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On Finite Blaschke Products Sharing Preimages of Sets

The role of complex analysis in complex dynamics in ICMS

Tuen Wai Ng
(joint work with Chiu Yin Tsang)

The University of Hong Kong

23 May 2013
Problem

Given two compact sets $E_1, E_2 \subset \mathbb{D}$, how to characterize all the finite Blaschke products $B_1, B_2$ satisfying

$$B_1^{-1}(E_1) = B_2^{-1}(E_2)?$$

Will solve this problem when $E_1$ and $E_2$ are connected compact sets of positive hyperbolic capacity.
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$B$ is a finite Blaschke product of degree $n$ if

$$B(z) = e^{i\theta} \frac{z - z_1}{1 - \overline{z}_1 z} \cdot \frac{z - z_2}{1 - \overline{z}_2 z} \cdot \ldots \cdot \frac{z - z_n}{1 - \overline{z}_n z},$$

where $z_i \in \mathbb{D}$ and $\theta \in \mathbb{R}$.

- Fatou (1923) proved that $B : \mathbb{D} \to \mathbb{D}$ is analytic and $n$-valent (i.e., every point in $\mathbb{D}$ has precisely $n$ preimages in $\mathbb{D}$ counted with multiplicity) iff $B$ is a finite Blaschke product of degree $n$.

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These two kinds of finite maps share many similar properties and hence we can establish a dictionary between these two kinds of finite maps.

Let \( f : X \to X \) be a finite map with \( \deg f > 1 \), \( X = \mathbb{C}, \mathbb{D} \).

**Definition**

A polynomial/finite Blaschke product \( f \) is said to be *prime* if there do not exist two polynomials/finite Blaschke products \( f_1, f_2 \) with \( \deg f_1, \deg f_2 \geq 2 \) s.t.

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    f(z) = f_1[f_2(z)].
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Otherwise, \( f \) is called *composite*.

Given a polynomial/finite Blaschke product \( f \), we can factorize it as a composition of prime polynomials/finite Blaschke products only, and this factorization will be called a *prime factorization*. 
Finite Blaschke Products vs Polynomials

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Definition

The Chebyshev polynomial \( T_n(z) \) is a polynomial of degree \( n \), defined by

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T_n(\cos \theta) = \cos n\theta.
\]

To define Chebyshev Blaschke products:

- \( \cos \theta \) is replaced by \( cd(u, \tau) := \frac{cn(u, \tau)}{dn(u, \tau)} \) for \( \tau \in \mathbb{R}_+i \).
- Note that \( cd(u, \tau) = sn(u + \frac{\omega_1}{2}, \tau) \).

\[ \therefore cd \text{ is an elliptic function with the periods } 2\omega_1 \text{ and } \omega_2, \text{ where} \]

\[
\omega_1(\tau) = \pi \vartheta_3^2(0, \tau) = \pi (1 + 2q + 2q^4 + \cdots)^2, \quad q = e^{\pi i \tau}
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- The elliptic modulus \( k(\tau) = \frac{\vartheta_2^2(0, \tau)}{\vartheta_3^2(0, \tau)} \), \( \sqrt{k(\tau)} := \frac{\vartheta_2(0, \tau)}{\vartheta_3(0, \tau)} \).
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$$f_{n,\tau}(\sqrt{k(\tau) \cd(u \omega_1(\tau), \tau)}) = \sqrt{k(n\tau) \cd(nu \omega_1(n\tau), n\tau)}.$$ 

For example,

$$f_{1,\tau}(z) = z;$$

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</tr>
<tr>
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</tr>
<tr>
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Let $B_1, B_2$ be finite Blaschke products, $\deg B_1 = d_1$, $\deg B_2 = d_2$, $d_1 \leq d_2$, and $E_1, E_2 \subset \mathbb{D}$ be connected compact sets of positive hyperbolic capacity s.t.

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For a compact subset $E \subset \mathbb{D}$, the hyperbolic capacity $\text{cap}_h(E)$ can be defined in a similar way of the logarithmic capacity (by replacing the Euclidean metric $|z - \zeta|$ by the pseudohyperbolic metric

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Theorem (M. Tsuji, 1947)

For each compact set $E \subset \mathbb{D}$ with $\text{cap}_h(E) > 0$, there exists a unique measure $\mu^h_E$ s.t. $V^h_E = I_h(\mu^h_E)$.

Such a measure $\mu^h_E$ is called the *hyperbolic equilibrium measure* for $E$.

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Let $E \subset \mathbb{D}$ be compact and let $\mu^h_E$ be the hyperbolic equilibrium measure for $E$. Then its potential $u^h_{\mu^h_E}$ has the following properties:

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For each compact set $E \subset \mathbb{D}$ with $\text{cap}_h(E) > 0$, there exists a unique measure $\mu_E^h$ s.t. $V_E^h = I_h(\mu_E^h)$.

