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Abstract—This paper addresses the global $\mathcal{H}_\infty$ pinning synchronization problem for a class of directed networks with aperiodic sampled-data communications. Important yet challenging issues of how many and which nodes should be pinned for realizing global synchronization in a fixed directed network without external disturbances are first discussed. By using a combined tool from the input-delay approach and free-weighting matrices technique, some sufficient synchronizability conditions are then derived for such networks. Furthermore, a multi-step algorithm is designed to estimate the upper bound of the maximum allowable sampling intervals for achieving synchronization. Theoretical results are then extended to global $\mathcal{H}_\infty$ pinning synchronization in fixed and switched directed networks with external disturbances, showing that a finite $\mathcal{H}_\infty$ performance index can be guaranteed under some suitable conditions. Finally, numerical simulations are performed to demonstrate the effectiveness of the analytical results.

Index Terms—Complex network, $\mathcal{H}_\infty$ control, pinning synchronization, sampled-data communication, spanning tree.

I. INTRODUCTION

The past few years have witnessed a strong upsurge of the study of complex networks in various fields, ranging from physics to mathematics and also to control engineering [1]–[5]. Typical examples of complex networks include the Internet, biological neural networks, smart grids, and various human social networks [5].

II. PROBLEM STATEMENT AND PRELIMINARIES

Compared with a single system, analysis on the dynamical behaviors of complex networks is more challenging since their dynamics are coupled with the topological evolution. Investigations on the dynamics of complex networks could not only help scientists understand the emergent mechanisms for many intriguing collective behaviors, such as synchronization and swarming, but also help engineers effectively apply complex networks theory to practical problems, for example, controlling the spread of diseases over computer networks. One of the most important collective behaviors in complex networks is the network synchronization, which is a timekeeping dynamical behavior in which the states of all nodes in the network converge towards the same trajectory. Much effort has been devoted to studying synchronization in complex networks in the last decade. Consequently, many profound results have been reported in the literature (see the recent survey paper [6] for more details). In [7], [8], local synchronization was investigated by using the master stability function method, while in [9]–[11], global synchronization was addressed based on the Lyapunov function method. For the case that the network under consideration cannot achieve synchronization by itself, one may apply some control inputs to the network to guarantee synchronization [6]. However, it is practically impossible to implement one controller to each node in a large-scale network. One practical way to solve this problem is to control only a small fraction of the nodes by applying some carefully designed local feedback injections, which is known as pinning synchronization control [12]–[20]. Note that active research on pinning synchronization is still ongoing today [21]–[23].

With the rapid developments of data communication technologies and high-performance computers, a designed continuous-time feedback controller is usually implemented in a digital form for better reliability, flexibility and cost-effectiveness. However, most of the aforementioned results on synchronization control of complex networks assume that the information transmission among the nodes is continuous, implying that the coupling law of the network is executed by analog signals. Furthermore, the communication topology among some real networks may be directed and time-varying. And, moreover, the microscopic synchronization principles of directed networks are less understood than those of undirected networks. From these observations, and based on the above-mentioned works, this paper focuses on pinning synchronization for complex directed networks with sampled-data communications and time-varying topologies. Particularly, the paper aims to deal with the following three issues. First, for an arbitrarily given network, it tries to find out at least how many and which nodes should be pinned for achieving global pinning synchronization. Second, it tries to provide some simple yet less-conservative criteria for achieving global pinning synchronization, and to estimate the upper bound of the maximum allowable sampling intervals for...
synchronization in both fixed and switched networks. Third, it
tries to analyze the $H_{\infty}$ performance for sampled-data pinning
synchronization of directed networks in the presence of external
disturbances. As will be shown, global $H_{\infty}$ pinning synchro-
nization of such networks with external disturbances can be
guaranteed if some nodes are carefully selected to pin, and some
sufficient criteria in terms of linear matrix inequalities (LMIs)
are satisfied. One favorable property of the present criteria is
that the dimensions of the involved LMIs are independent
of those of each node’s states. This feature is very desirable for
the case when nodes’ dimensions are high. The effectiveness of
the theoretical analysis is demonstrated by performing numerical
simulations on coupled Chua’s circuits as well as some other
dynamical systems.

The rest of the paper is structured as follows. In Section II,
notations and some preliminaries on algebraic graph theory are
provided. Problem formulation is then given in Section III.
In Section IV, the main theoretical results are presented. In
Section V, numerical simulations are performed for illustration.
Concluding remarks are finally drawn in Section VI.

II. NOTATIONS AND PRELIMINARIES

Notations and some basic preliminaries on algebraic graph
theory are presented in this section.

A. Notations

Let $\mathbb{R}^n$ and $\mathbb{R}^{n \times n}$ be respectively the sets of $n$-dimensional
column real vectors and $n \times n$ real matrices. $I^n = [0, +\infty)$ is the
$n$-dimensional square integrable function space over $[0, +\infty)$. $\mathbf{0}_n$ represents the $n$-dimensional column vector with all
elements being 0. $I_{n \times n}(\mathbf{0}_n)$ is the $n \times n$ identity (zero) matrix. De-
note the set of positive natural numbers by $\mathbb{N}$. Matrix inequality
$A > 0$ (resp., $A \geq 0$) means that $A$ is positive definite (resp.,
positive semi-definite). Symbols $\otimes$, $\cdot$ and $\cdot_2$ denote the
Kronecker product of matrices, Euclidean norm and $L_2$ norm,
respectively. Notation $\mathbf{diag}\{A_1, \ldots, A_n\}$ represents a block-di-
gonal matrix with matrices $A_i$, $i = 1, \ldots, n$, being its diagonal
elements.

B. Algebraic Graph Theory

Let $G(V, E, A)$ be a weighted directed graph of order $N$, 
where $V = \{1, \ldots, N\}$ is the set of nodes, $E \subseteq V \times V$ is the
set of edges, and $A = [a_{ij}]_{N \times N}$ is the weighted adjacency matrix.
A directed edge $d_{ij}$ is denoted by the ordered pair of nodes
$(j, i)$, and $d_{ij} \in E$ if and only if $a_{ij} > 0$. In the sequel, denote
$G(V, E, A)$ by $G(A)$ if no confusion will occur. The in-degree
of node $i$ is defined as $D_i = \sum_{j=1}^{N} a_{ji}$, and $D_i$ is the
in-degree of node $i$. A directed path on $G(A)$ from node $z_1$ to
node $z_k (s > 1)$ is a sequence of ordered edges of the form
$(z_{s+1}, z_{k})$, $k = 1, 2, \ldots, s - 1$. A directed
graph is strongly connected if there exists at least one
directed path between any pair of distinct nodes [24]. A directed
circuit contains a directed spanning tree if there exists a node
r, called a root, such that there exists a directed path from this
circuit to every other node $v$, i.e., $v$ is reachable from $r$ [24]. The Lapla-
cian matrix $\tilde{L} = \tilde{L}_{ij} \in \mathbb{R}^{N \times N}$ of $G(A)$ is defined with
$\tilde{L}_{ij} = -a_{ij}$, $i \neq j$, and $\tilde{L}_{ii} = \sum_{j=1}^{N} a_{ij}$ for $i = 1, 2, \ldots, N$.

