<table>
<thead>
<tr>
<th>Title</th>
<th>H₂ pinning synchronization of directed networks with aperiodic sampled-data communications</th>
</tr>
</thead>
<tbody>
<tr>
<td>Author(s)</td>
<td>Wen, G; Yu, W; Chen, MZ; Yu, X; Chen, G</td>
</tr>
<tr>
<td>Citation</td>
<td>IEEE Transactions on Circuits and Systems I: Regular Papers,  2014, v. 61 n. 11, p. 3245-3255</td>
</tr>
<tr>
<td>Issued Date</td>
<td>2014</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/10722/217094">http://hdl.handle.net/10722/217094</a></td>
</tr>
<tr>
<td>Rights</td>
<td>IEEE Transactions on Circuits and Systems I: Regular Papers. Copyright © IEEE.; ©2014 IEEE. Personal use of this material is permitted. Permission from IEEE must be obtained for all other uses, in any current or future media, including reprinting/republishing this material for advertising or promotional purposes, creating new collective works, for resale or redistribution to servers or lists, or reuse of any copyrighted component of this work in other works.; This work is licensed under a Creative Commons Attribution-NonCommercial-NoDerivatives 4.0 International License.</td>
</tr>
</tbody>
</table>


**Abstract**—This paper addresses the global $\mathcal{H}_\infty$ pinning synchronization problem for a class of directed networks with aperiodic sampled-data communications. Important yet challenging issues of how many and which nodes should be pinned for realizing global synchronization in a fixed directed network without external disturbances are first discussed. By using a combined tool from the input-delay approach and free-weighting matrices technique, some sufficient synchronizability conditions are then derived for such networks. Furthermore, a multi-step algorithm is designed to estimate the upper bound of the maximum allowable sampling intervals for achieving synchronization. Theoretical results are then extended to global $\mathcal{H}_\infty$ pinning synchronization in fixed and switched directed networks with external disturbances, showing that a finite $\mathcal{H}_\infty$ performance index can be guaranteed under some suitable conditions. Finally, numerical simulations are performed to demonstrate the effectiveness of the analytical results.

**Index Terms**—Complex network, $\mathcal{H}_\infty$ control, pinning synchronization, sampled-data communication, spanning tree.

I. INTRODUCTION

THERE past few years have witnessed a strong upsurge of the study of complex networks in various fields, ranging from physics to mathematics and also to control engineering [1]–[5]. Typical examples of complex networks include the Internet, biological neural networks, smart grids, and various human social networks [5].

Manuscript received March 04, 2014; revised May 22, 2014; accepted June 05, 2014. Date of publication July 17, 2014; date of current version October 24, 2014. This research was supported by the National Nature Science Foundation of China under Grants 61304168 and 61322302, the Natural Science Foundation of Jiangsu Province of China under Grants BK20130595 and BK2011581, the Research Fund for the Doctoral Program of Higher Education of China under Grants 20130092120030 and 20110092120024, the Fundamental Research Funds for the Central Universities of China, the Hong Kong Research Grants Council under the GRF Grant CityU 1120/14, and the Discovery Scheme under Grant DIP140100544. This paper was recommended by Associate Editor M. Porfiri.

G. Wen is with the Department of Mathematics, Southeast University, Nanjing, 210096, China (e-mail: wenguanghui@gmail.com).

W. Yu is with the School of Electrical and Computer Engineering, RMIT University, Melbourne, VIC 3001, Australia. He is also with the Faculty of Engineering, King Abdulaziz University, Jeddah, 21589, Saudi Arabia (e-mail: wenwuay@gmail.com).

M. Z. Q. Chen is with the Department of Mechanical Engineering, The University of Hong Kong, Pokfulam, Hong Kong SAR, China (e-mail: mzchen@hku.hk).

X. Yu is with the School of Electrical and Computer Engineering, RMIT University, Melbourne VIC 3001, Australia. He is also with the School of Automation, Southeast University, Nanjing, 210096, China (e-mail: x.yu@rmit.edu.au).

G. Chen is with the Department of Electronic Engineering, City University of Hong Kong, Hong Kong SAR, China (e-mail: egchen@cityu.edu.hk).

Color versions of one or more of the figures in this paper are available online at http://ieeexplore.ieee.org.

Digital Object Identifier 10.1109/TCSI.2014.2334871

Compared with a single system, analysis on the dynamical behaviors of complex networks is more challenging since their dynamics are coupled with the topological evolution. Investigations on the dynamics of complex networks could not only help scientists understand the emergent mechanisms for many intriguing collective behaviors, such as synchronization and swarming, but also help engineers effectively apply complex networks theory to practical problems, for example, controlling the spread of diseases over computer networks. One of the most important collective behaviors in complex networks is the network synchronization, which is a timekeeping dynamical behavior in which the states of all nodes in the network converge towards the same trajectory. Much effort has been devoted to studying synchronization in complex networks in the last decade. Consequently, many profound results have been reported in the literature (see the recent survey paper [6] for more details). In [7], [8], local synchronization was investigated by using the master stability function method, while in [9]–[11], global synchronization was addressed based on the Lyapunov function method. For the case that the network under consideration can not achieve synchronization by itself, one may apply some control inputs to the network to guarantee synchronization [6]. However, it is practically impossible to implement one controller to each node in a large-scale network. One practical way to solve this problem is to control only a small fraction of the nodes by applying some carefully designed local feedback injections, which is known as pinning synchronization control [12]–[20]. Note that active research on pinning synchronization is still ongoing today [21]–[23].

With the rapid developments of data communication technologies and high-performance computers, a designed continuous-time feedback controller is usually implemented in a digital form for better reliability, flexibility and cost-effectiveness. However, most of the aforementioned results on synchronization control of complex networks assume that the information transmission among the nodes is continuous, implying that the coupling law of the network is executed by analog signals. Furthermore, the communication topology among some real networks may be directed and time-varying. And, moreover, the microscopic synchronization principles of directed networks are less understood than those of undirected networks. From these observations, and based on the above-mentioned works, this paper focuses on pinning synchronization for complex directed networks with sampled-data communications and time-varying topologies. Particularly, the paper aims to deal with the following three issues. First, for an arbitrarily given network, it tries to find out at least how many and which nodes should be pinned for achieving global pinning synchronization. Second, it tries to provide some simple yet less-conservative criteria for achieving global pinning synchronization, and to estimate the upper bound of the maximum allowable sampling intervals for
synchronization in both fixed and switched networks. Third, it tries to analyze the $\mathcal{H}_\infty$ performance for sampled-data pinning synchronization of directed networks in the presence of external disturbances. As will be shown, global $\mathcal{H}_\infty$ pinning synchronization of such networks with external disturbances can be guaranteed if some nodes are carefully selected to pin, and some sufficient criteria in terms of linear matrix inequalities (LMIs) are satisfied. One favorable property of the present criteria is that the dimensions of the involved LMIs are independent of those of each node’s states. This feature is very desirable for the case when nodes’ dimensions are high. The effectiveness of the theoretical analysis is demonstrated by performing numerical simulations on coupled Chua’s circuits as well as some other dynamical systems.

The rest of the paper is structured as follows. In Section II, notations and some preliminaries on algebraic graph theory are provided. Problem formulation is then given in Section III. In Section IV, the main theoretical results are presented. In Section V, numerical simulations are performed for illustration. Concluding remarks are finally drawn in Section VI.

II. NOTATIONS AND PRELIMINARIES

Notations and some basic preliminaries on algebraic graph theory are presented in this section.