Such a measure $\mu_E^h$ is called the hyperbolic equilibrium measure for $E$.

Theorem (M. Tsuji, 1947)

Let $E \subset \mathbb{D}$ be compact and let $\mu_E^h$ be the hyperbolic equilibrium measure for $E$. Then its potential $u_{\mu_E^h}^h$ has the following properties:

(a) $u_{\mu_E^h}^h(z) \leq V_E^h$ in $\mathbb{D}$ and

(b) $u_{\mu_E^h}^h(z) = V_E^h$ quasi-everywhere (q.e.) on $E$, i.e., except for a set of capacity zero.
Application of the Hyperbolic Equilibrium Measure

**Theorem (A)**

Let $B_1$ and $B_2$ be finite Blaschke products of degrees $d_1 \geq 1$ and $d_2 \geq 1$ respectively, and let $E_1, E_2 \subset \mathbb{D}$ be compact. Suppose that $\text{cap}_h(E_1), \text{cap}_h(E_2) > 0$ and $\Omega := B_1^{-1}(E_1) = B_2^{-1}(E_2)$. Then

\[
\frac{u_{\mu_{E_1}}^h \circ B_1(z)}{d_1} = \frac{u_{\mu_{E_2}}^h \circ B_2(z)}{d_2}, \quad \text{for all } z \in \overline{\mathbb{D}}.
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$$\frac{u^h_{\mu_{E_1}} \circ B_1(z)}{d_1} = \frac{u^h_{\mu_{E_2}} \circ B_2(z)}{d_2}, \quad \text{for all } z \in \overline{\mathbb{D}}.$$
Pullback Measure

**Definition**

Given a probability measure $\mu$ on $E$, the *pullback measure* $B^*\mu$ is the probability measure on $B^{-1}(E)$ s.t. for all holomorphic functions $f$ on $B^{-1}(E)$,

$$
\int_{B^{-1}(E)} f(\xi) d(B^*\mu)(\xi) = \int_E \sum_{\xi \in B^{-1}(\{\zeta\})} f(\xi) d\mu(\zeta),
$$

where the summation is over all the roots of $B(\xi) - \zeta$ and a root of multiplicity $m$ is repeated $m$ times. Indeed,

$$
B^*\mu(B^{-1}(E_0)) = \int_{E_0} \sum_{\xi \in B^{-1}(\{\zeta\})} 1 d\mu(\zeta) = d \cdot \mu(E_0), \quad E_0 \subset E.
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Equilibrium Measure on $B^{-1}(E)$

**Proposition**

Let $B$ be a finite Blaschke product of degree $d$ and $\Omega = B^{-1}(E)$. Suppose $\text{cap}_h(E) > 0$. If $\mu^h_E$ is the equilibrium measure on $E$, then the equilibrium measure $\mu^h_\Omega$ on $\Omega$ is

$$\frac{B^* \mu^h_E}{d}$$
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$$

To prove this theorem, we need the following lemma.

**Lemma**

Let $\mu$ be a finite Borel measure on $\mathbb{D}$ with compact support. Then

$$
\Delta u_{\mu}^h = -2\pi \mu,
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where $\Delta$ is the generalized Laplacian.
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Proof of Theorem A

Let $\mu^h_{E_j}$ be the hyperbolic equilibrium measure of $E_j$.

- $d_1^{-1}B_1^*\mu^h_{E_1}$ and $d_2^{-1}B_2^*\mu^h_{E_2}$ are hyperbolic equilibrium measures of $\Omega$.
- By the uniqueness, we get $d_1^{-1}B_1^*\mu^h_{E_1} = d_2^{-1}B_2^*\mu^h_{E_2}$.
- $\frac{u^h_{\mu^h_{E_i}}}{d_i} = \frac{u^h_{\mu^h_{B_i\cdot E_i}}}{d_i}$, $i = 1, 2$.
- By Lemma,

\[\Delta u^h_{\mu^h_{B_1^* E_1}} = -2\pi d_1^{-1}B_1^*\mu^h_{E_1} = -2\pi d_2^{-1}B_2^*\mu^h_{E_2} = \Delta u^h_{\mu^h_{B_2^* E_2}}.\]

- Let $\Psi = \frac{u^h_{\mu^h_{B_1^* E_1}}}{d_1} - \frac{u^h_{\mu^h_{B_2^* E_2}}}{d_2}$ on $\overline{D}$. As $\Delta \Psi = 0$, $\Psi$ is harmonic on $D$.

As $\frac{u^h_{\mu^h_{B_i^* E_i}}}{d_i} \equiv 0$ on $\partial D$, $\Psi \equiv 0$ on $\partial D$.