By introducing a new node, labeled as $N + 1$, and some
directed edges in the form of $(N + 1, i)$, $i \in V$, to $G(V, E, A)$,
one may obtain a graph $G'(V, E, \tilde{A})$ with order of $N + 1$. Here,
$G'(V, E, \tilde{A})$ is called the augmented graph of $G(V, E, A)$. Let $\tilde{L}$
be the Laplacian matrix of $G'(V, \tilde{E}, \tilde{A})$, it can be verified that

$$\tilde{L} = \begin{bmatrix} \tilde{L} & -\varphi \\ \mathbf{0}_N & 0 \end{bmatrix}, \text{ where } \varphi = (\varphi_1, \varphi_2, \ldots, \varphi_N)^T \text{ with } \varphi_i > 0$$

if $(N + 1, i) \in \tilde{E}$ and $\varphi_i = 0$ otherwise, $i = 1, 2, \ldots, N$, and
$L = \tilde{L} + \mathbf{d} \mathbf{a} \{p_1, p_2, \ldots, p_N\}$.

III. PROBLEM FORMULATION

Consider the following dynamical network consisting of $N$
nodes with aperiodic sampled-data-based diffusive couplings:

$$\dot{x}_i(t) = f(x_i(t), t) + c \sum_{j=1}^{N} a_{ij} \left( x_j(t_k) - x_i(t_k) \right) + \omega_i(t), \quad t \in [t_k, t_{k+1}), \quad k \in \mathbb{N},$$

(1)

where $x_i(t) = [x_{i1}(t), x_{i2}(t), \ldots, x_{in}(t)]^T \in \mathbb{R}^n$ is the state
vector of the $i$-th node, $1 \leq i \leq N$; $f : \mathbb{R}^n \times [0, +\infty) \rightarrow \mathbb{R}^n$ is a
nonlinear function satisfying the global Lipschitz condition

$$\|f(y, t) - f(\hat{y}, t)\| \leq \varrho \|y - \hat{y}\|, \quad \forall y, \hat{y} \in \mathbb{R}^n.$$

(2)

for some given $\varrho > 0$; $c > 0$ is the coupling strength;
$A = [a_{ij}]_{N \times N}$ is the adjacency matrix of the communication
topology $G(A)$ of the network; $\omega_i(t) \in L^2_2[0, +\infty)$ are external
disturbances.

For the case of $\omega_i(t) = \mathbf{0}_N, i = 1, 2, \ldots, N$, the global pinning
synchronization is said to be achieved if, under pinning
control, the states of all nodes in network (1) converge to a pre-
scribed trajectory $s(t)$ in the sense of $\lim_{t \rightarrow +\infty} \|x_i(t) - s(t)\| = 0$
for any given initial conditions, where $s(t)$ is generated by

$$s(t) = f(s(t), t),$$

(3)

for some $s(t) \in \mathbb{R}^n$. For notational brevity, let $e(t) = [e_1(t), e_2(t), \ldots, e_N(t)]^T$ with $e_i(t) = x_i(t) - s(t)$, $i = 1, 2, \ldots, N$. The global $H_{\infty}$ pinning
synchronization with performance index $\gamma > 0$ is said to be achieved if, under
pinning control, the following two conditions are satisfied:

i) For $\omega(t) \equiv \mathbf{0}_N$, global pinning synchronization in net-
work (1) with target system (3) can be achieved.

ii) For a given scalar $\gamma > 0$ and initial condition $e(0) = \mathbf{0}_N$, the worst-case norm of synchronization error vector
$e(t)$ over all admissible exogenous disturbances $\omega(t)$, defined
by

$$\gamma_\omega = \sup_{\omega(t) \in L^2_2[0, +\infty)} \frac{\|e(t)\|_2}{\|\omega(t)\|_2},$$

(4)

is less than $\gamma$, i.e., $\gamma_\omega < \gamma$.

Remark 1: The notion of $H_{\infty}$ pinning synchronization is bor-
rrowed from the idea of $H_{\infty}$ control for dynamical systems in the
context of modern control theory [25]. The classic $H_{\infty}$ control
problem can be described as how to construct some stabilizing
controllers, such that the closed-loop systems are internal stable
and the $H_{\infty}$ norm of the transfer matrix from exogenous sig-
nals to the performance variable is less than a prespecified
positive number. Within the context of pinning synchronization in
complex networks, the synchronization error is always taken
to be a performance variable [26], [27]. However, for the con-
venience of analysis, it is always assumed in the literature on
$H_{\infty}$ pinning synchronization of complex networks that the information
exchange only happens at some discrete time instants. It is thus practically important to
study how to realize an $H_{\infty}$ pinning synchronization with sam-
ples-data communications.
In the present paper, some sampled-data-based negative feedback injections will be employed to control network (1). The closed-loop network under pinning control is described by:

\[
\dot{x}_i(t) = f(x_i(t), t) + c\sum_{j=1}^{N} a_{ij}(x_j(t_k) - x_i(t_k)) + \omega_i(t),
\]

\[
-r_i(x_i(t_k) - s_i(t_k)), \quad t \in [t_k, t_{k+1}), \quad k \in \mathbb{N},
\]

in which the feedback gain \( p_i > 0 \) if node \( i \) is pinned, and \( p_i = 0 \) otherwise, \( i = 1, 2, \ldots, N \). Taking the target system (3) as a virtual node labeled as \( \bar{N} + 1 \) in network (5) and defining \( \bar{e}_i(t) = x_i(t) - s_i(t), i = 1, 2, \ldots, N \), one has

\[
\bar{e}_i(t) = F(x(t); s(t)) - c(\bar{L} \otimes I_n)\bar{e}_i(t) + \omega(t),
\]

where

\[
F(x; s) = [f_1(x_1; s_1), \ldots, f_N(x_N; s_N)]^T, \quad f_i(x_i; s_i) = \frac{d}{dt}x_i(t_k) - f(s_i(t), t), \quad i = 1, 2, \ldots, N,
\]

\[
\bar{L} = \mathcal{L} + \text{diag}(p_1, p_2, \ldots, p_N), \quad \omega(t) = \sum_{i=1}^{N} \omega_i(t) = \sum_{i=1}^{N} \omega_i(t).
\]

Let \( \bar{L} = [\bar{L}_{i,j}] \in \mathbb{R}^{N \times N} \) is the Laplacian matrix of the augmented graph \( \bar{G}(\bar{A}) \).