A. Notations

Let $\mathbb{R}^n$ and $\mathbb{R}^{n \times n}$ be respectively the sets of $n$-dimensional column real vectors and $n \times n$ real matrices. $I_f^n [0, +\infty)$ is the $n$-dimensional square integrable function space over $[0, +\infty)$. $\mathbb{R}_n$ represents the $n$-dimensional column vector with all elements being 0. $I_n$($O_n$) is the $n \times n$ identity (zero) matrix. Denote the set of positive natural numbers by $\mathbb{N}$. Matrix inequality $A > 0$ (resp., $A \geq 0$) means that $A$ is positive definite (resp., positive semi-definite). Symbols $\otimes$, $\cdot$ and $\cdot L_f^2$ denote the Kronecker product of matrices, Euclidean norm and $L_2$ norm, respectively. Notation $d:\text{ag}\{A_1, \cdots, A_n\}$ represents a block-diagonal matrix with matrices $A_i$, $i = 1, \cdots, n$, being its diagonal elements.

B. Algebraic Graph Theory

Let $\mathcal{G}(\mathcal{V}, \mathcal{E}, \mathcal{A})$ be a weighted directed graph of order $N$, where $\mathcal{V} = \{1, \cdots, N\}$ is the set of nodes, $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is the set of edges, and $\mathcal{A} = [a_{ij}]_{N \times N}$ is the weighted adjacency matrix. A directed edge $d_{ij}$ is denoted by the ordered pair of nodes $(j, i)$, and $d_{ij} \in \mathcal{E}$ if and only if $a_{ij} > 0$. In the sequel, denote $\mathcal{G}(\mathcal{V}, \mathcal{E}, \mathcal{A})$ by $\mathcal{G}(\mathcal{A})$ if no confusion will occur. The in-degree of node $i$ is defined as $D_i = \sum_{j=1}^{N} a_{ij}$ for $i = 1, \cdots, N$. A directed path on $\mathcal{G}(\mathcal{A})$ from node $z_1$ to node $z_k$ (with $s > 1$) is a sequence of ordered edges of the form $(z_{s+1}, z_k)$, $k = 1, 2, \cdots, s - 1$. A directed graph is strongly connected if there exists at least one directed path between any pair of distinct nodes [24]. A directed graph contains a directed spanning tree if there exists a node $r$, called a root, such that there exists a directed path from this node to every other node $v_i$, i.e., $v$ is reachable from $r$ [24]. The Laplacian matrix $L = [l_{ij}]_{N \times N}$ of $\mathcal{G}(\mathcal{A})$ is defined with $l_{ij} = -a_{ij}$, $i \neq j$, and $l_{ii} = \sum_{j=1}^{N} a_{ij}$ for $i = 1, 2, \cdots, N$.

By introducing a new node, labeled as $i$, $i \not\in \mathcal{V}$, to $\mathcal{G}(\mathcal{V}, \mathcal{E}, \mathcal{A})$, one may obtain a graph $\mathcal{G}(\mathcal{V}, \mathcal{E}, \mathcal{A})$ with order of $N + 1$. Here, $\mathcal{G}(\mathcal{V}, \mathcal{E}, \mathcal{A})$ is called the augmented graph of $\mathcal{G}(\mathcal{V}, \mathcal{E}, \mathcal{A})$, and $L$ be the Laplacian matrix of $\mathcal{G}(\mathcal{V}, \mathcal{E}, \mathcal{A})$, it can be verified that $\tilde{L} = \begin{bmatrix} L & -\varphi \\ \mathbf{0}_N & 0 \end{bmatrix}$, where $\varphi = (\varphi_1, \varphi_2, \cdots, \varphi_N)^T$ with $\varphi_i > 0$ if $(N + 1, i) \in \mathcal{E}$ and $\varphi_i = 0$ otherwise, $i = 1, 2, \cdots, N$, and $\tilde{L} = L + d:\text{ag}\{p_1, p_2, \cdots, p_N\}$.

III. PROBLEM FORMULATION

Consider the following dynamical network consisting of $N$ nodes with aperiodic sampled-data-based diffusive couplings:

$$\dot{x}_i(t) = f(x_i(t), t) + e\Sigma_{j=1}^{N} a_{ij} (x_j(t_k) - x_i(t_k)) + \omega_i(t),$$

$$t \in [t_k, t_{k+1}), k \in \mathbb{N},$$

(1)

where $x_i(t) = [x_{1i}(t), x_{2i}(t), \cdots, x_{Ni}(t)]^T \in \mathbb{R}^n$ is the state vector of the $i$-th node, $1 \leq i \leq N$; $f : \mathbb{R}^n \times [0, +\infty) \rightarrow \mathbb{R}^n$ is a nonlinear function satisfying the global Lipschitz condition

$$\|f(y, t) - f(\tilde{y}, t)\| \leq \varphi_y \|y - \tilde{y}\|, \quad \forall y, \tilde{y} \in \mathbb{R}^n.$$  

for some given $\varphi > 0$; $e > 0$ is the coupling strength; $A = [a_{ij}]_{N \times N}$ is the adjacency matrix of the communication topology $G(A)$ of the network; $\omega_i(t) \in L_2^\infty [0, +\infty)$ are external disturbances.

For the case of $\omega_i(t) = 0$, $1 \leq i \leq N$, the global pinning synchronization is said to be achieved if, under pinning control, the states of all nodes in network (1) converge to a prescribed trajectory $s(t)$ in the sense of $\lim_{t \rightarrow \infty} \|x_i(t) - s(t)\| = 0$ for any given initial conditions, where $s(t)$ is generated by

$$s(t) = f(s(t), t),$$

(3)

for some $s(t) \in \mathbb{R}^n$. For notational brevity, let $e(t) = [e_1(t), e_2(t), \cdots, e_N(t)]^T$ with $e_i(t) = x_i(t) - s(t)$, $i = 1, 2, \cdots, N$. The global $\mathcal{H}_\infty$ pinning synchronization with performance index $\gamma > 0$ is said to be achieved if, under pinning control, the following two conditions are satisfied:

i) For $\omega(t) \equiv 0$, $\gamma > 0$ and initial condition $e(0) = 0$, the worst-case norm of synchronization error vector $e(t)$ over all admissible exogenous disturbances $\omega(t)$, defined by

$$\gamma_w = \sup_{\omega(t) \in L_2^\infty [0, +\infty)} \|e(t)\|_{L_2},$$

(4)

is less than $\gamma$, i.e., $\gamma_w < \gamma$.

Remark 1: The notion of $\mathcal{H}_\infty$ pinning synchronization is borrowed from the idea of $\mathcal{H}_\infty$ control for dynamical systems in the context of modern control theory [25]. The classic $\mathcal{H}_\infty$ control problem can be described as how to construct some stabilizing controllers, such that the closed-loop systems are internal stable and the $\mathcal{H}_\infty$ norm of the transfer matrix from exogenous signals to the performance variable is less than a prescribed positive number. Within the context of pinning synchronization in complex networks, the synchronization error is always taken as the performance variable [26], [27]. However, for the convenience of analysis, it is always assumed in the literature on $\mathcal{H}_\infty$ pinning synchronization of complex networks that the information can be transmitted continuously [26], [27]. This indicates that each node needs to share its state information with the neighbors continuously. However, there exist some practical situations where such information exchange only happens at some discrete time instants. It is thus practically important to study how to realize an $\mathcal{H}_\infty$ pinning synchronization with sampled-data communications.
In the present paper, some sampled-data-based negative feedback injections will be employed to control network (1). The closed-loop network under pinning control is described by:

\[
\dot{x}_i(t) = f(x_i(t), t) + c\sum_{j=1}^{N} a_{ij} (x_j(t_k) - x_i(t_k)) + \omega_i(t),
\]

\[
- r_{pi}(x_i(t_k) - x_i(t_k)), \quad t \in [t_k, t_{k+1}), \quad k \in \mathbb{N},
\]

