from testing uniform glass beads on the same apparatus (Yang & Gu, 2013). In that earlier study the difference was found to be within 10%, with BE measurements being slightly larger. This raises the question of what the possible reason is for the differing observations. All specimens of glass beads in the earlier study were prepared by dry tamping, whereas in the current study all specimens were prepared by moist tamping. One may therefore speculate that the effect of the testing method might be coupled with the sample preparation method.

To verify this, a set of Toyoura sand specimens were prepared by dry tamping, then saturated and subjected to BE and RC testing using the same apparatus. The test results are shown in Fig. 10. The dry tamping method was not used to prepare specimens of mixtures to avoid uncertainty with the possible effect of segregation. It is striking to note that the BE and RC measurements become comparable for specimens prepared by dry tamping, with the former being slightly larger. The difference observed on Figs 9 and 10 is understandable if one recalls the effect of sample preparation that has been observed on the large-strain behaviour of sands (e.g. Miura & Toki, 1982; Sze & Yang, 2014). Generally, the dry tamping method tends to produce samples with an anisotropic fabric because of the gravitational deposition of grains, whereas samples produced by the moist tamping method tend to be more isotropic because of the capillary effect (Sze & Yang, 2014). A detailed discussion of the issue is beyond the scope of this paper, but further work along this line is worthwhile.

**Effect of fines on \( G_0 \) values**

Given that \( G_0 \) values are dependent on both confining stress and void ratio, it is important to take account of these two factors in quantifying the effect of fines. In doing so, for a...
given confining stress the values of \( G_0 \) are first divided by a void ratio function \( F(e) \) to remove the influence of void ratio, and then presented as a function of fines content, as shown in Fig. 11. For ease of comparison, RC test data are plotted in Fig. 11(a) and BE test data are plotted in Fig. 11(b). Although several void ratio functions are available in the literature, the following one has received wide recognition (Iwasaki & Tatsuoka, 1977; Yamashita et al., 2009) and is adopted here:

\[
F(e) = \frac{(2.17 - e)^2}{1 + e}
\]  

(3)

For either RC or BE tests, the void ratio-corrected \( G_0 \) values decrease approximately linearly with increasing fines content at a given confining stress. The rate of reduction at high confining stress tends to be slightly greater than that at low confining stress, but at a specific confining stress the reduction rate measured by RC tests appears to be similar to that measured by BE tests. It is worth noting that the observed reduction of \( G_0 \) caused by the addition of fines is not due to lower stiffness of the fines compared with the base sand. Laboratory experiments conducted on pure silica fines indicate that under otherwise similar conditions, the crushed silica fines have relatively higher stiffness than the base sand (see Table 3).

Furthermore, the void ratio-corrected \( G_0 \) values are plotted as a function of confining stress that is also normalised by a reference stress in Fig. 12. For the purpose of comparison, RC test data are presented in Fig. 12(a) and BE test data are given in Fig. 12(b). In each plot, the two trend lines represent the case of clean sand (upper bound) and the case of highest fines content tested (lower bound), and the range in between them indicates the effect of varying fines content. The stress dependence of \( G_0 \) is immediately evident in both plots, and this dependence can be represented by a power law as given in equation (1). For each case of fines content, the stress exponent \( n \) and the coefficient \( A \) can be determined by regression, and their values are summarised in Table 4. The high coefficients of determination suggest that the empirical equation with the void ratio function in equation (3) works reasonably well for both clean sand and sand–fines mixtures. In particular, the data obtained have several important features: (a) the stress exponent is not sensitive to the presence of fines; (b) the reduction of \( G_0 \) is mainly reflected by the coefficient \( A \) in the way that it decreases with increasing fines content; and (c) the BE and RC tests tend to yield a similar stress exponent.

By plotting values of \( A \) as a function of fines content, a fairly good correlation is obtained (Fig. 13). Using the RC data as an example, the correlation can be given in an exponential form as follows:

\[
A = 95.39e^{-FC}
\]  

(4)

where \( A \) is in MPa and FC is in decimal. Note that at FC = 0, the coefficient \( A \) takes the value for clean sand (Table 4). Combining equations (1), (3) and (4) yields a simple model for estimating \( G_0 \) values for clean sand and sand–fines mixtures. As an example, Fig. 14(a) shows the calculated \( G_0 \) values plotted against the measured ones from RC tests, indicating a reasonably good agreement between them.