Let \( d_k(t) = t - t_k, \) for \( t \in [t_k, t_{k+1}), k \in \mathbb{N} \). One then gets that \( t_k = t - d_k(t) \) with \( 0 \leq d_k(t) < h, \) for \( t \in [t_k, t_{k+1}), k \in \mathbb{N} \). Then, the error system (6) can be rewritten as the following retarded functional differential equation:

\[
\dot{\bar{e}}(t) = F(x(t); s(t)) - c(\bar{L} \otimes I_n)\bar{e}(t - d_k(t)) + \omega(t),
\]

where

\[
t \in [t_k, t_{k+1}), k \in \mathbb{N}.
\]

The initial condition of (7) is set as \( \bar{e}(\theta) \equiv 0 \), for all \( \theta \in [-h, 0] \). It is not hard to verify that \( \bar{e}(t) \equiv 0_{N_N} \) is a fixed point of the error dynamical system (7). Noticeably, under the condition \( \omega(t) \equiv 0_{N_N}, \) global pinning synchronization in network (7) will be achieved if and only if the trivial solution \( \bar{e}(t) \equiv 0_{N_N} \) is a globally attractive fixed point of (7).

Before ending this subsection, two assumptions are made.

**Assumption 1:** There is a constant \( h > 0 \) such that \( t_{k+1} - t_k \leq h, \) for \( k \in \mathbb{N} \).

**Assumption 2:** The augmented graph \( \bar{G}(\bar{A}) \) contains a directed spanning tree with node \( N + 1 \) as the root.

**IV. MAIN RESULTS**

The main theoretical results are established and analyzed in this section.

**A. Selective Pinning Strategy**

In this subsection, a graph search algorithm with linear time complexity is provided for choosing the nodes to be pinned in network (5). The important issues of at least how many and which nodes should be pinned such that Assumption 2 holds will be addressed.

**Algorithm 1**

Let \( \bar{G}(\bar{A}) \) be the communication topology of directed network (1). Then, Assumption 2 will hold if the \( r_0 \) nodes searched by the following procedures are selected and pinned.

1) Use Tarjan’s algorithm [29] to find all the nodes with zero in-degree and strongly connected components of \( \bar{G}(\bar{A}) \). Suppose that there are \( \kappa_1 \{\kappa_1 \geq 0\} \) nodes with zero in-degree, labeled as \( v_1, v_2, \ldots, v_{\kappa_1}, \) and \( \kappa_2 \{\kappa_2 \geq 0\} \) strongly connected components, represented by \( \bar{G}(V_1, E_1, A_1), \bar{G}(V_2, E_2, A_2), \ldots, \bar{G}(V_{\kappa_2}, E_{\kappa_2}, A_{\kappa_2}) \) in \( \bar{G}(\bar{A}) \). Let \( r_i = 0, \) for \( i = 0, 1, \ldots, \kappa_2, \) and \( g = 1 \).

2) All the \( \kappa_1 \) nodes with zero in-degree should be selected and pinned. Then, update the value of \( r_0 \) by \( r_0 = r_0 + \kappa_1 \).

3) Check the condition \( \kappa_2 \neq 0 \)? If it does not hold, stop; else go to step 4).

4) Check whether there exists at least one node in \( V_g \) which is reachable from a node belonging to the node set \( V \setminus V_g \). If it holds, go to step 5; otherwise, go to step 6).

5) Check the following condition: \( g < \kappa_2 \)? If it holds, let \( g = g + 1 \) and re-perform step 4; else stop.

6) Arbitrarily select one node in \( V_g \) to be pinned, update the value of \( r_0 \) by \( r_0 = r_0 + 1 \). Check the following condition: \( g < \kappa_2 \)? If it holds, let \( g = g + 1 \) and go to step 4; else stop.

It can be checked that there exists at least one node in \( \bar{G}(\bar{A}) \) which is not reachable by node \( N + 1 \) if there are less than \( r_0 \) nodes in \( \bar{G}(\bar{A}) \) that are selected and pinned. Furthermore, it is worth noting that the complexity of Algorithm 1 is \( O(N + |E'|) \), where \( |E'| \) is the number of the directed edges in \( \bar{G}(\bar{A}) \). Noticeably, under the condition that target system (3) possesses a globally attractive solution \( \bar{s}(t) \), the global pinning synchronization in network (5) can be achieved asymptotically even when there is no coupling between any pair of neighboring nodes in \( \bar{G}(\bar{A}) \). In the sequel, it is assumed that target system (3) does not possess a globally attractive solution. Then, it is not hard to verify that global pinning synchronization in network (5) cannot be ensured if the augmented graph \( \bar{G}(\bar{A}) \) does not contain any directed spanning tree.
B. Pinning Synchronization of Directed Complex Networks With Aperiodic Sampled-Data Communications

In this subsection, global pinning synchronization for directed network (5) with \( \omega(t) \equiv 0_N \), is first studied. The \( H_\infty \) pinning synchronization for directed network (5) in the presence of external disturbances is then addressed.

Based on the discussions in Section II, one can establish the following theorem which summarizes the main results on global pinning synchronization for directed network (5) with \( \omega(t) \equiv 0_N \).

Theorem 1: Suppose that Assumptions 1 and 2 hold, and \( \omega(t) \equiv 0_N \). Then, global pinning synchronization in directed network (5) can be achieved if there exist a scalar \( \tau > 0 \), positive definite matrices \( P, Q \in \mathbb{R}_{+}^{N \times N} \), positive semi-definite matrix \( X \in \mathbb{R}_{+}^{N \times N} \), and \( N_i \in \mathbb{R}_{+}^{N \times N} \), \( i = 1, 2, 3 \), such that

\[
\Lambda = \begin{bmatrix}
\Lambda_{11} + \tau P^2 & \Lambda_{12} & \Lambda_{13} \\
* & \Lambda_{22} & \Lambda_{23} \\
* & * & \Lambda_{33} - \tau I_N
\end{bmatrix} < 0, \tag{8}
\]

\[
\Xi = \begin{bmatrix}
X_{11} & X_{12} & X_{13} & N_1 \\
X_{12} & X_{22} & X_{23} & N_2 \\
* & X_{33} & X_{33} & N_3 \\
* & * & * & Q
\end{bmatrix} > 0, \tag{9}
\]

where \( \Lambda_{11} = N_1 + N_1^T + hX_{11} \), \( \Lambda_{12} = -cP \tilde{E} + N_2^T - N_1 + hX_{12} \), \( \Lambda_{13} = P + N_3^T + hX_{13} \), \( \Lambda_{22} = -N_2 - N_2^T + hX_{22} + 2ch\tilde{Q}^T \tilde{Q} \), \( \Lambda_{23} = -N_2 - N_2^T + hX_{23} - ch\tilde{Q}^T \tilde{Q} \), \( \tilde{E} \) is defined in (6), and \( \Lambda_{33} = hX_{33} + hQ \), with \( X_{11} \in \mathbb{R}_{+}^{N \times N} \), \( i = 1, 2, 3 \), and \( X = \begin{bmatrix} X_{11} & X_{12} & X_{13} \\
* & X_{22} & X_{23} \\
* & * & X_{33} \end{bmatrix} \in \mathbb{R}_{+}^{N \times N} \).