- $\Psi \equiv 0$ on $\overline{D}$, which proves the Theorem A.
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- Let $\Psi = u^h_{B_1^*\mu^h_{E_1}} - u^h_{B_2^*\mu^h_{E_2}}$ on $\overline{\mathbb{D}}$. As $\Delta \Psi = 0$, $\Psi$ is harmonic on $\mathbb{D}$.
- As $u^h_{B_i^*\mu^h_{E_i}} \equiv 0$ on $\partial \mathbb{D}$, $\Psi \equiv 0$ on $\partial \mathbb{D}$.
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By Lemma,

$$\Delta u^h_{B_1^*\mu^h_{E_1}} = -2\pi d_1^{-1}B_1^*\mu^h_{E_1} = -2\pi d_2^{-1}B_2^*\mu^h_{E_2} = \Delta u^h_{B_2^*\mu^h_{E_2}}.$$

Let $\Psi = u^h_{B_1^*\mu^h_{E_1}} - u^h_{B_2^*\mu^h_{E_2}}$ on $\overline{D}$. As $\Delta \Psi = 0$, $\Psi$ is harmonic on $D$.

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Proof of Theorem A

Let $\mu_{E_j}^h$ be the hyperbolic equilibrium measure of $E_j$.

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Let $\mu^h_{E_j}$ be the hyperbolic equilibrium measure of $E_j$.

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Invariants of a Finite Blaschke Product near $\partial \mathbb{D}$

- Study the continuous function $u : \partial \mathbb{D} \rightarrow \partial \mathbb{D}$ which is invariant for a finite Blaschke product $B$, i.e.,

$$B \circ u = B \text{ on } \partial \mathbb{D}.$$ 

- In fact, these functions form a cyclic group.

**Theorem (Cassier & Chalendar (2000))**

Let $B$ be a finite Blaschke product of degree $d \geq 1$. The set of the continuous functions $u : \partial \mathbb{D} \rightarrow \partial \mathbb{D}$ s.t. $B \circ u = B$ is a cyclic group (for the composition) of order $d$, say $\{u_1, \cdots, u_d\}$. 

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Tuen Wai Ng (joint work with Chiu Yin Tsang) (HKUST)
Analytic Extension of $u_k$

Moreover, each $u_k$ can be extended analytically to a neighborhood of $\partial \mathbb{D}$.

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Let $B$ be a finite Blaschke product of degree $d \geq 1$ and denote by $M = \max\{|\alpha| : B(\alpha) = 0\}$. Then each of the $d$ continuous function $u_k$ on $\partial \mathbb{D}$ ($1 \leq k \leq d$) s.t. $B \circ u_k = B$ has an analytic extension $\tilde{u}_k$ in the annulus $A = \{z \in \mathbb{C} : M < |z| < 1/M\}$ which still satisfies $B \circ \tilde{u}_k = B$.

Denote the extension $\tilde{u}_1$ by $u_B$. Then $u_B$ is a conformal map in a small neighborhood of $\partial \mathbb{D}$ s.t.

1. $B \circ u_B^k = B$ ($1 \leq k \leq d$),
2. $u_B^d = id$, and
3. $u_B, u_B^2, \ldots, u_B^{d-1}, id$ are all distinct.
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The $u_B$ will give information about the factorizations of $B$

**Theorem (B)**

Let $B_1$ and $B_2$ be two finite Blaschke products of degrees $d_1 \geq 1$ and $d_2 \geq 1$, and let $d = \gcd(d_1, d_2)$.

(a) If $\Phi$ is a finite Blaschke product s.t. $\Phi \circ u_{B_1} = \Phi$ in the neighborhood of $\partial \mathbb{D}$, then there exists a finite Blaschke product $B$ s.t.

$$
\Phi = B \circ B_1.
$$

(b) If $u_{B_1}^{\circ k_1 d_1/d} = u_{B_2}^{\circ k_2 d_2/m}$ ( $\gcd(k_j, d) = 1$), then there exist finite Blaschke products $B, \tilde{B}_1, \tilde{B}_2$ (deg $B = d$) s.t.

$$
B_1 = \tilde{B}_1 \circ B, \quad B_2 = \tilde{B}_2 \circ B.
$$

(c) If $d = 1$ and if $u_{B_1} \circ u_{B_2} = u_{B_2} \circ u_{B_1}$, there exist finite Blaschke products $\Phi$ (deg $\Phi = d_1 d_2$), $B_1^*, B_2^*$ s.t.