(5)

where the feedback gain \( p_i > 0 \) if node \( i \) is pinned, and \( p_i = 0 \) otherwise, \( i = 1, 2, \ldots, N \). Taking the target system (3) as a virtual node labeled as \( N+1 \) in network (5) and defining \( e_i(t) = x_i(t) - s(t), i = 1, 2, \ldots, N \), one has

\[
\dot{e}_i(t) = F(x(t); s(t)) - c(\hat{L} \otimes I_n) e(t_k) + \omega(t),
\]

(6)

where \( e(t) = [e_1^T(t), e_2^T(t), \ldots, e_N^T(t)]^T, t \in [t_k, t_{k+1}), F(x(t); s(t)) = [f_1(x(t_k); s(t)), f_2(x_2(t_k); s(t)), \ldots, f_N(x_N(t_k); s(t))]^T, \)

\[
f(x(t_k); s(t)) - f(x_i(t_k)) - f(s(t_k)), \quad i = 1, 2, \ldots, N, \quad \hat{L} = \mathcal{L} + \text{diag}(p_1, p_2, \ldots, p_N), \quad \omega(t) = \omega_1^T(t), \omega_2^T(t), \ldots, \omega_N^T(t),
\]

Let \( \hat{L} = \begin{bmatrix} 0_n & -p \end{bmatrix} \) is the Laplacian matrix of the augmented graph \( G(\hat{A}) \).

Let \( d_k(t) = t - t_k, \) for \( t \in [t_k, t_{k+1}), k \in \mathbb{N} \). One then gets that \( t_k = t - d_k(t) \) with \( 0 \leq d_k(t) < h, \) for \( t \in [t_k, t_{k+1}), k \in \mathbb{N} \). Then, the error system (6) can be rewritten as the following retarded functional differential equation:

\[
\dot{e}(t) = F(x(t); s(t)) - c(\hat{L} \otimes I_n) e(t-d_k(t)) + \omega(t),
\]

(7)

where \( t \in [t_k, t_{k+1}), k \in \mathbb{N} \). The initial condition of (7) is set as \( e(t) = 0 \) for all \( t \in [-h, 0] \). It is not hard to verify that \( e(t) \equiv 0 \) is a fixed point of the error dynamical system (7). Noticeably, under the condition \( \omega(t) \equiv 0 \), global pinning synchronization in network (7) will be achieved if and only if the trivial solution \( e(t) \equiv 0 \) is a globally attractive fixed point of (7).

Before ending this subsection, two assumptions are made.

Assumption 1: There is a constant \( h > 0 \) such that \( t_{k+1} - t_k \leq h, \) for \( k \in \mathbb{N} \).

Assumption 2: The augmented graph \( G(\hat{A}) \) contains a directed spanning tree with node \( N+1 \) as the root.

IV. MAIN RESULTS

The main theoretical results are established and analyzed in this section.

A. Selective Pinning Strategy

In this subsection, a graph search algorithm with linear time complexity is provided for choosing the nodes to be pinned in network (5). The important issues of at least how many and which nodes should be pinned such that Assumption 2 holds will be addressed.

Algorithm 1

Let \( G(\hat{A}) \) be the communication topology of directed network (1). Then, Assumption 2 will hold if the \( r_0 \) nodes searched by the following procedures are selected and pinned.

1) Use Tarjan’s algorithm [29] to find all the nodes with zero in-degree and strongly connected components of \( G(\hat{A}) \). Suppose that there are \( \kappa_1 (\kappa_1 \geq 0) \) nodes with zero in-degree, labeled as \( v_1, v_2, \ldots, v_{\kappa_1} \), and \( \kappa_2 \) (\( \kappa_2 \geq 0 \)) strongly connected components, represented by \( G(V_1, E_1, A_1), G(V_2, E_2, A_2), \ldots, G(V_{\kappa_2}, E_{\kappa_2}, A_{\kappa_2}) \) in \( G(\hat{A}) \). Set \( r_i = 0 \), for \( i = 0, 1, \ldots, \kappa_2 \), and \( g = 1 \).

2) All the \( \kappa_1 \) nodes with zero in-degree should be selected and pinned. Then, update the value of \( r_0 \) by \( r_0 = r_0 + \kappa_1 \).

3) Check the condition \( \kappa_2 \neq 0 \)? If it does not hold, stop; else go to step 4).

4) Check whether there exists at least one node in \( V_g \) which is reachable from a node belonging to the node set \( V \setminus V_g \). If it holds, go to step 5); otherwise, go to step 6).

5) Check the following condition: \( g < \kappa_2 \)? If it holds, let \( g = g + 1 \) and re-perform step 4); else stop.

6) Arbitrarily select one node in \( V_g \) to be pinned, update the value of \( r_0 \) by \( r_0 = r_0 + 1 \). Check the following condition: \( g < \kappa_2 \)? If it holds, let \( g = g + 1 \) and go to step 4); else stop.

It can be verified that there exists at least one node in \( G(\hat{A}) \) which is not reachable by node \( N+1 \) if there are less than \( r_0 \) nodes in \( G(\hat{A}) \) that are selected and pinned. Furthermore, it is worth noting that the complexity of Algorithm 1 is \( O(N + |E^r|) \), where \( E^r \) is the number of the directed edges in \( G(\hat{A}) \). Noticeably, under the condition that target system (3) possesses a globally attracting solution \( \tilde{x}(t) \), the global pinning synchronization in network (5) can be achieved asymptotically even when there is no coupling between any pair of neighboring nodes in \( G(\hat{A}) \). In the sequel, it is assumed that target system (3) does not possess a globally attractive solution. Then, it is not hard to verify that global pinning synchronization in network (5) can not be ensured if the augmented graph \( G(\hat{A}) \) does not contain any directed spanning tree. Next, we demonstrate how to use Algorithm 1 to find the nodes to be pinned in a given directed graph such that the augmented graph contains a directed spanning tree. Suppose that the network topology \( G(\hat{A}) \) contains 14 nodes as shown in Fig. 1. According to step 1) of Algorithm 1, one has that there are two nodes in \( G(\hat{A}) \) with zero in-degree, nodes 5 and 14 (see the nodes highlighted by blue shading in Fig. 1), and two strongly connected components (see the subgraphs highlighted by pink shading in Fig. 1). According to step 2) of Algorithm 1, one knows that nodes 5 and 14 should be pinned. By steps 3)–6) of Algorithm 1, one obtains that two distinct nodes respectively selected from node sets \{1,2,3,4\} and \{9,10,11\} should be pinned. Then, select nodes 4 and 9 to be pinned. One may observe that the augmented network \( G(\hat{A}) \) contains a directed spanning tree rooted at node 15.
B. Pinning Synchronization of Directed Complex Networks With Aperiodic Sampled-Data Communications

In this subsection, global pinning synchronization for directed network (5) with $\omega(t) \equiv 0$, $i = 1, 2, \ldots, N$, is first studied. The $H_{\infty}$ pinning synchronization for directed network (5) in the presence of external disturbances is then addressed.

Based on the discussions in Section II, one can establish the following theorem which summarizes the main results on global pinning synchronization for directed network (5) with $\omega(t) \equiv 0$, $N \in \mathbb{R}^{N \times N}$. For notational brevity, asterisk '*' in a symmetric matrix denotes the entry implied by symmetry.