Proof: Construct the following piecewise differentiable Lyapunov-Krasovskii functional for error system (7):

\[
V(t, e_t) = e^T(t)(P \otimes I_N)e(t) + (t_{k+1} - t) \int_{t_k}^{t} e^T(s)(Q \otimes I_N)e(s)ds, \tag{10}
\]

where \( P > 0 \), \( Q > 0 \), \( t \in [t_k, t_{k+1}) \), \( k \in \mathbb{N} \). Then, taking the time derivative of \( V(t, e_t) \) along the trajectory of (7) for \( t \in [t_k, t_{k+1}) \) and \( k \in \mathbb{N} \) gives

\[
\dot{V}(t, e_t) = 2e^T(t)(P \otimes I_N)e(t) + (t_{k+1} - t)e^T(t)(Q \otimes I_N)e(t) - \int_{t_k}^{t} e^T(s)(Q \otimes I_N)e(s)ds. \tag{11}
\]

From the Newton-Leibnitz formula, the following equation holds for any given matrices \( N_i \in \mathbb{R}_{+}^{N \times N} \), \( i = 1, 2, 3 \):

\[
2 \begin{bmatrix} e^T(t)(N_1 \otimes I_N) + e^T(t - d_k(t)) (N_2 \otimes I_N) + F^T(x(t); s(t)) \\
\alpha(N_3 \otimes I_N) \end{bmatrix}
\int_{t_{d_k(t)}}^{t} e^T(s)ds = 0. \tag{12}
\]

On the other hand, for any given positive semi-definite matrix \( X = \begin{bmatrix} X_{11} & X_{12} & X_{13} \\
* & X_{22} & X_{23} \\
* & * & X_{33} \end{bmatrix} \in \mathbb{R}_{+}^{N \times N} \), the following inequality holds:

\[
\eta_1^T(t)(X \otimes I_N)\eta_1(t) - \int_{t_{d_k(t)}}^{t} \eta_1^T(t)(X \otimes I_N)\eta_1(t)ds \geq 0, \tag{13}
\]

where \( \eta_1(t) = [e^T(t), e^T(t - d_k(t))] \), \( F^T(x(t); s(t)) \). It thus follows from (10)–(12) that

\[
\dot{V}(t, e_t) \leq 2e^T(t)(P \otimes I_N)e(t) + he^T(t)(Q \otimes I_N)e(t) - \int_{t_{d_k(t)}}^{t} e^T(s)(Q \otimes I_N)e(s)ds + 2c e^T(t)(N_1 \otimes I_N) + e^T(t - d_k(t)) (N_2 \otimes I_N)
+ F^T(x(t); s(t)) (N_3 \otimes I_N)
\times \left[ e(t) - e(t - d_k(t)) - \int_{t_{d_k(t)}}^{t} e(s)ds \right] + \eta_1^T(t)
\times (X \otimes I_N)\eta_1(t) - \int_{t_{d_k(t)}}^{t} \eta_1^T(t)(X \otimes I_N)\eta_1(t)ds. \tag{14}
\]

Some calculations give that

\[
he^T(t)(Q \otimes I_N)e(t)
= c^2h e^T(t - d_k(t)) (\tilde{E}Q \tilde{E} \otimes I_N)e(t - d_k(t))
+ he^T(x(t); s(t)) (Q \otimes I_N)F(x(t); s(t))
- 2c e^T(t - d_k(t)) (\tilde{E}Q \otimes I_N)e(t - d_k(t)). \tag{15}
\]

Substituting (14) into (13) gives

\[
\dot{V}(t, e_t) \leq \eta_1^T(t)(\Lambda \otimes I_N)\eta_1(t)
- \int_{t_{d_k(t)}}^{t} \eta_1^T(t)(\Xi \otimes I_N)\eta_2(t, s)ds, \tag{16}
\]

with \( \eta_2(t, s) = [e^T(t), e^T(t - d_k(t))] \).

Theorem 1 is proved.
According to (2), it is sufficient to show that $\dot{V}(t, e_t) < 0$ if there exists a positive scalar $\tau > 0$ such that

$$\eta_i^2(t)(\tilde{A} \otimes I_n)\eta_i(t) - \int_{t-d[\tau]}^{t} \eta_i^2(t,s)(\Xi \otimes I_n)\eta_i(t,s)ds + \tau \rho^2 \dot{e}^T(t)e(t) - \tau F^T(x(t); e(t))F(x(t); e(t)) < 0.$$ (15)

Let $\Lambda = \tilde{A} + \begin{bmatrix} \tau \rho^2 & O & O \\ O & O & O \\ * & * & -\tau I_N \end{bmatrix}$. Noticeably, for each $t \in \{t_k, t_{k+1}\}$ and $k \in \mathbb{N}$, one has $\dot{V}(t, e_t) < 0$ if $\exists \gamma > 0$ and $\tilde{A} < 0$.

Furthermore, it can be verified that $\lim_{t \to +\infty} V(t, e_t) = V(t, e_t) e^{-\tau(t_{k+1})} (P \otimes I_n) e(t_{k+1})$, for all $k \in \mathbb{N}$. By using some similar arguments as those in the standard proof of Lyapunov-Krasovskii stability theory [30], one gets that $|e(t)|$ will converge to zero asymptotically under conditions (8) and (9), which indicates that global pinning synchronization of directed complex network (5) with $\omega(t) = 0_{N_n}$ is achieved.

Suppose that the conditions given in Theorem 1 can be ensured, i.e., global synchronization can be ensured for some given sampling interval $h = h_0 > 0$. It is interesting to further study the maximum allowable sampling interval $h_{\text{max}}$ guaranteeing pinning synchronization in Theorem 1. For this purpose, the following algorithm is provided.

**Algorithm 2**

The maximum allowable sampling interval $h_{\text{max}}$ guaranteeing pinning synchronization in Theorem 1 can be estimated by the following procedures:

1. Set $h_{\text{max}} = h_0$ and step size $\kappa = \kappa_0$, where $\kappa_0 > 0$ is sufficiently small compared to $h_0$.
2. Search matrices $P > 0$, $Q > 0$, $X > 0$, $N_i$, $i = 1, 2, 3$, and scalar $\tau > 0$ such that LMIs (8) and (9) hold. If the conditions are satisfied, set $h_{\text{max}} = h_{\text{max}} + \kappa$ and re-perform step 2). Otherwise, stop and let $h_{\text{max}}$ be the maximum allowable sampling interval.