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(c) If $d = 1$ and if $u_{B_1} \circ u_{B_2} = u_{B_2} \circ u_{B_1}$, there exist finite Blaschke products $\Phi$ (deg $\Phi = d_1 d_2$), $B_1^*, B_2^*$ s.t.

$$\Phi = B_1^* \circ B_2 = B_2^* \circ B_1.$$
The $u_B$ will give information about the factorizations of $B$

**Theorem (B)**

Let $B_1$ and $B_2$ be two finite Blaschke products of degrees $d_1 \geq 1$ and $d_2 \geq 1$, and let $d = \gcd(d_1, d_2)$.

(a) If $\Phi$ is a finite Blaschke product s.t. $\Phi \circ u_{B_1} = \Phi$ in the neighborhood of $\partial \mathbb{D}$, then there exists a finite Blaschke product $B$ s.t.

$$\Phi = B \circ B_1.$$ 

(b) If $u_{B_1}^{\circ k_1 d_1 / d} = u_{B_2}^{\circ k_2 d_2 / m}$ ($\gcd(k_j, d) = 1$), then there exist finite Blaschke products $B$, $\tilde{B}_1$, $\tilde{B}_2$ (deg $B = d$) s.t.

$$B_1 = \tilde{B}_1 \circ B, \ B_2 = \tilde{B}_2 \circ B.$$ 

(c) If $d = 1$ and if $u_{B_1} \circ u_{B_2} = u_{B_2} \circ u_{B_1}$, there exist finite Blaschke products $\Phi$ (deg $\Phi = d_1 d_2$), $B_1^*$, $B_2^*$ s.t.

$$\Phi = B_1^* \circ B_2 = B_2^* \circ B_1.$$
How to get $g_1 \circ B_1 = g_2 \circ B_2$?

By Theorem A, we have

$$
\frac{u^h_{\mu_{E_1}} \circ B_1(z)}{d_1} = \frac{u^h_{\mu_{E_2}} \circ B_2(z)}{d_2}, \text{ for all } z \in \overline{D}.
$$

For $i = 1, 2$, let $K_i$ be the component of $D \setminus E_i$ which borders on $\partial D$. Since $E_i$ is connected, $K_i$ is doubly connected and there exists a biholomorphisic function $\varphi_i$ from $K_i$ onto $\{\rho_i < |w| < 1\}$ s.t. $\varphi_i(\partial D) = \partial D$.

Note that $u^h_{\mu_{E_i}}(z) = -\log |\varphi_i(z)|$ for all $z \in K_i$.

$$
\frac{\log |\varphi_1 \circ B_1(z)|}{d_1} = \frac{\log |\varphi_2 \circ B_2(z)|}{d_2} \text{ and } |z^{d_2} \circ \varphi_1 \circ B_1(z)| = |z^{d_1} \circ \varphi_2 \circ B_2(z)|
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- For any $z \in \mathbb{D}$ sufficiently close to $\partial \mathbb{D}$,

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- $\varphi_i$ can extend continuously and homeomorphically to $\partial \mathbb{D}$ and

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  can also be defined on $\partial \mathbb{D}$.

- Try to show that $u_{B_1}$ and $u_{B_2}$ satisfy conditions in (b) or (c) of Theorem B.
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Pakovich’s approach for $\Omega = B_1^{-1}(E_1) = B_2^{-1}(E_2)$

Let $\mathcal{B}_n$ denote the set of all finite Blaschke products of degree $n$ and let $E \subset \mathbb{D}$ be compact.

**Definition**

A finite Blaschke product $\tilde{B} \in \mathcal{B}_n$ is called a minimal Blaschke product of degree $n$ for $E$ if $\|\tilde{B}\|_E = \min_{B \in \mathcal{B}_n} \|B\|_E$.

**Theorem (Walsh (1952))**

(Existence and location of zeros) A minimal Blaschke product $\tilde{B}$ exists and its zeros lie in the convex hull of $E$ with respect to the hyperbolic geometry in $\mathbb{D}$.
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The counter-part of the two conjectures below are known to be true for polynomials.

**Conjecture (A)**

*Such a minimal Blaschke product of degree $n$ is unique up to multiplication by $e^{i\theta}$ when $|E| \geq n$.*

**Conjecture (B)**

*Let $T$ be a minimal Blaschke product of degree $m$ for $E$. Then for any finite Blaschke product $B$ of degree $n$, $T \circ B$ is a minimal Blaschke product of degree $mn$ for $B^{-1}(E)$.*
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- For $i = 1, 2$, let $C_i$ be a minimal Blaschke product of degree $d_i/d$ for $E_i$, where $d = \gcd(d_1, d_2)$.
- By conjecture B, $C_i \circ B_i$ is a minimal Blaschke product of degree $d_1d_2/d$ for $\Omega = B_i^{-1}(E_i)$.
- Suppose $|\Omega| \geq \operatorname{lcm}(d_1, d_2) = d_1d_2/d$, by Conjecture A, we have
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