**Theorem 1:** Suppose that Assumptions 1 and 2 hold, and $\omega(t) \equiv 0$, $N \in \mathbb{R}^{N \times N}$. Then, global pinning synchronization in directed network (5) can be achieved if there exist a scalar $\tau > 0$, positive definite matrices $P, Q \in \mathbb{R}^{N \times N}$, positive semi-definite matrix $X \in \mathbb{R}^{N \times N}$, and $N_i \in \mathbb{R}^{N \times N}$, $i = 1, 2, 3$, such that

$$\begin{align*}
\Lambda &= \begin{bmatrix} 
\Lambda_{11} + \tau \rho^2 & \Lambda_{12} & \Lambda_{13} \\
* & \Lambda_{22} & \Lambda_{23} \\
* & * & \Lambda_{33} - \tau I_N 
\end{bmatrix} < 0, \\
\Xi &= \begin{bmatrix} 
X_{11} & X_{12} & X_{13} \\
* & X_{22} & X_{23} \\
* & * & X_{33} 
\end{bmatrix} > 0,
\end{align*}$$

where $\Lambda_{11} = N_1 + N_T^T + hX_{11}$, $\Lambda_{12} = -cP \tilde{\mathcal{L}} + N_T^T - N_1 + hX_{12}$, $\Lambda_{13} = P + N_T^T + hX_{13}$, $\Lambda_{22} = -N_2 - N_T^T + hX_{22}$, $\Lambda_{23} = -N_T^T + hX_{12} + c^2h\tilde{\mathcal{L}}^T Q \tilde{\mathcal{L}}$, $\Lambda_{33} = -N_3 - N_T^T + c^2h\tilde{\mathcal{L}}^T Q \tilde{\mathcal{L}}$ is defined in (6), and $\Lambda_{23} = hX_{12} + hQ$, with $X_i \in \mathbb{R}^{N \times N}$, $i, j = 1, 2, 3$, and $X = \begin{bmatrix} X_{11} & X_{12} & X_{13} \\
* & X_{22} & X_{23} \\
* & * & X_{33} \end{bmatrix} \in \mathbb{R}^{N \times N}$.

**Proof:** Construct the following piecewise differentiable Lyapunov-Krasovskii functional for error system (7):

$$V(t, e_t) = e^T(t) (P \otimes I_N) e(t) + (t_{k+1} - t) \int_{t_k}^{t} e^T(s) (Q \otimes I_N) e(s) ds.$$

where $P > 0$, $Q > 0$, $t \in [t_k, t_{k+1})$, $k \in \mathbb{N}$. Then, taking the time derivative of $V(t, e_t)$ along the trajectory of (7) for $t \in [t_k, t_{k+1})$ and $k \in \mathbb{N}$ gives

$$\dot{V}(t, e_t) = 2e^T(t) (P \otimes I_N) e(t) + (t_{k+1} - t) e^T(t) (Q \otimes I_N) e(t)$$

$$- \int_{t_k}^{t} e^T(s) (Q \otimes I_N) e(s) ds.$$

From the Newton-Leibnitz formula, the following equation holds for any given matrices $N_i \in \mathbb{R}^{N \times N}$, $i = 1, 2, 3$:

$$2 \left[ e^T(t) (N_i \otimes I_N) e(t) + e^T(t) \left( t - d_k(t) \right) (N_3 \otimes I_N) e(t) + F^T (x(t); s(t)) \right]$$

$$= \begin{bmatrix} 
X_{11} & X_{12} & X_{13} \\
* & X_{22} & X_{23} \\
* & * & X_{33} 
\end{bmatrix} \begin{bmatrix} 
N_{11} & N_{12} \\
* & N_{22} \\
* & * 
\end{bmatrix} = 0.$$

On the other hand, for any given positive semi-definite matrix $X = \begin{bmatrix} X_{11} & X_{12} & X_{13} \\
* & X_{22} & X_{23} \\
* & * & X_{33} \end{bmatrix} \in \mathbb{R}^{3N \times 3N}$, the following inequality holds:

$$h \eta_1^T(t) (X \otimes I_N) \eta_1(t) - \int_{t-d_k(t)}^{t} \eta_1^T(s) (Q \otimes I_N) \eta_1(s) ds \geq 0,$$

where $\eta_1(t) = [e^T(t), e^T(t - d_k(t)), F^T (x(t); s(t))]^T$. It thus follows from (10)–(12) that

$$\dot{V}(t, e_t) \leq 2e^T(t) (P \otimes I_N) e(t) + h e^T(t) (Q \otimes I_N) e(t)$$

$$+ \int_{t-d_k(t)}^{t} e^T(s) (Q \otimes I_N) e(s) ds + 2 \left[ e^T(t) (N_1 \otimes I_N) e(t) + e^T(t - d_k(t)) (N_2 \otimes I_N) e(t) + F^T (x(t); s(t)) \right]$$

$$\times \left[ e(t) - e(t - d_k(t)) - \int_{t-d_k(t)}^{t} e(s) ds \right] + h \eta_1^T(t)$$

$$\times (X \otimes I_N) \eta_1(t) - \int_{t-d_k(t)}^{t} \eta_1^T(s) (X \otimes I_N) \eta_1(s) ds.$$

Some calculations give that

$$h e^T(t) (Q \otimes I_N) e(t)$$

$$= c^2 h e^T(t - d_k(t)) (\tilde{\mathcal{L}}^T Q \tilde{\mathcal{L}} \otimes I_N) e(t - d_k(t))$$

$$+ h F^T (x(t); s(t)) (Q \otimes I_N) F (x(t); s(t))$$

$$- 2 c e^T(t - d_k(t)) (\tilde{\mathcal{L}}^T Q \otimes I_N) F (x(t); s(t)).$$

Substituting (14) into (13) gives

$$\dot{V}(t, e_t) \leq \eta_1^T(t) (\Lambda \otimes I_N) \eta_1(t)$$

$$- \int_{t-d_k(t)}^{t} \eta_2^T(s) (\Xi \otimes I_N) \eta_2(s) ds,$$

where $\eta_2(t, s) = [e^T(t), e^T(t - d_k(t)), F^T (x(t); s(t)); e^T(s)]^T$.

$$\Lambda = \begin{bmatrix} 
\Lambda_{11} & \Lambda_{12} & \Lambda_{13} \\
* & \Lambda_{22} & \Lambda_{23} \\
* & * & \Lambda_{33} 
\end{bmatrix}, \quad \Xi = \begin{bmatrix} 
X_{11} & X_{12} & X_{13} \\
* & X_{22} & X_{23} \\
* & * & X_{33} 
\end{bmatrix}.$$

$$\Lambda_{11} = N_1 + N_T^T + hX_{11}, \Lambda_{12} = -cP \tilde{\mathcal{L}} + N_T^T - N_1 + hX_{12},$$

$$\Lambda_{13} = P + N_T^T + hX_{13}, \Lambda_{22} = -N_2 - N_T^T + hX_{22} + c^2h\tilde{\mathcal{L}}^T Q \tilde{\mathcal{L}},$$

$$\Lambda_{23} = -N_T^T + hX_{12} + c^2h\tilde{\mathcal{L}}^T Q \tilde{\mathcal{L}}.$$
According to (2), it is sufficient to show that 
\[
\dot{V}(t, e_t) < 0
\]
if there exists a positive scalar \( \tau > 0 \) such that
\[
\eta_1^T(t)(\dot{\Lambda} \otimes I_n)\eta_1(t) - \int_{t-d(t)}^t \eta_2^T(t, s)(\Xi \otimes I_n)\eta_2(t, s)ds \\
+ \tau \rho^2 e^T(t)e(t) - \tau F^T(x(t); e(t))F(x(t); e(t)) < 0.
\]  
(15)

Let \( \Lambda = \bar{\Lambda} + \begin{bmatrix} \tau \rho^2 & O & O \\ O & O & * \\ * & * & -\tau I_N \end{bmatrix} \). Noticeably, for each \( t \in \{k, t_{k+1}\} \) and \( k \in \mathbb{N} \), one has \( \dot{V}(t, e_t) < 0 \) if \( \Xi \geq 0 \) and \( \bar{\Lambda} < 0 \). Furthermore, it can be verified that \( \lim_{t \to \infty} V(t, e_t) = V(t, e_t) - e^T(t_{k+1})(P \otimes I_n)e(t_{k+1} + 1) \), for all \( k \in \mathbb{N} \). By using some similar arguments as those in the standard proof of Lyapunov-Krasovskii stability theory [30], one gets that \( |e(t)| \) will converge to zero asymptotically under conditions (8) and (9), which indicates that global pinning synchronization of directed complex network (5) with \( \omega(t) = 0_{N_n} \) is achieved.