Note that to obtain a less-conservative estimation on the maximum allowable sampling interval, the free-weighting matrices technique was employed in the proof of Theorem 1. It is also worth noting that the dimensions of the LMIs (8) and (9) are independent of those of the nodes’ states in network (5). This ‘decoupling’ feature will be more desirable when each node is a high-dimensional system. Alternatively, one may get the following corollary where the dimensions of the synchronization criteria are dependent on those of the nodes’ states. Generally speaking, the synchronization conditions given in the following corollary will be less conservative than those given in Theorem 1. However, it will be seen that solving the LMIs given in the following corollary is challenging.

**Corollary 1:** Suppose that Assumptions 1 and 2 hold, and $\omega(t) = 0_{N_n}$. Then, global pinning synchronization in directed network (5) with external disturbances can be ensured under some suitable conditions.

$$V(t, e_t) = \rho^2 \dot{e}^T(t)P\dot{e}(t) + (t_{k+1} - t) \int_{t_k}^{t} \dot{e}^T(s)Q\dot{e}(s)ds,$$ (18)

with $P > 0$, $Q > 0$, $t \in \{t_k, t_{k+1}\}$, $k \in \mathbb{N}$. The corollary can be proved by following the steps in the proof of Theorem 1.

**Remark 2:** Compared with the proof of Corollary 1, a special kind of Lyapunov-Krasovskii functional was employed in proving Theorem 1. Though the synchronization conditions given in Theorem 1 have less complexity, they may be conservative in estimating the maximum allowable sampling interval for achieving synchronization. However, the numerical studies indicate that the conservativeness introduced by employing a special kind of Lyapunov-Krasovskii functional in the proof of Theorem 1 is not severe (see Table I in Example 1).

Based on Theorem 1, one may get the following theorem, which states that global $\mathcal{H}_\infty$ pinning synchronization for network (5) with external disturbances can be ensured under some suitable conditions.

**Theorem 2:** Suppose that Assumptions 1 and 2 hold. Then, global $\mathcal{H}_\infty$ pinning synchronization with performance index $\gamma > 0$ in directed network (5) can be achieved if there exist a scalar $\gamma > 0$, positive definite matrices $S, T \in \mathbb{R}^{N_n \times N_n}$, positive semi-definite matrix $Y \in \mathbb{R}^{N_n \times N_n}$, and $W_i \in \mathbb{R}^{N_n \times N_n}$, $i = 1, \cdots, 4$, such that

$$\Psi = \begin{bmatrix} \Psi_{12} & \Psi_{13} & \Psi_{14} \\ * & \Psi_{22} & \Psi_{23} \\ * & * & \Psi_{33} \end{bmatrix} \leq 0,$$ (19)

$$\Omega = \begin{bmatrix} Y_{11} & Y_{12} & Y_{13} & Y_{14} & W_1 \\ * & Y_{22} & Y_{23} & Y_{24} & W_2 \\ * & * & Y_{33} & Y_{34} & W_3 \\ * & * & * & Y_{44} & W_4 \\ * & * & * & * & T \end{bmatrix} \geq 0,$$ (20)

where $\Psi_{11} = W_1 + W_2^T + hY_{11}$, $\Psi_{12} = -cS\tilde{L} + W_1^T - W_1 + hY_{12}$, $\Psi_{13} = -S + W_3^T + hY_{13}$, $\Psi_{14} = -S + W_4^T + hY_{14}$, $\Psi_{22} = -hY_{23} + c^2h\tilde{L}^T\tilde{L}$, $\Psi_{23} = -W_2^T + hY_{23} + c^2h\tilde{L}^T\tilde{L}$, $\Psi_{33} = -W_3^T + hY_{23} + c^2h\tilde{L}^T\tilde{L}$, $\tilde{L}$ is defined in (6), $\Psi_{34} = -W_3^T + hY_{34}$, $\Psi_{34} = hY_{34} + hT$, $\Psi_{34} = -hY_{34}$, with $\Omega_{ij} \in \mathbb{R}^{N_n \times N_n}$, $i, j = 1, 2, 3, 4$, and $Y = \begin{bmatrix} Y_{11} & Y_{12} & Y_{13} & Y_{14} \\ Y_{12} & Y_{22} & Y_{23} & Y_{24} \\ Y_{13} & Y_{23} & Y_{33} & Y_{34} \\ Y_{14} & Y_{24} & Y_{34} & Y_{44} \end{bmatrix} \in \mathbb{R}^{4N_n \times 4N_n}$. 

\(\Xi = \begin{bmatrix} X_{11} & X_{12} & X_{13} & N_1 \\ * & X_{22} & X_{23} & N_2 \\ * & * & X_{33} & N_3 \\ * & * & * & Q \end{bmatrix} \geq 0,\) (17)
Proof: According to Theorem 1, it follows from conditions (19) and (20) that global pinning synchronization in directed network (5) with \( \omega(t) = 0 \) can be achieved.

Next, global \( H_{\infty} \) pinning synchronization problem with performance index \( \gamma \) is studied. For \( \forall \theta \in [t_k, t_{k+1}) \) and an arbitrarily given \( k \in \mathbb{N} \), define

\[
J_\theta = \int_0^\theta \left( \gamma^{-1}e^T(t)e(t) - \gamma \omega^T(t)\omega(t) \right) \, dt,
\]

where \( e(t) = -\tilde{e}(t) \), \( \omega(t) = -\tilde{\omega}(t) \), and \( e(t) \) and \( \omega(t) \) are defined in (19) and (20), respectively. By the zero initial condition \( e(t) \equiv 0 \) for \( t \in [0, \theta] \), one gets

\[
J_\theta = \int_0^\theta \left( \gamma^{-1}e^T(t)e(t) - \gamma \omega^T(t)\omega(t) \right) \, dt,
\]

\[
\leq \int_0^\theta \left( \gamma^{-1}e^T(t)e(t) - \gamma \omega^T(t)\omega(t) \right) \, dt.
\]

Furthermore, from the Newton-Leibnitz formula, the following equation holds for any given matrices \( W_i \in \mathbb{R}^{N \times N} \), \( i = 1, \ldots, 4 \):

\[
\int_{t_k}^t e^T(s) \cdot T \times e^T(s) \, ds = 0.
\]

Based on the above analysis and by arguments similar to the proof of Theorem 1, one has

\[
\tilde{V}(t,e_i) \leq \tilde{\eta}_2^T(t)\tilde{e}(t) - \int_{t-d_k(t)}^{t} \tilde{\eta}_2^T(s)\Omega \tilde{e}(s) \, ds,
\]

where \( \tilde{\eta}_2(t) = -[e^T(t), \omega^T(t)] \), \( e^T(t) \) and \( \omega^T(t) \) are defined in (19), \( \Omega \) is defined in (20), and \( \tilde{\eta}_2(t,s) = [e^T(t), \omega^T(t)] \).