The applicability of empirical equations developed from experiments on clean sands to sand–fines mixtures is an interesting concern. The classical Hardin’s equation (Hardin & Richart, 1963; Hardin & Black, 1966) is one that is
commonly used as a first approximation to estimate $G_0$ values. For angular sands the equation is given as

$$G_0 = 3.9 \frac{(2.97 - e)^2}{1 + e} \sigma^0.5$$

(5)

where $G_0$ is in MPa and $\sigma'$ in kPa. The comparison of calculated $G_0$ values with measured ones given in Fig. 14(b) indicates that Hardin’s equation tends to overestimate $G_0$ values of the mixtures, particularly at a large confining stress and with a high fines content. It is to be mentioned that the comparison shown in Fig. 14 is not intended to claim that the proposed equation is superior to Hardin’s equation; rather, it is to suggest that care should be exercised in the direct use of Hardin’s equation for sand–fines mixtures.

**MICROMECHANICAL CONSIDERATIONS**

Exploring the underlying mechanism for the reduction of $G_0$ caused by the presence of fines is of considerable interest. The existing explanation seems to suggest that the fines in a sand–fines mixture are positioned in the voids formed by sand grains and do not develop effective contacts with sand grains and with a high fines content. It is to be mentioned that the void ratio significantly underestimates $G_0$ values plotted against the measured ones. Evidently, the use of the skeleton void ratio significantly underestimates $G_0$ values of the mixtures even at a low percentage of fines (FC = 5% and 10%), and the discrepancy becomes larger as the quantity of fines increases. This indicates that the concept of the granular void ratio does not work well.

To explore the micro-scale mechanism of small-strain stiffness of granular materials, Gu & Yang (2013) conducted a series of numerical experiments on a regular packing of spheres with different diameter tolerances by using the discrete-element method (DEM). An important finding of their study is that, at an approximately constant void ratio, the $G_0$ of the packing increases as the coordination number increases (Fig. 16). In the context of micromechanics, the coordination number is a key index describing the arrangement of discrete particles in an assembly under a given confinement, and it is defined as the average contact number per particle. Drawing on this grain-scale analysis, it is hypothesised that the reduction of $G_0$ caused by the addition

$$e_s = e + FC \frac{\sigma'}{1 - FC}$$

(6)

where FC is fines content in decimal. With this density index, the consequence of inclusion of fines becomes an increase in the skeleton void ratio and thus a decrease in $G_0$ values.

The above concept was followed by Rahman et al. (2014) in formulating a constitutive model for sand–fines mixtures, in which they proposed that the empirical equation for clean sand can be directly used for sand–fines mixtures as long as the usual void ratio in the equation is replaced by the skeleton void ratio or its modified form. The validity of the proposal can be examined using the experimental data obtained from the current study. In doing so, an empirical equation is first established from RC test data on clean Toyoura sand as a reference

$$G_0 (\text{MPa}) = 95.39 \frac{(2.17 - e)^2}{1 + e} \left(\frac{\sigma'}{\sigma_p}\right)^{0.37}$$

Then the $G_0$ values for Toyoura sand mixed with different quantities of silica fines are calculated by substituting $e_s$ into equation (7). Fig. 15 shows the calculated $G_0$ values plotted against the measured ones. Evidently, the use of the skeleton void ratio significantly underestimates $G_0$ values of the mixtures even at a low percentage of fines (FC = 5% and 10%), and the discrepancy becomes larger as the quantity of fines increases. This indicates that the concept of the granular void ratio does not work well.

![Fig. 12. Void ratio-corrected shear modulus as a function of normalised confining stress: (a) RC tests; (b) BE tests](image-url)
of fines into clean sand, observed at an approximately constant void ratio, is mainly associated with the reduction in the coordination number. Note that this reduction in the coordination number differs from that associated with an increase in the void ratio (e.g. Chang et al., 1991), which has been well recognised.