Suppose that the conditions given in Theorem 1 can be ensured, i.e., global synchronization can be ensured for some given sampling interval \( h = h_0 > 0 \). It is interesting to further study the maximum allowable sampling interval \( h_{\text{max}} \) guaranteeing pinning synchronization in Theorem 1. For this purpose, the following algorithm is provided.

**Algorithm 2**

The maximum allowable sampling interval \( h_{\text{max}} \) guaranteeing pinning synchronization in Theorem 1 can be estimated by the following procedures:

1. Set \( h_{\text{max}} = h_0 \) and step size \( \kappa = \kappa_0 \), where \( \kappa_0 > 0 \) is sufficiently small compared to \( h_0 \).
2. Search matrices \( P > 0, Q > 0, X > 0, N_i, i = 1, 2, 3 \), and scalar \( \tau > 0 \) such that LMIs (8) and (9) hold. If the conditions are satisfied, set \( h_{\text{max}} = h_{\text{max}} + \kappa \) and re-perform step 2). Otherwise, stop and let \( h_{\text{max}} \) be the maximum allowable sampling interval.

Note that to obtain a less-conservative estimation on the maximum allowable sampling interval, the free-weighting matrices technique was employed in the proof of Theorem 1. It is also worth noting that the dimensions of the LMIs (8) and (9) are independent of those of the nodes’ states in network (5). This ‘decoupling’ feature will be more desirable when each node is a high-dimensional system. Alternatively, one may get the following corollary where the dimensions of the synchronization criteria are dependent on those of the nodes’ states. Generally speaking, the synchronization conditions given in the following corollary will be less conservative than those given in Theorem 1. However, it will be seen that solving the LMIs given in the following corollary is challenging.

**Corollary 1:** Suppose that Assumptions 1 and 2 hold, and \( \omega(t) = 0_{N_n} \). Then, global pinning synchronization in directed network (5) with external disturbances can be ensured under some suitable conditions.

**Theorem 2:** Suppose that Assumptions 1 and 2 hold. Then, global \( \mathcal{H}_\infty \) pinning synchronization with performance index \( \gamma > 0 \) in directed network (5) can be achieved if there exist a scalar \( \varsigma > 0 \), positive definite matrices \( S, T \in \mathbb{R}^{N \times N} \), positive semi-definite matrix \( X \in \mathbb{R}^{N \times N} \), and \( W_1 \in \mathbb{R}^{N \times N} \), \( i = 1, \ldots, 4 \), such that

\[
\Psi = \begin{bmatrix} \Psi_{11} + \varsigma \rho^2 + \gamma^{-1}I_N & \Psi_{12} & \Psi_{13} & \Psi_{14} \\ * & \Psi_{22} & \Psi_{23} & \Psi_{24} \\ * & * & \Psi_{33} - \gamma I_N & \Psi_{34} \\ * & * & * & -\gamma I_N \end{bmatrix} < 0,
\]

(19)

\[
\Omega = \begin{bmatrix} Y_{11} & Y_{12} & Y_{13} & Y_{14} & W_1 \\ * & Y_{22} & Y_{23} & Y_{24} & W_2 \\ * & * & Y_{33} & Y_{34} & W_3 \\ * & * & * & Y_{44} & W_4 \\ * & * & * & * & T \end{bmatrix} > 0,
\]

(20)

where \( \Psi_{11} = W_1 + W_2^T + hY_{11}, \Psi_{12} = -cS\hat{L} + W_2^T - W_1 + hY_{12}, \Psi_{13} = S + W_3^T + hY_{13}, \Psi_{14} = -W_2^T + hY_{14}, \Psi_{22} = -Y_{12}^T + hY_{24} + c\hat{L}^T T^T, \Psi_{23} = -W_3^T + hY_{32}, \hat{L}^T \hat{L}, \hat{L} \) is defined in (6). \( \Psi_{24} = -W_2^T + hY_{24}, \Psi_{33} = Y_{23} + h\hat{L}, \Psi_{34} = hY_{34}, \Psi_{44} = hY_{44} \), with \( Y_{ij} \in \mathbb{R}^{N \times N} \), \( i, j = 1, 2, 3, 4 \), and \( Y = Y_{11} Y_{12} Y_{13} Y_{14} + Y_{22} Y_{23} Y_{24} W_1 + Y_{33} Y_{34} W_3 + Y_{44} W_4 \)
Proof: According to Theorem 1, it follows from conditions (19) and (20) that global pinning synchronization in directed network (5) with \( \omega(t) = 0 \) can be achieved.

Next, global \( H_\infty \) pinning synchronization problem with performance index \( \gamma \) is studied. For \( \theta \in [t_k, t_{k+1}) \) and an arbitrarily given \( k \in \mathbb{N} \), define

\[
J_\theta = \int_0^\theta (\gamma^{-1}e^T(t)e(t) - \gamma \omega^T(t)\omega(t)) \, dt,
\]

(21)

where \( \omega(t) = [\omega_1(t), \omega_2(t), \ldots, \omega_N(t)]^T \) with \( \omega_i(t) \in L^2_2[0, +\infty) \). By the zero initial condition \( e(t) = 0 \), for \( t \in [-h, 0] \), one gets

\[
J_\theta = \int_0^\theta (\gamma^{-1}e^T(t)e(t) - \gamma \omega^T(t)\omega(t) + \dot{V}(t)) \, dt,
\]

(22)

where

\[
V(t, e_i) = e^T(t)(S \otimes I_n)e(t) + (t_{k+1} - t)
\times \int_{t_k}^t e^T(s)(T \otimes I_n)e(s) \, ds.
\]

Furthermore, from the Newton-Leibnitz formula, the following equation holds for any given matrices \( W_i \in \mathbb{R}^{N \times N} \), \( i = 1, \ldots, 4 \):

\[
2 \int e^T(t)(W_1 \otimes I_n) + e^T(t - d_k(t))(W_2 \otimes I_n)
+ F^T(x(t); s(t))(W_3 \otimes I_n) + \omega^T(t)(W_4 \otimes I_n)
\times \left[ e(t) - e(t - d_k(t)) - \int_{t-d_k(t)}^t \dot{e}(s) \, ds \right] = 0.
\]

(23)

Based on the above analysis and by arguments similar to the proof of Theorem 1, one has

\[
\dot{V}(t, e_i) \leq \Phi \tilde{\eta}_i(t)(\tilde{\Omega} \otimes I_n) \tilde{\eta}_i(t)

- \int_{t-d_k(t)}^{t} \tilde{\eta}_i(t, s)(\Omega \otimes I_n) \tilde{\eta}_i(t, s) \, ds,
\]