On the other hand, one obtains from (2) that \( \varsigma e^T(t)e(t) - \varsigma F^T(x(t);e(t))F(x(t);e(t)) > 0 \) for all \( \varsigma > 0 \). It thus follows from conditions (19) and (20) that \( J_\theta < 0 \), i.e.,

\[
\int_0^\theta e^T(r)e(r) \, dr < \varsigma^2 \int_0^\theta \omega^T(s)\omega(s) \, ds,
\]

for \( \forall \theta \in [t_k, t_{k+1}) \) and \( k \in \mathbb{N} \). By noticing that \( \omega(t) \in L_2[0, \infty) \), integrating the above inequality from \( \theta = 0 \) to \( \infty \) yields

\[
\frac{1}{2} \int_0^\infty e^T(r)e(r) \, dr < \frac{\varsigma^2}{2} \int_0^\infty \omega^T(s)\omega(s) \, ds.
\]

This indicates that the global \( H_{\infty} \) pinning synchronization problem of directed complex network (5) with a prescribed performance index \( \gamma \) is indeed achieved. The proof is thus completed.

**Remark 3:** For a prescribed \( \gamma > 0 \), global \( H_{\infty} \) synchronization with a disturbance rejection level less than \( \gamma \) can be verified by checking the conditions in Theorem 2. However, it is practically important to know the allowable smallest disturbance rejection level \( \gamma_{\text{min}} \) for synchronization in network (5). Note that it is very hard or even impossible to calculate \( \gamma_{\text{min}} \) theoretically. But it can be numerically estimated by solving the following optimization problem:

\[
\min_{\gamma} \quad \gamma
\]

subject to:

\[
\begin{bmatrix}
\Psi_{11} + \varsigma \rho^2 + \gamma^{-1}I_{4N} & \Psi_{12} & \Psi_{13} & \Psi_{14} \\
* & \Psi_{22} & \Psi_{23} & \Psi_{24} \\
* & * & \Psi_{33} & \Psi_{34} \\
* & * & * & \Psi_{44}
\end{bmatrix} < 0
\]

\[
\begin{bmatrix}
\Psi_{11} & \Psi_{12} & \Psi_{13} & \Psi_{14} \\
* & \Psi_{22} & \Psi_{23} & \Psi_{24} \\
* & * & \Psi_{33} & \Psi_{34} \\
* & * & * & \Psi_{44}
\end{bmatrix} > 0,
\]

where \( \Psi \) and \( \Omega \) are respectively given in (19) and (20), \( \varsigma > 0 \), matrices \( S > 0, T > 0, Y \geq 0, \) and \( W_i, i = 1, \ldots, 4, \) with appropriate dimensions.

In Theorem 2, the dimensions of LMIs (19) and (20) are independent of those of the nodes' states in network (5). Alternatively, one may get the following corollary where the dimensions of the \( \gamma_{\text{min}} \) synchronization criteria are dependent on those of the nodes' states.

**Corollary 2:** Suppose that Assumptions 1 and 2 hold. Then, global \( H_{\infty} \) pinning synchronization with performance index \( \gamma > 0 \) in directed network (5) can be achieved if there exist a scalar \( \varsigma > 0 \), positive definite matrices \( \Psi, \Omega \in \mathbb{R}^{4N \times 4N} \), positive semi-definite matrix \( \Psi = \mathbb{R}^{4N \times 4N} \), and \( \Psi_i \in \mathbb{R}^{N \times N}, i = 1, \ldots, 4, \) such that

\[
\begin{bmatrix}
\Psi_{11} & \Psi_{12} & \Psi_{13} & \Psi_{14} \\
* & \Psi_{22} & \Psi_{23} & \Psi_{24} \\
* & * & \Psi_{33} & \Psi_{34} \\
* & * & * & \Psi_{44}
\end{bmatrix} > 0.
\]
C. Extension to Global $\mathcal{H}_\infty$ Pinning Synchronization of Switching Directed Networks With Sampled-Data Communications

Global $\mathcal{H}_\infty$ pinning synchronization for fixed directed network with sampled-data communications has been investigated in previous subsections. However, in reality, the network topology may be time-varying due to technological limitations of sensors, external disturbances or communication channel failures. As pointed out in [31], analysis and synthesis on the dynamical behaviors of complex networks with time-varying network structures is very important. From this observation, global $\mathcal{H}_\infty$ pinning synchronization in switching directed networks with sampled-data communications is further studied in this subsection.

Let $\mathcal{G}(\mathcal{A}^t_{(s)})$, $s \geq 0$, be the set of all possible topologies. Suppose that there exists an infinite sequence of uniformly bounded non-overlapping time intervals $[t_k, t_{k+1})$, $s \in \mathbb{N}$, with $t_1 = 0$, $0 < \tau_0 < t_{z+1} - t_z$, and $\tau_0$ being a specified positive constant. For each $z \in \mathbb{N}$, the underlying topology is time-invariant for all $t \in [t_z, t_{z+1})$. Here, $\tau_0$ is called the dwell time. For the convenience of expression, a switching signal $\mathcal{e}(t) : [0, +\infty) \rightarrow \{1, \ldots, q\}$ is introduced to characterize the topology's evolution.

Let $\mathcal{G}(\mathcal{A}^t_{(s)})$ be the topology of network (1) at time $t \geq 0$. On the other hand, the coupling force between any pair of neighboring nodes is generated by employing sampling technique and a zero-order hold circuit, i.e., the coupling force acting on each node $x_i(t)$ is time-invariant for all $t \in [t_k, t_{k+1})$, $k \in \mathbb{N}$. This indicates that for each time interval $[t_k, t_{k+1})$, $k \in \mathbb{N}$, the interaction among the nodes is determined only by the coupling strength $c$ and the communication topology $\mathcal{G}(\mathcal{A}^t_{(s)})$. Thus, the closed-loop network under pinning control can be described as:

\[
\dot{x}_i(t) = f(x_i(t), t) + c\sum_{j=1}^{N_{ij}} a_{ij}^{(s)} [x_j(t_k) - x_i(t_k)] + w_i(t),
\]