The hypothesis is schematically illustrated in Fig. 17, where three idealised packings are given to represent three cases: case (a), Fig. 17(a), is for clean sand without fines, case (b), Fig. 17(b), is for sand with a small amount of fines, and case (c), Fig. 17(c), for sand with a relatively large amount of fines. Note that all three packings have the same solid fraction and hence the same void ratio, but they possess different coordination numbers. For the clean sand case, the coordination number is the highest and hence its $G_0$ is the largest, whereas in case (c) the coordination number is the least and correspondingly its $G_0$ is the smallest. To verify the hypothesis about the effect of fines on the coordination number, a series of three-dimensional DEM simulations of random assemblies of spherical particles of coarse and fine sizes under triaxial loading have been conducted. In the simulations the mean size of coarse particles was set to be 1032 μm whereas the mean size of fine particles was set as 245 μm, giving the size ratio of 4·21. This ratio is comparable with that of Toyoura sand–fines mixtures (4, see Table 1). The simulation results, shown in Fig. 18, confirm that, at a given void ratio, the coordination number tends to decline as the quantity of fine particles increases, suggesting that the decrease in the coordination number is a sound micro-scale mechanism for the reduction of $G_0$ observed at the macro-scale. Note that in calculating the coordination number, particles with zero or only one contact have been excluded as they make no contribution to the stable state of stress (Thornton, 2000). Readers are referred to Luo & Yang (2013) where some additional interesting results were given.

### Table 4. Best-fit parameters for shear modulus measurements

<table>
<thead>
<tr>
<th>FC: %</th>
<th>Test method</th>
<th>Fitting parameters*</th>
<th>$R^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>BE</td>
<td>114·38 0·38 1·00</td>
<td></td>
</tr>
<tr>
<td></td>
<td>RC</td>
<td>95·39 0·37 0·99</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>BE</td>
<td>105·96 0·39 0·99</td>
<td></td>
</tr>
<tr>
<td></td>
<td>RC</td>
<td>88·83 0·39 0·99</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>BE</td>
<td>100·52 0·40 0·99</td>
<td></td>
</tr>
<tr>
<td></td>
<td>RC</td>
<td>85·92 0·37 0·96</td>
<td></td>
</tr>
<tr>
<td>20</td>
<td>BE</td>
<td>89·30 0·39 0·95</td>
<td></td>
</tr>
<tr>
<td></td>
<td>RC</td>
<td>73·42 0·37 0·95</td>
<td></td>
</tr>
<tr>
<td>30</td>
<td>BE</td>
<td>85·89 0·40 0·97</td>
<td></td>
</tr>
<tr>
<td></td>
<td>RC</td>
<td>68·14 0·38 0·96</td>
<td></td>
</tr>
</tbody>
</table>

* $A$ (in MPa) and $n$ are the two parameters in equation (1) with the void ratio function in equation (3).

### Fig. 13. Variation of coefficient $A$ with fines content

### Fig. 14. Predicted plotted against measured shear modulus values: (a) proposed equation; (b) Hardin equation

UNIFIED CHARACTERISATION OF $G_0$

In the current literature, the common approach to characterising $G_0$ values for sand is to account separately for the influence of void ratio and confining stress, as for example expressed in equation (1). When fines are present in a clean sand, this empirical approach will lead to a set of trend curves in the $G_0/\bar{F}(\epsilon)-(\sigma/p_a)$ plane, as shown in Fig. 12, with each curve corresponding to a mixture at a specific
percentage of fines. Exploring whether a unified characterisation of $G_0$ can be developed for both clean sand and its mixtures in a theoretical framework is of considerable interest.

In the study of the mechanical behaviour of sands with particular reference to the liquefaction phenomenon (e.g. Casagrande, 1971; Poulos et al., 1985; Verdugo & Ishihara, 1996; Yang, 2002), it has been well recognised that various aspects of the behaviour can be characterised in the framework of critical state soil mechanics, which defines a unique critical state locus in the void ratio plane such that the locus serves as a boundary separating the initial states of sand into contractive and dilative regions (Schofield & Wroth, 1968; Been & Jefferies, 1985; Wood, 1990). The nature of the critical state locus implies that the behaviour of sand can be more closely related to the proximity of its current state to the critical state locus. A state parameter ($\psi$), defined as the difference between the void ratio at the current state and the void ratio at the critical state under the same mean effective stress (Been & Jefferies, 1985), has been found useful in capturing various aspects of the stress–strain–strength behaviour of sand (e.g. Jefferies, 1993; Gajo & Wood, 1999; Yang & Li, 2004). Notably, a state parameter-based platform has also been established to analyse the cyclic strength of sand under both symmetric and non-symmetric loading conditions (Yang & Sze, 2011).

Hence, an attempt is made here to explore whether the state dependence of $G_0$ can be better characterised using this state parameter. The critical state loci of the mixtures (FC = 0%, 5% and 20%) were carefully determined in the earlier study of Yang & Wei (2012), as shown in the four plots in Fig. 19. On each plot the states of the specimens tested in the current study are superimposed, showing a wide spectrum of states ranging from very loose to very dense with reference to the critical state.