(24)

where

\[
\tilde{\eta}_i(t) = [e^T(t), e^T(t - d_k(t)), F^T(x(t); s(t)), \omega^T(t)]^T,
\]

\[
\tilde{\eta}_i(t, s) = [e^T(t), e^T(t - d_k(t)), F^T(x(t); s(t)), \omega^T(t), \dot{e}^T(s)]^T,
\]

\[
\tilde{\Psi} = \begin{bmatrix}
\Psi_{11} & \Psi_{12} & \Psi_{13} & \Psi_{14} \\
* & \Psi_{22} & \Psi_{23} & \Psi_{24} \\
* & * & \Psi_{33} & \Psi_{34} \\
* & * & * & \Psi_{44}
\end{bmatrix},
\]

\[
\Psi_{11} = \begin{bmatrix}
Y_{11} & Y_{12} & Y_{13} & Y_{14} \\
* & Y_{22} & Y_{23} & Y_{24} \\
* & * & Y_{33} & Y_{34} \\
* & * & * & Y_{44}
\end{bmatrix},
\]

(25)

where \( Y_{ij}, i, j = 1, \ldots, 4 \), are defined in (19), \( \Omega \) is defined in (20).

On the other hand, one obtains from (2) that

\[
\phi^{T}(x(t); e(t))F^T(x(t); e(t)) > 0 \quad \text{for all } \zeta > 0.
\]

It thus follows from conditions (19) and (20) that \( J_\theta < 0 \), i.e.,

\[
\int_0^\theta e^T(r)e(r) \, dr < \gamma^2 \int_0^\theta \omega^T(s)\omega(s) \, ds,
\]

(25)

for \( \theta \in [t_k, t_{k+1}) \) and \( k \in \mathbb{N} \). By noticing that \( \omega(t) \in L^2_2[0, +\infty) \), integrating the above inequality from \( \theta = 0 \) to \( \infty \) yields

\[
\int_0^\infty e^T(r)e(r) \, dr < \gamma^2 \int_0^\infty \omega^T(s)\omega(s) \, ds.
\]

This indicates that the global \( H_\infty \) pinning synchronization problem of directed complex network (5) with a prescribed performance index \( \gamma \) is indeed achieved. The proof is thus completed.

Remark 3: For a prescribed \( \gamma > 0 \), global \( H_\infty \) synchronization with a disturbance rejection level less than \( \gamma \) can be verified by checking the conditions in Theorem 2. However, it is practically important to know the allowable smallest disturbance rejection level \( \gamma_{\text{min}} \) for synchronization in network (5). Note that it is very hard or even impossible to calculate \( \gamma_{\text{min}} \) theoretically. But it can be numerically estimated by solving the following optimization problem:

\[
\text{minimize } \gamma \quad \text{subject to } \psi < 0, \Omega \geq 0,
\]

where \( \psi \) and \( \Omega \) are respectively given in (19) and (20), \( \zeta > 0 \), matrices \( \psi > 0, \Omega > 0 \), and \( \psi_i, i = 1, \ldots, 4 \), with appropriate dimensions.

In Theorem 2, the dimensions of LMIs (19) and (20) are independent of those of the nodes’ states in network (5). Alternatively, one may get the following corollary where the dimensions of the \( H_\infty \) synchronization criteria are dependent on those of the nodes’ states.

Corollary 2: Suppose that Assumptions 1 and 2 hold. Then, global \( H_\infty \) pinning synchronization with performance index \( \gamma > 0 \) in directed network (5) can be achieved if there exist a scalar \( \zeta > 0 \), positive definite matrices \( S, T \in \mathbb{R}^{N \times N} \), positive semi-definite matrix \( \Psi \in \mathbb{R}^{4N \times 4N} \), and \( \psi_i, i = 1, \ldots, 4 \), such that

\[
\begin{bmatrix}
\Psi_{11} + \zeta \rho^2 + \gamma^{-1}I_{4N} & \Psi_{12} & \Psi_{13} & \Psi_{14} \\
\Psi_{22} & \Psi_{23} & \Psi_{24} & \\
\Psi_{33} & \Psi_{34} & \Psi_{44} & \Psi_{44}
\end{bmatrix} > 0,
\]

(26)

\[
\begin{bmatrix}
\Psi_{11} & \Psi_{12} & \Psi_{13} & \Psi_{14} & \Psi_{22} & \Psi_{23} & \Psi_{24} & \\
\Psi_{33} & \Psi_{34} & \Psi_{44} & \Psi_{44} & \Psi_{44} & \Psi_{44} & \Psi_{44} & \Psi_{44} & \Psi_{44} & \Psi_{44} & \Psi_{44}
\end{bmatrix} > 0.
\]

(27)
C. Extension to Global $\mathcal{H}_\infty$ Pinning Synchronization of Switching Directed Networks With Sampled-Data Communications

Global $\mathcal{H}_\infty$ pinning synchronization for fixed directed network with sampled-data communications has been investigated in previous subsections. However, in reality, the network topology may be time-varying due to technological limitations of sensors, external disturbances or communication channel failures. As pointed out in [31], analysis and synthesis on the dynamical behaviors of complex networks with time-varying network structures is very important. From this observation, global $\mathcal{H}_\infty$ pinning synchronization in switching directed networks with sampled-data communications is further studied in this subsection.

Let $\{G(A^i), \ldots, G(A^q)\}$, $q > 2$, be the set of all possible topologies. Suppose that there exists an infinite sequence of uniformly bounded non-overlapping time intervals $[t_k, t_{k+1})$, $k \in \mathbb{N}$, with $t_1 = 0$, $0 < \tau_0 < t_{k+1} - t_k$, and $\tau_0$ being a specified positive constant. For each $i \in \mathbb{N}$, the underlying topology is time-invariant for all $t \in [t_k, t_{k+1})$. Here, $\tau_0$ is called the dwell time. For the convenience of expression, a switching signal $\sigma(t) : [0, +\infty) \to \{1, \ldots, q\}$ is introduced to characterize the topology's evolution.

Let $G(A^{(i)}(t))$ be the topology of network (1) at time $t \geq 0$. On the other hand, the coupling force between any pair of neighboring nodes is generated by employing sampling technique and a zero-order hold circuit, i.e., the coupling force acting on each node $i$ is time-invariant for all $t \in [t_k, t_{k+1})$, $k \in \mathbb{N}$, this indicates that for each time interval $[t_k, t_{k+1})$, $k \in \mathbb{N}$, the interaction among the nodes is determined only by the coupling strength $\epsilon$ and the communication topology $G(A^{(i)}(t))$. Let $\{G(A^1), \ldots, G(A^q)\}$ be the set of all possible augmented network topologies. Thus, the closed-loop network under pinning control can be described as:

$$
\dot{x}_i(t) = f(x_i(t), t) + c_N \sum_{j=1}^{N} a_{ij}(t) \{x_j(t) - x_i(t)\} + \omega_i(t) - c_N p_i^{(i)} \{x_i(t_k) - s_i(t_k)\}, \quad t \in [t_k, t_{k+1}),
$$

where $c_N p_i^{(i)} > 0$ if node $i$ is pinned at time $t_k$, and $c_N p_i^{(i)} = 0$ otherwise, $i = 1, 2, \ldots, N$. Here, $A^{(i)}(t) = -[a_{ij}(t)]$ is the adjacency matrix of graph $G(A^{(i)}(t))$, $k \in \mathbb{N}$. Taking the target system (3) as a virtual node labeled as $N + 1$ in the considered network, one then gets

$$
\dot{e}(t) = F(x(t); s(t)) - c(\tilde{E}^{(i)}(t) \otimes I_N) e(t - d_k(t)) + \omega(t), \quad t \in [t_k, t_{k+1}), \quad k \in \mathbb{N},
$$

where $e(\theta) \equiv e(t)$ for all $\theta = [-h, 0]$, $\tilde{E}^{(i)}(t) = L^{(i)}(t) + \text{diag}(P^{(i)}(t), P^{(i)}(t), \ldots, P^{(i)}(t))$, $L^{(i)}(t)$ is the Laplacian matrix of $G(A^{(i)}(t))$. Furthermore, one has that $\tilde{E}^{(i)}(t) = \begin{bmatrix} \tilde{E}^{(i)}(t) & -P^{(i)}(t) \\ 0^T & 0 \end{bmatrix}$ is the Laplacian matrix of the augmented graph $G(\tilde{A}^{(i)}(t))$. Here, $\tilde{P}^{(i)}(t_k) = (P^{(i)}(t_k), P^{(i)}(t_k), \ldots, P^{(i)}(t_k)) \in \mathbb{R}^N$.

