

# The Classical $n$ -Port Resistive Synthesis Problem

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## Introduction

An  $n$ -port resistive network is an  $n$ -port circuit consisting of only passive resistors, which is an important class of passive networks. An  $n$ -port resistive network is usually characterized by its *impedance* or *admittance* matrix. Since there are no reactive elements, passivity and reciprocity imply that the impedance and admittance matrices of  $n$ -port resistive networks must be nonnegative definite if they exist [21]. Since no transformers are present, there are further constraints. The realizability problem of  $n$ -port resistive networks was an active topic and was widely investigated from the 1950s to the 1970s. Recently, the invention of a new mechanical element named inerter [25] has revived research into passive network synthesis. Investigation on  $n$ -port resistive network synthesis can be a critical step towards solving transformerless realizations of multi-port passive networks, and its results can be directly applied to minimal realizations of one-port mechanical and electrical networks based on element extraction [11, 12]. Therefore, the significance of this topic has become apparent again.

**Problem:** What are testable necessary and sufficient conditions for a real symmetric  $n \times n$  matrix to be realizable as the admittance (resp. impedance) of an  $n$ -port resistive network?

In [24], Tellegen has shown that *paramountcy* is a necessary and sufficient condition for any second-order or third-order real symmetric matrix to be realizable as the impedance (resp. admittance) of a two-port or three-port resistive network. Since Tellegen's proof is in Dutch and there is no English version, [10, Appendix A] presents a full and better structured reworking of Tellegen's discussion.

A question arose whether the condition of paramountcy can be generalized to the case of  $n > 3$ . Utilizing the graph theory, Cederbaum [8] first showed that if a matrix  $Y_n \in \mathbb{S}^n$  (resp.  $Z_n \in \mathbb{S}^n$ ) is the admittance (resp. impedance) of an  $n$ -port resistive network, then  $Y_n$  (resp.  $Z_n$ ) must be paramount. In [9], Cederbaum presented a paramount matrix that cannot be realized as either the impedance or the admittance of an  $n$ -port resistive network and presented a matrix that is realizable as the impedance but not the admittance of an  $n$ -port resistive network. Hence, when  $n > 3$ , paramountcy is only a necessary but not a sufficient condition for the realization, and realizability conditions of admittances and impedances are not the same.

Investigation on the synthesis of  $n$ -port resistive networks when  $n > 3$  has primarily focused on the admittance synthesis. For the realizability of admittances as  $n$ -port resistive networks, the least number of terminals is  $(n + 1)$ . Brown *et al.* [5–7, 14] obtained the necessary and sufficient conditions for  $Y_n \in \mathbb{S}^n$  to be realizable as admittances of  $n$ -port resistive networks with  $(n + 1)$  terminals when the port graph  $\mathcal{G}_p$  is a *path tree* or a *Lagrangian tree*. Systematic approaches to determine the possible port graphs for realizability are available in [1–4, 14, 16], and the corresponding realizability can be tested by transforming the port graph into a Lagrangian tree based on the discussion in [7]. Unlike the  $(n + 1)$ -terminal case, the realizability of admittances as  $n$ -port resistive networks with more than  $(n + 1)$  terminals for  $n > 3$  has not

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been completely solved, although a series of results have been derived [13, 15, 17–20, 22, 23]. Specifically, Chen *et al.* [13] recently derived some new results on the realization problem of  $n$ -port resistive networks containing  $2n$  terminals, which can be an important step towards solving the realizability with more than  $(n + 1)$  terminals. However, there are still three challenges in completely solving this problem.

The first challenge of solving this problem is to deal with parameters in realizability conditions with more than  $(n + 1)$  terminals when  $n > 3$ . Although some results for the realizability of admittances with  $(n + 2)$  and  $2n$  terminals have been presented in [23] and [13], the conditions are based on the existence of a set of parameters, which are not directly testable. How to eliminate the parameters or to establish a systematic procedure of testing the existence of parameters for  $n > 3$  is far from being solved.

The second challenge is the complexity of different cases for the realizability when  $n > 3$ . The number of possible topological connections becomes increasingly large when  $n$  increases. It is difficult to establish a unified framework for the discussion when  $n$  is a large number.

The third challenge is the synthesis of impedances when  $n > 3$ . As shown in [9], the necessary and sufficient conditions for the realizability of impedances and admittances as  $n$ -port resistive networks are different when  $n > 3$ . Therefore, it is necessary to discuss them separately. At present there are few investigations available on realizability of impedances as  $n$ -port resistive networks when  $n > 3$ . It is also difficult to generalize the methodology of investigation on the admittance synthesis to the impedance case.

## Conclusion

In summary, the classical  $n$ -port resistive synthesis problem is solved in the case of  $n \leq 3$ , but there exist significant challenges when  $n > 3$ .

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