For a given confining stress, say 100 kPa, the values of $G_0$ obtained from RC tests for specimens of clean sand and mixtures are presented as a function of the state parameter in Fig. 20(a). It is very encouraging to note that regardless of fines content, a unique trend line emerges that can fit all data points fairly well. Similar results are obtained for other cases of confining stress, as shown in Figs 20(b)–20(d). All these plots indicate that $G_0$ tends to decrease approximately linearly with an increasing state parameter, meaning that as the specimen becomes loose its $G_0$ reduces – this is certainly a reasonable trend.

Furthermore, by taking account of the stress dependence and introducing a state parameter function, $F(\psi)$, a general expression for characterising $G_0$ is proposed as follows

$$G_0 = A_\psi F(\psi) \left( \frac{\sigma'_{\psi}}{p_a} \right)^m = A_\psi \left( a - \psi \right)^2 \left( \frac{\sigma'_{\psi}}{p_a} \right)^m$$

(8)

where $A_\psi$, $a$ and $m$ are parameters that can be determined by regression analysis. For example, using RC test data obtained for samples of FC = 0%, 5%, 10% and 20%, the general expression in equation (8) can be further given as

$$G_0 \text{ (MPa)} = 41.33 \left( \frac{1.36 - \psi}{1 + \psi} \right)^2 \left( \frac{\sigma'_{\psi}}{p_a} \right)^{0.4}$$

(9)

In Fig. 21 the experimental data points are plotted together with the trend line represented by equation (9) in the plane of $G_0/F(\psi)-\sigma'_{\psi}/p_a$. A unified characterisation...
Fig. 19. States of specimens with reference to critical state locus: (a) FC = 0%; (b) FC = 5%; (c) FC = 10%; (d) FC = 20%

Fig. 20. Variation of shear modulus with state parameter at different confining stresses: (a) $\sigma' = 100$ kPa; (b) $\sigma' = 200$ kPa; (c) $\sigma' = 400$ kPa; (d) $\sigma' = 500$ kPa
of $G_0$ values for both clean sand and sand–fines mixtures is achieved satisfactorily.

The general expression proposed in equation (8) is not trivial, but rather it is of significance in several aspects. First, it provides a rational approach for characterising the state-dependent $G_0$ in a unified way. Second, because it is anchored with the state parameter – which has been shown in previous studies to be useful in describing various aspects of the large-strain behaviour of granular soils – the expression provides theoretical insights into the various empirical methods that involve correlations between the small-strain stiffness property and the large-strain response, for example the shear wave-based method for liquefaction evaluation (Ishihara, 1996; Stokoe et al., 1999).

Given the complexity of the effect of fines, further work to validate the unified approach by using experimental data on granular soils of varying size, shape and mineralogy is worthwhile. For gap-graded binary mixtures, the size ratio between coarse and fine grains has long been recognised as an important factor affecting soil behaviour such as piping (e.g. Skempton & Brogan, 1994; Shire et al., 2014) and static liquefaction (Wei, 2012). An effort has thus been made to carry out similar testing series on mixtures of Fujian sand and crushed silica fines to examine the effect of size ratio on $G_0$. Compared with Toyoura sand, Fujian sand is also a uniform quartz sand with sub-angular to sub-rounded grains, but it has a larger mean size ($D_{10} = 282.0 \mu m$, $D_{50} = 397 \mu m$, $C_S = 1.532$), leading to mixtures with a larger size ratio (7.35). Fig. 22(a) shows $G_0$ values plotted against void ratios at three different percentages of fines (FC = 0%, 5% and 10%), measured using the RC method under the confining pressure of 100 kPa. Similarly, at a given void ratio, $G_0$ declines as the quantities of fines increases; but the reduction rate appears to be larger than that for Toyoura sand, and this is thought to be mainly associated with the effect of size ratio. Using the critical state loci defined by Yang & Wei (2012) for mixtures of Fujian sand and silica fines and using $\psi$ as the state variable, the three trend lines in Fig. 22(a) tend to merge into a single line regardless of fines content (Fig. 22(b)), showing that $G_0$ decreases with an increasing state parameter.