To derive the main results of this subsection, the following assumption is introduced.

**Assumption 3:** For each $i$, $1 \leq i \leq q$, the augmented graph $G(\tilde{A}^i)$ contains a directed spanning tree with node $N + 1$ being the root.

From the above analysis, one gets the following two theorems, which summarize the main results of this subsection.

**Theorem 3:** Suppose that Assumptions 1 and 3 hold, and $\omega(t) \equiv 0_{N \times N}$. Then, global pinning synchronization in directed network (28) can be achieved if there exist a scalar $\tau > 0$, positive definite matrices $P, Q \in \mathbb{R}^{N \times N}$, positive semi-definite matrix $X \in \mathbb{R}^{N \times N}$, and $N_i \in \mathbb{R}^{N \times N}$, $i = 1, 2, 3$, such that

$$
\begin{bmatrix}
\Lambda_1 + \tau P^2 & \Lambda_{12} & \Lambda_{13} \\
* & \Lambda_{22} & \Lambda_{23} \\
* & * & \Lambda_{33} - \tau I_N
\end{bmatrix} < 0, \quad (30)
$$

where $\Lambda_1 = N_1 + N_2^T + hX_{11}$, $\Lambda_{12} = -cP\tilde{C}^i + N_2^T - N_1 + hX_{12}$, $\Lambda_{13} = P + N_2^T + hX_{13}$, $\Lambda_{22} = -N_2 - N_2^T + hX_{22} + c^2h(\tilde{C}^i)^TQ\tilde{C}^i$, $\Lambda_{23} = -N_2^T + hX_{23} - ch(\tilde{C}^i)^TQ$, $\Lambda_{33} = hX_{33} + hQ$, $\tilde{C}^i = M\tilde{C}^iM^T, \tilde{C}^i$ is the Laplacian matrix of $G(\tilde{A}^i), i = 1, \ldots, q, M = [I_N, 0_N], X_{ij} \in \mathbb{R}^{N \times N}, i, j = 1, 2, 3, X = [X_{11}, X_{12}, X_{13}] \in \mathbb{R}^{N \times N}$.

**Proof:** Construct the following common piecewise differentiable Lyapunov-Krasovskii functional for the error system (29):

$$
V(t, e_t) = e^T(t)(P \otimes I_N)e(t) + \int_{t_k}^{t} \dot{e}^T(s)(Q \otimes I_N)\dot{e}(s)ds,
$$

where $P > 0, Q > 0, t \in [t_k, t_{k+1}), k \in \mathbb{N}$. Then, the theorem can be proved by following the steps in the proof of Theorem 1.

Furthermore, one can get the following theorem on global $\mathcal{H}_\infty$ pinning synchronization of directed network (28) with a switching directed topology. The detailed proof is omitted for brevity.

**Theorem 4:** Suppose that Assumptions 1 and 3 hold. Then, global $\mathcal{H}_\infty$ pinning synchronization with performance index $\gamma > 0$ in directed network (28) can be achieved if there exist a scalar $\gamma > 0$, positive definite matrices $S, T \in \mathbb{R}^{N \times N}$, positive semi-definite matrix $Y \in \mathbb{R}^{N \times N}$, and $W_i \in \mathbb{R}^{N \times N}$, $i = 1, \ldots, 4$, such that

$$
\begin{bmatrix}
\Psi_{11} + \gamma P^2 & \gamma I_N \\
* & \Psi_{12} & \Psi_{13} & \Psi_{14} \\
* & * & \Psi_{22} & \Psi_{23} & \Psi_{24} \\
* & * & * & \Psi_{33} - \gamma I_N & \Psi_{34} \\
* & * & * & * & \Psi_{44} - \gamma I_N
\end{bmatrix} < 0,
$$

$$
\begin{bmatrix}
Y_{11} & Y_{12} & Y_{13} & Y_{14} & W_i \\
* & W_{22} & W_{23} & W_{24} & * \\
* & * & W_{33} & Y_{34} & W_3 \\
* & * & * & W_{44} & W_4 \\
* & * & * & * & Q
\end{bmatrix} \geq 0,
$$

where $W_1 = W_{11} + hY_{11}$, $\Psi_{12} = -cS\tilde{C}^i + W_{12} - W_{11} + hY_{12}$, $\Psi_{13} = S + W_{12} + hY_{13}$, $\Psi_{14} = S + W_{14}^T + hY_{14}$, $\Psi_{22} = -W_{22} - W_{22}^T + hY_{22} + c^2h(\tilde{C}^i)^TQ\tilde{C}^i$, $\Psi_{23} = -W_{22}^T + hY_{23} - ch(\tilde{C}^i)^TQ\tilde{C}^i$, $\Psi_{33} = hY_{33} + hT$, $\Psi_{34} = hY_{34}$, $\tilde{C}^i = M\tilde{C}^iM^T, \tilde{C}^i$ is the Laplacian matrix of $G(\tilde{A}^i)$.
Remark 4: In the present work, the dynamical nodes in the considered network are assumed to have a homogeneous time-varying sampling rate, i.e., the sampler embedded in each node works at the same time instants: $t_k, k \in \mathbb{N}$. It is more interesting but challenging to further study how to achieve global $H_\infty$ pinning synchronization in fixed or switched directed network with heterogeneous time-varying sampling rates. Furthermore, the synchronization criteria provided in the present paper are dependent on the solvability of some high dimensional LMIs, which are thus inapplicable for complex networks of huge size.

V. NUMERICAL SIMULATIONS

In this section, some numerical examples are given to illustrate the effectiveness of the theoretical analysis. For the convenience of simulation, the periodic sampled-data technique is adopted.