in which the pinning gain $a_{ij}^{(s)} > 0$ if node $i$ is pinned at time $t_k$, and $a_{ij}^{(s)} = 0$ otherwise, $i = 1, 2, \ldots, N$. Here, $\mathcal{A}^{(s)}_{(t_k)} = [\delta_{ij}^{(s)}]_{N \times N}$ is the adjacency matrix of graph $\mathcal{G}(\mathcal{A}^{(s)}_{(t_k)})$, $k \in \mathbb{N}$. Taking the target system (3) as a virtual node labeled as $N + 1$ in the considered network, one then gets

\[
\dot{e}(t) = F(x(t); s(t)) - c(\tilde{L}^{(s)}(t) \otimes I_N) e(t - d_k(t)) + \omega(t),
\]

where $e(\theta) \equiv e(t)$ for all $\theta = [-h, 0]$, $\tilde{L}^{(s)}(t) = \mathcal{L}^{(s)}(t) + \text{diag}(p_1^{(s)}(t), p_2^{(s)}(t), \ldots, p_N^{(s)}(t))$, $\mathcal{L}^{(s)}(t)$ is the Laplacian matrix of $\mathcal{G}(\mathcal{A}^{(s)}_{(t_k)})$. Furthermore, one has that $\tilde{L}^{(s)}(t_k) = \begin{bmatrix} \mathcal{L}^{(s)}(t_k) & -p^{(s)}(t_k) \\ 0^T & 0 \end{bmatrix}$ is the Laplacian matrix of the augmented graph $\mathcal{G}(\mathcal{A}^{(s)}_{(t)})$. Here, $p^{(s)}(t_k) = (p_1^{(s)}(t_k), p_2^{(s)}(t_k), \ldots, p_N^{(s)}(t_k)) \in \mathbb{R}^N$.

To derive the main results of this subsection, the following assumption is introduced.

Assumption 3: For each $i$, $1 \leq i \leq q$, the augmented graph $\mathcal{G}(\mathcal{A}^i)$ contains a directed spanning tree with node $N + 1$ being the root.

From the above analysis, one gets the following two theorems, which summarize the main results of this subsection.

Theorem 3: Suppose that Assumptions 1 and 3 hold, and $\omega(t) \equiv 0_{N \times N}$. Then, global pinning synchronization in directed network (28) can be achieved if there exist a scalar $\tau > 0$, positive definite matrices $P, Q \in \mathbb{R}^{N \times N}$, and $N_1 \in \mathbb{R}^{N \times N}$, $i = 1, 2, 3$, such that

\[
\begin{align*}
A_i &= \begin{bmatrix} A_{i1} + \tau P^2 & A_{i2} & A_{i3} \\ * & A_{i2} & A_{i2} \\ * & * & A_{i3} - \tau I_N \end{bmatrix} < 0, \\
\Xi &= \begin{bmatrix} X_{i1} & X_{i2} & X_{i3} \\ * & X_{i2} & X_{i3} \\ * & * & X_{i3} \end{bmatrix} N_1 \\
&\geq 0,
\end{align*}
\]

where $A_{i1} = N_1 + N_i^T + hX_{i1}$, $A_{i2} = -cP\mathcal{L}^i + N_i^T - N_1 + hX_{i2}$, $A_{i3} = P + N_i^T + hX_{i3}$, $A_{i22} = -N_2 - N_i^T + hX_{i22} + c^2 h(\mathcal{L}^i)^T Q^i$, $A_{i33} = -N_3^T + hX_{i33} - c^2 h(\mathcal{L}^i)^T Q^i$.

Proof: Construct the following common piecewise differentiable Lyapunov-Krasovskii functional for the error system (29):

\[
V(t, e(t)) = e^T(t)(P \otimes I_N) e(t) + \int_{t_k}^{t} \dot{e}^T(s)(Q \otimes I_N) \dot{e}(s) ds,
\]

where $P > 0$, $Q > 0$, $t \in [t_k, t_{k+1})$, $k \in \mathbb{N}$. Then, the theorem can be proved by following the steps in the proof of Theorem 1.

Furthermore, one can get the following theorem on global $\mathcal{H}_\infty$ pinning synchronization of directed network (28) with a switching directed topology. The detailed proof is omitted for brevity.

Theorem 4: Suppose that Assumptions 1 and 3 hold. Then, global $\mathcal{H}_\infty$ pinning synchronization with performance index $\gamma > 0$ in directed network (28) can be achieved if there exist a scalar $\tau > 0$, positive definite matrices $P, T \in \mathbb{R}^{N \times N}$, positive semi-definite matrix $Y \in \mathbb{R}^{N \times N}$, and $W_i \in \mathbb{R}^{N \times N}$, $i = 1, \ldots, 4$, such that

\[
\begin{bmatrix} \psi_{11} + \gamma \rho^2 + \gamma^{-1} I_N & \psi_{12} & \psi_{13} & \psi_{14} \\ * & \psi_{22} & \psi_{23} & \psi_{24} \\ * & \psi_{33} - \gamma I_N & \psi_{34} \\ * & * & * & \psi_{44} \end{bmatrix} < 0
\]

\[
\begin{bmatrix} Y_{11} & Y_{12} & Y_{13} & W_i \\ * & Y_{22} & Y_{23} & W_i \\ * & * & Y_{33} & W_i \\ * & * & * & W_i \end{bmatrix} \geq 0,
\]

where $\psi_{ij}, Y_{ij}, W_i \in \mathbb{R}^{N \times N}$, $i = 1, \ldots, 4, j = 1, 2, 3, 4$.
Remark 4: In the present work, the dynamical nodes in the considered network are assumed to have a homogeneous time-varying sampling rate, i.e., the sampler embedded in each node works at the same time instants: \( t_k, k \in \mathbb{N} \). It is more interesting but challenging to further study how to achieve global \( H_\infty \) pinning synchronization in fixed or switched directed network with heterogeneous time-varying sampling rates. Furthermore, the synchronization criteria provided in the present paper are dependent on the solvability of some high dimensional LMIs, which are thus inapplicable for complex networks of huge size.

V. NUMERICAL SIMULATIONS

In this section, some numerical examples are given to illustrate the effectiveness of the theoretical analysis. For the convenience of simulation, the periodic sampled-data technique is adopted.

Example 1: Consider a fixed directed network (5) whose network topology \( \mathcal{G}(\mathcal{A}) \) is shown in Fig. 2 with weighted edges. It can be seen from Fig. 2 that \( \mathcal{G}(\mathcal{A}) \) contains a directed spanning tree with the target system as the root. In simulations, each node on the considered network is assumed to be a Chua’s circuit [32]–[34]. Recently, synchronization of networked Chua’s circuits has received an increasing attention [11], [35], [36]. The dimensionless equation of Chua’s circuit can be rewritten as

\[
\begin{align*}
\dot{x} &= \alpha (y - x - \mu(x)), \\
\dot{y} &= x - y + z, \\
\dot{z} &= -\beta y - \theta z,
\end{align*}
\] (34)

with \((x, y, z) \in \mathbb{R}^3, \mu(x) = bx + (1/2)(a - b)(x + 1) - x - 1\).