One more particular concern is whether the proposed approach works for natural silty sands with continuous grading. Experimental studies that contain adequate information for interpretation in this respect are lacking in the literature. Huang et al. (2004) reported test data on shear wave velocity ($V_s$) for a natural silty sand with different quantities of fines, measured at a confining pressure of 100 kPa by using bender elements installed in a triaxial device. They also conducted a series of monotonic loading tests leading to the information on critical states, but the $V_s$ data were analysed using the conventional method of Hardin & Richart (1963). Fig. 23(a) shows the measured $V_s$ data as a function of void ratio at different fines contents. For a given void ratio, $V_s$ decreases with increasing fines content. By converting $V_s$ to $G_0$ and then plotting the data against the calculated state parameters, it is very encouraging to notice that a unique trend line fitting all data points can be drawn, regardless of fines content, and the trend line also suggests a reduction of $G_0$ as the state parameter increases.

The state parameter-based approach appears to work reasonably well for both gap-graded and continuously graded mixtures. This finding should be expected since the approach is established in the critical state framework, with particular reference to the critical state locus in the compression space. As observed in many experimental studies, the critical state locus tends to change its position with changes in soil grain characteristics (e.g. gradation and particle shape), and this change will consequently lead to changes in the state parameter for a given void ratio and hence changes in $G_0$ and $V_s$.
SUMMARY AND CONCLUSIONS

This paper presents a study where the aim was to investigate how the addition of fines alters the shear wave velocity ($V_s$) and associated stiffness ($G_0$) of sand through well-controlled laboratory experiments in conjunction with analysis and interpretation at the macro and micro scale. The main findings resulting from the study are summarised as follows:

(a) The RC and BE tests consistently show that for the range of fines content (0–30%) considered, the value of $G_0$ tends to decrease continuously as the quantity of fines is increased. By removing the influence of the void ratio, the rates of reduction due to the addition of fines appear to be similar and do not show a notable dependence on the confining stress.

(b) Both RC and BE tests yield a similar stress dependence for $G_0$ and the stress exponent does not appear to be sensitive to changes in fines content. The reduction of $G_0$ is mainly reflected by the coefficient $A$ in the way that its value decreases exponentially with increasing fines content, and the size ratio between coarse and fine particles may play an important role in the variation of $A$ with fines content.

(c) The effect of testing method on $G_0$ appears to be coupled with sample reconstitution methods or associated sample fabrics. For samples prepared by the moist tamping method, $G_0$ values measured by BE testing are notably greater than those measured by RC testing, whereas for samples prepared by the dry tamping method, the BE and RC measurements tend to become comparable.

(d) A new approach that allows the unified characterisation of $G_0$ values for both clean sand and sand–fines mixtures is established in the framework of critical state soil mechanics. Anchored with a state parameter with reference to the critical state locus, the approach provides important insights into the various empirical correlations that involve $V_s$ or $G_0$ in geotechnical engineering practice.

(e) The micro-scale mechanism for the observed reduction of $G_0$ is considered to be associated with the decrease of the coordination number caused by the presence of fines at an approximately constant void ratio. The existing explanation that fines act as voids in a sand–fines mixture is shown to be unsupported by the experimental data.

(f) Given the simultaneous use of RC and BE techniques and the broad range of states covered, the experimental data sets provide a useful reference for the validation and calibration of numerical simulations and theoretical developments in the area. Future work towards validation of the unified approach using physical and/or numerical experiments on different materials is worthwhile.

ACKNOWLEDGEMENTS

The financial support provided by the Natural Science Foundation of China (NSFC) through the Overseas Investigator Award (no. 51428901) and by the University of Hong Kong through the Seed Funding for Basic Research scheme is gratefully acknowledged. The lead author is also thankful to Shanghai Jiao Tong University for the Distinguished Visiting Professorship during the course of this research.

NOTATION

$A$ coefficient in equation (1)
$A_p$ coefficient in equation (8)
$a$ parameter in equation (8)
$c_u$ coefficient of uniformity
$D_{50}$ grain size at which 10% of sample is finer
$D_{60}$ mean particle size
$D_{90}$ grain size at which 60% of sample is finer
$e$ void ratio after consolidation
$e_s$ skeleton void ratio
$F(e)$ void ratio function
$F(\psi)$ state parameter function
$f_{in}$ frequency of input signal in BE test
$f_0$ resonant frequency
$G_0$ small-strain shear stiffness
$G_i$ specific gravity
$I$ mass polar moment of inertia of specimen
$I_0$ mass polar moment of inertia of added mass
$L$ height of specimen
$m, n$ stress exponents
$P'$ mean effective stress
$p_s$ reference stress
$q$ deviatoric stress
$V_s$ shear wave velocity
$\beta$ parameter in equation (2)
$e_a$ axial strain
$\sigma'$ mean effective stress
$\psi$ state parameter