Example 1: Consider a fixed directed network (5) whose network topology $\mathcal{G}(\hat{A})$ is shown in Fig. 2 with weighted edges. It can be seen from Fig. 2 that $\mathcal{G}(\hat{A})$ contains a directed spanning tree with the target system as the root. In simulations, each node on the considered network is assumed to be a Chua’s circuit [32]–[34]. Recently, synchronization of networked Chua’s circuits has received an increasing attention [11], [35], [36]. The dimensionless equation of Chua’s circuit can be rewritten as

$$\begin{align*}
\dot{x} & = -\alpha (y - x - \mu(x)), \\
\dot{y} & = x - y + z, \\
\dot{z} & = -\beta y - \theta z, \\
\end{align*}$$

(34)

with $(x, y, z) \in \mathbb{R}^3$, $\mu(x) = bx + (1/2)(a-b)(x+1-x-1)$. In simulations, the dynamics of each node are determined by the equation of Chua’s circuit (34) with $\alpha = -1.3018$, $\beta = -0.0135$, $\theta = -0.0297$, $a = 0.1091$, and $b = -0.5700$. The trajectory of the target system is given by (3) with $s(0) = (0.7, 0.1, 0.65)^T$ is shown in Fig. 3. Take $\chi(x, y, z) = (\alpha(y - x - \mu(x)), x - y + z, -\beta y - \theta z)^T, x, y, z \in \mathbb{R}$. Some calculations give

$$\begin{align*}
\chi(x, y, z) & = \begin{pmatrix} 0.5598 & -1.3018 & 0 \\
1.0000 & -0.0000 & 1.0000 \\
0 & 0.0135 & 0.0297 \\
\end{pmatrix} \begin{pmatrix} x \\
y \\
z \end{pmatrix} + \bar{\chi}(x, y, z), \\
\end{align*}$$

where $\bar{\chi}(x, y, z) = -((\alpha(a-b)/2)(x+1-x-1), 0, 0)^T$. Then, $||\chi(\xi) - \bar{\chi}(\xi)|| < 2.98899|\xi - \xi|^2$ for all $\xi, \xi \in \mathbb{R}^3$. Set $\varepsilon = 1$. It follows from Algorithm 2 that global pinning synchronization can be achieved with a maximum allowable sampling interval $h_{\text{max}} = 0.09504$. The state trajectories of the nodes are given in Fig. 4, which indicates that the global pinning synchronization in directed network (5) with sampling interval $h = 0.085$ and $\omega(t) \equiv 0.15$ is achieved. The relationship between the maximum allowable sampling interval $h_{\text{max}}$ and the coupling strength $c$ is numerically obtained and depicted in Fig. 5. Interestingly, it is found that $h_{\text{max}}$ increases firstly and then decreases, when enlarging the coupling strength $c$. Use $\text{Error}(t) = \sqrt{\sum_{j=1}^{5} |x_j - x_j(t)|^2}$ to denote the synchronization error of the network. Fig. 6 indicates that a faster convergence rate will be yielded when enlarging the sampling interval.
Furthermore, the maximum allowable sampling intervals $h_{\text{max}}$ determined by Theorem 1 and Corollary 1 versus different coupling coupling strength $c$ are given in Table I, from which it can be seen that the results provided in Theorem 1 is not very conservative.

Next, the global $H_\infty$ pinning synchronization in directed network (5) is studied numerically. Let the coupling strength $c = 1$. According to Theorem 2, the global $H_\infty$ pinning synchronization with performance index $\gamma = 1.12$ in directed network (5) can be guaranteed. Choose $\omega_i(t) = (\sin(it), \cos(it), \sin(t))^T$, for $0 \leq t \leq 3$, and $\omega_i(t) = 0_3$ for $t > 3$, $i = 1, \ldots, 3$. The state trajectories of the nodes are given in Fig. 7. The energy trajectories of $e(t)$ and $\omega(t)$ are shown in Fig. 8, which indicates that the global $H_\infty$ pinning synchronization control problem is solved.

**Example 2:** In this example, the global $H_\infty$ pinning synchronization problem of switched directed network (28) is numerically studied. The network topology is assumed to switch back and forth between $G(A^1)$ and $G(A^2)$ per 0.12 second. The possible topologies $G(A^1)$ and $G(A^2)$ are shown in Fig. 9 where the weights are indicated on the edges. It can be verified that Assumption 3 holds. Let $f(x_i(t), t) = (0.5 \sin(x_{i1}(t)), 0.5 \cos(x_{i2}(t)))^T$, for $i = 1, \ldots, 6$. Set $c = 0.5$. It can be then obtained from Theorem 3 that the maximum allowable sampling interval for achieving global pinning synchronization of switched directed network (28) is $h_{\text{max}} = 0.10132$. The state trajectories of the switched directed network (28) with coupling strength $c = 0.5$ and sampling interval $h = 0.10$ are shown in Fig. 10. The relationship between $h_{\text{max}}$ and $c$ is numerically depicted in Fig. 11. Use $\text{Error}(t) = \sqrt{\sum_{j=1}^{3} ||x_j - x_0(t)||^2}$ to denote the synchronization error of the network. Fig. 12 indicates that a faster convergence rate will be yielded when enlarging the sampling interval. Furthermore, it can be obtained from Theorem 3 that global $H_\infty$ pinning synchronization with performance index $\gamma = 0.45$ in switched directed network (28) can be guaranteed. Choose $\omega_i(t) = (2 \sin(it), 2 \cos(it))^T$, for
input-delay approach, Lyapunov-Krasovskii stability analysis, and LMI technique, some sufficient conditions for achieving global pinning synchronization in fixed directed networks have been derived and discussed. The results are then extended to global $H_{\infty}$ pinning synchronization with external disturbances and switching topologies. Here, it is assumed that the dynamic nodes in the present network model have a common homogeneous sampling rate, that is, only global $H_{\infty}$ pinning synchronization for switched directed networks with synchronous sampling rate is addressed in the present paper. The global $H_{\infty}$ pinning synchronization problem for directed complex networks with asynchronous sampled-data communication and switched directed topology is still in its infancy. It remains to be seen how to solve such a challenging problem in the future.

ACKNOWLEDGMENT

The authors would like to thank the Associate Editor and all the anonymous reviewers for their constructive suggestions, which have led to the improvement of the presentation of this paper.

REFERENCES

Wenwu Yu (S’07–M’12) received the B.Sc. degree in information and computing science and the M.Sc. degree in applied mathematics from the Department of Mathematics, Southeast University, Nanjing, China, in 2004 and 2007, respectively, and the Ph.D. degree from the Department of Electronic Engineering, City University of Hong Kong, Hong Kong, China, in 2010.

Currently, he is a Professor in the Department of Mathematics, Southeast University, China. His research interests include nonlinear dynamics and control, complex networks and systems.

Michael Z. Q. Chen (M’08) received the B.Eng. degree in electrical and electronic engineering from Nanyang Technological University, Singapore, and the Ph.D. degree in control engineering from Cambridge University, Cambridge, U.K.

He is currently an Assistant Professor in the Department of Mechanical Engineering at the University of Hong Kong. His research interests include passive network synthesis, mechanical control, complex networks, and smart grid.

Xinghuo Yu (M’92–SM’98–F’08) received the B.Eng. and M.Eng. degrees from the University of Science and Technology of China, Hefei, China, in 1982 and 1984, respectively, and the Ph.D. degree from Southeast University, Nanjing, China in 1988.

He is now with RMIT University, Melbourne, Australia, where he is currently the Founding Director of the RMIT Platform Technologies Research Institute. His research interests include variable structure control and nonlinear control, complex and intelligent systems and industrial applications.

Guanrong Chen (M’89–SM’92–F’08) received the M.Sc. degree in computer science from the Sun Yat-sen University, Guangzhou, China, in 1981, and the Ph.D. degree in applied mathematics from Texas A&M University, College Station, TX, USA, in 1987.

Currently, he is a Chair Professor and the Founding Director of the Center for Chaos and Complex Networks at the City University of Hong Kong, prior to which he was a tenured Full Professor in the University of Houston, TX. He is a Fellow of the IEEE (Jan. 1997), with research interests in chaotic dynamics, complex networks and nonlinear controls.

Guanghi Wen (S’11–M’13) received the Ph.D. degree in mechanical systems and control from Peking University, China, in 2012.

From September 2012 to January 2013, he was a Research Associate and Post-Doctor in the University of New South Wales at Australian Defence Force Academy, Australia. Currently, he is a Lecturer in the Department of Mathematics, Southeast University, China. His research focuses on cooperative control of multi-agent systems.