In simulations, the dynamics of each node are determined by the equation of Chua’s circuit (34) with \( \alpha = -1.3018, \beta = -0.0135, \theta = -0.0297, a = 0.1091, \) and \( b = -0.5700 \). The trajectory of the target system is given by (3) with \( s(0) = (0.7, 0.1, 0.65)^T \) is shown in Fig. 3. Take \( \chi(x, y, z) = (\alpha y - x - \mu(x)), x - y + z, -\beta y - \theta z \)^T, \( x, y, z \in \mathbb{R} \). Some calculations give

\[
\chi(x, y, z) = \begin{pmatrix} 0.5598 & -1.3018 & 0 \\ 1.0000 & -0.0000 & 1.0000 \\ 0 & 0.0135 & 0.0297 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \tilde{\chi}(x, y, z),
\]

where \( \tilde{\chi}(x, y, z) = -\left(\alpha(a - b)/2\right)(x + 1) - x - 1\).

Then, \( ||\chi(\xi) - \chi(\tilde{\xi})|| \leq 2.99899||\xi - \tilde{\xi}|| \) for all \( \xi, \tilde{\xi} \in \mathbb{R}^3 \). Set \( \epsilon = 1 \). It follows from Algorithm 2 that global pinning synchronization can be achieved with a maximum allowable sampling interval \( h_{max} = 0.09504 \). The state trajectories of the nodes are given in Fig. 4, which indicates that the global pinning synchronization in directed network (5) with sampling interval \( h = 0.085 \) and \( \omega(t) \equiv 0 \) is achieved. The relationship between the maximum allowable sampling interval \( h_{max} \) and the coupling strength \( c \) is numerically obtained and depicted in Fig. 5. Interestingly, it is found that \( h_{max} \) increases firstly and then decreases, when enlarging the coupling strength \( c \). Use \( \text{Error}(t) = \sqrt{\sum_{j=1}^5 |x_j - x_6(t)|^2} \) to denote the synchronization error of the network. Fig. 6 indicates that a faster convergence rate will be yielded when enlarging the sampling interval.
TABLE I

<table>
<thead>
<tr>
<th>Coupling strength $c$</th>
<th>$h_{\text{max}}$ Theorem 1</th>
<th>$h_{\text{max}}$ Corollary 1</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.85</td>
<td>0.07122</td>
<td>0.07122</td>
</tr>
<tr>
<td>1.00</td>
<td>0.09904</td>
<td>0.09904</td>
</tr>
<tr>
<td>1.25</td>
<td>0.08311</td>
<td>0.08311</td>
</tr>
<tr>
<td>1.40</td>
<td>0.07601</td>
<td>0.07601</td>
</tr>
<tr>
<td>1.55</td>
<td>0.06980</td>
<td>0.06985</td>
</tr>
</tbody>
</table>

Furthermore, the maximum allowable sampling intervals $h_{\text{max}}$ determined by Theorem 1 and Corollary 1 versus different coupling coupling strength $c$ are given in Table I, from which it can be seen that the results provided in Theorem 1 is not very conservative.

Next, the global $H_{\infty}$ pinning synchronization in directed network (5) is studied numerically. Let the coupling strength $c = 1$. According to Theorem 2, the global $H_{\infty}$ pinning synchronization with performance index $\gamma = 1.12$ in directed network (5) can be guaranteed. Choose $\omega_i(t) = (\sin(it), \cos(it), \sin(t))$ for $0 \leq t \leq 3$, and $\omega_i(t) = 0$ for $t > 3$, $i = 1, \ldots, 5$.

Example 2: In this example, the global $H_{\infty}$ pinning synchronization problem of switched directed network (28) is numerically studied. The network topology is assumed to switch back and forth between $G(A_1)$ and $G(A_2)$ per 0.12 second. The possible topologies $G(A_1)$ and $G(A_2)$ are shown in Fig. 9 where the weights are indicated on the edges. It can be verified that Assumption 3 holds. Let $f(x_i(t), t) = (0.5 \sin(x_{1i}(t)), 0.5 \cos(x_{2i}(t)), x_{3i}(t))$ for $i = 1, \ldots, 6$. Set $c = 0.5$. It can be then obtained from Theorem 3 that the maximum allowable sampling interval for achieving global pinning synchronization of switched directed network (28) is $h_{\text{max}} = 0.10132$. The state trajectories of the switched directed network (28) with sampling interval $h = 0.10$ are shown in Fig. 10. The relationship between $h_{\text{max}}$ and $c$ is numerically depicted in Fig. 11. Use $Error(t) = \sqrt{\sum_{j=1}^{6}||x_j - x_6(t)||^2}$ to denote the synchronization error of the network. Fig. 12 indicates that a faster convergence rate will be yielded when enlarging the sampling interval. Furthermore, it can be obtained from Theorem 3 that global $H_{\infty}$ pinning synchronization with performance index $\gamma = 0.45$ in switched directed network (28) can be guaranteed. Choose $\omega_i(t) = (2 \sin(it), 2 \cos(it))$ for
Fig. 11. Maximum allowable sampling interval $h_{\text{max}}$ versus the coupling strength $c$ in Example 2.

Fig. 12. Synchronization error $\epsilon(t)$ versus the sampling interval $h$ in Example 2.

Fig. 13. Energy trajectories of $\omega(t)$ and $\omega(t)$ in Example 2.

$0 \leq t \leq 4$, and $\omega(t) - 0.6$ for $t > 4$. The energy trajectories of $r(t)$ and $\omega(t)$ are shown in Fig. 13, which indicate that the global $H_{\infty}$ pinning synchronization control problem is solved. It can be observed from Fig. 13 that the estimation of the $H_{\infty}$ performance index, i.e., $\gamma = 0.45$, is not very conservative.

VI. CONCLUSIONS

Global $H_{\infty}$ pinning synchronization in fixed and switched directed complex networks with aperiodic sampled-data communications has been investigated. By using a combined tool from input-delay approach, Lyapunov-Krasovskii stability analysis, and LMI technique, some sufficient conditions for achieving global pinning synchronization in fixed directed networks have been derived and discussed. The results are then extended to global $H_{\infty}$ pinning synchronization with external disturbances and switching topologies. Here, it is assumed that the dynamic nodes in the present network model have a common homogeneous sampling rate, that is, only global $H_{\infty}$ pinning synchronization for switched directed networks with synchronous sampling rate is addressed in the present paper. The global $H_{\infty}$ pinning synchronization problem for directed complex networks with asynchronous sampled-data communication and switched directed topology is still in its infancy. It remains to be seen how to solve such a challenging problem in the future.

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