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Stabilizing Solution and Parameter Dependence of Modified Algebraic Riccati Equation With Application to Discrete-Time Network Synchronization

Michael Z. Q. Chen, Member, IEEE, Liangyin Zhang, Housheng Su, and Guanrong Chen, Fellow, IEEE

Abstract—This technical note deals with a modified algebraic Riccati equation (MARE) and its corresponding inequality and difference equation, which arise in modified optimal control and filtering problems and are introduced into the cooperative control problems recently. The stabilizing property of the solution to MARE is presented. Then, the uniqueness is proved for the almost stabilizing and positive semi-definite solution. Next, the parameter dependence of MARE is analyzed. An obtained parameter dependence result is finally applied to the study of semi-global synchronization of leader-following networks with discrete-time linear dynamics subject to actuator saturation.

Index Terms—Input saturation, parameter dependence, Riccati equation/inequality, stabilizing solution, synchronization.

I. INTRODUCTION

This technical note considers the following modified algebraic Riccati equation (MARE):

\[ P = A^T P A - \gamma A^T P B (B^T P B + R)^{-1} B^T P A + Q. \]

This kind of quadratic matrix equation and its corresponding inequality and difference equation have been studied in modified optimal control [1]–[3], modified filtering [4]–[7], and the control and estimation for networked systems [8]–[12]. The existence and uniqueness of a positive semi-definite solution are established for MARE in [9]. Recently, MARE and the modified algebraic Riccati inequality (MARI) are applied to the discrete-time cooperative control problems in [13]–[17].

The scalar parameter \( \gamma \) in MARE is called the characteristic parameter hereafter. If \( \gamma \neq 1 \), MARE cannot be transformed to ARE via scaling. However, some properties of the discrete-time ARE are still preserved for MARE even if \( \gamma \neq 1 \). If \( \gamma \) is greater than a critical value \( \gamma_c \) and some other conditions are satisfied, then there exists a unique positive semi-definite solution [12]. In this note, it is shown that this solution possesses the stabilizing property. Besides, a similar relationship between the solutions of ARE and of the algebraic Riccati inequality (ARI) is demonstrated for MARE and MARI. In addition, it is noted that the critical value \( \gamma_c \) non-decreasingly depends on the parameter matrices \( Q \) and \( R \). Furthermore, the monotonic dependence of MARE solution on parameter matrices is shown to be the same as that for ARE; and the MARE solution is found to be non-increasingly dependent on the characteristic parameter. To the best of our knowledge, both stabilizing property and parameter dependence have not been studied in detail for MARE before. The parameter dependence results are important for studying discrete-time network synchronization, including the event-triggered cooperative control of linear dynamical networks in [18] and the input-saturated synchronization in this note.

Noticeably, it is difficult to analyze the uniqueness of the almost stabilizing solution for MARE. When \( \gamma \neq 1 \), none of the existing methods in the literature for proving the uniqueness of an almost stabilizing solution to ARE is applicable to MARE. Nevertheless, for null controllable systems [19]–[23] containing single-integrator [24]–[26], double-integrator [26], [27], and multiple-integrator systems [28] as special cases, that is, \((A, B)\) is stabilizable and the spectral radius of \( A \) is not larger than 1, the uniqueness of the almost stabilizing and positive semi-definite solution of MARE with \( Q = 0 \) is demonstrated in this note. Consequently, a key parameter dependence result for MARE can be established. Specifically, under some assumptions, the solution converges monotonically to a zero matrix as \( Q \) approaches zero.

This key MARE parameter dependence result is then applied to synchronization of dynamical networks with linear dynamics subject to input saturation. We are concerned with the discrete-time problem for leader-following networks on undirected switching graphs. In this setting, the network coupling makes it more difficult than the continuous-time one discussed in [20], since the ARE results in [23] for a single system will not work for multi-agent systems (MAS). In light of this, the MARE is explored in this note. The obtained MARE counterparts of the ARE results in [23] enable one to design low-gain feedback laws for leader-following networks with discrete-time higher-order dynamics subject to input saturation, so as to achieve semi-global exponential synchronization.

Nomenclature: For \( X \in \mathbb{R}^{n \times p} \), its eigenvalues are denoted by \( \lambda_1(X), \lambda_2(X), \ldots, \lambda_p(X) \) satisfying that \( |\lambda_1(X)| \leq \cdots \leq |\lambda_p(X)| \); and \( \rho(X) = |\lambda_p(X)| \) denotes its spectral radius. A square matrix \( A \) is said to be Schur if \( \rho(A) < 1 \). \( I \) denotes an identity matrix with compatible dimension; \( \text{diag}\{\cdot\} \) denotes a diagonal matrix; and \( \otimes \) denotes the Kronecker product.

II. MODIFIED ALGEBRAIC RICCATI EQUATION

A. Preliminaries

Lemma 1 [9]: Let \( Q \in \mathbb{R}^{n \times n} \), \( R \in \mathbb{R}^{m \times m} \), \( Q > 0 \), and \( R > 0 \). Assume that the pair \((A, B)\) is stabilizable with \( \rho(A) \geq 1 \). Consider the modified algebraic Riccati equation (MARE)

\[ P = g_\gamma(P) \triangleq A^T P A - \gamma A^T P B (B^T P B + R)^{-1} B^T P A + Q \]
and the modified algebraic Riccati inequality (MARI)

\[ P \succeq g_\gamma(P) \quad \text{for a symmetric matrix } P \]

as well as the strict MARI

\[ P > g_\gamma(P) \quad \text{for a symmetric matrix } P. \]

Then, there exists a critical value \( \gamma_c \in [0, 1) \) satisfying that

\[ \gamma_c \triangleq \inf \{ \gamma \mid 3 P \succeq 0 \text{ solving MARI (3)} \} \]

and \( \gamma_c \geq \gamma_1(A) \triangleq 1 - 1/(\rho(A))^2 \geq 0 \). For any \( \gamma \in (\gamma_1(A), 1] \), every positive semi-definite solution (if it exists) to MARI (2) is positive definite. For any \( \gamma \in (\gamma_1, 1) \), MARE (1) has a unique positive semi-definite solution \( P \succeq 0 \), which is positive definite; furthermore, \( P \) is the limit of any sequence of matrices \( \{P_k\} \) defined by the following modified Riccati difference equations:

\[ P_{k+1} = g_\gamma(P_k), \quad k = 0, 1, 2, \ldots \]

for any initial \( P_0 \geq 0 \). The scalar parameter \( \gamma \) is referred to as the characteristic parameter of MARE (1) and of other corresponding Riccati equation/inequalities.

**Lemma 2 [12, Lemma 5.4 (b)]:** The critical value \( \gamma_c \) defined in (4) satisfies that

\[ 0 \leq \gamma \leq \gamma_c \leq \gamma_2(A), \quad \text{where } \gamma_1(A) = 1 - 1/(\rho(A))^2 \]

and \( \gamma_2(A) \triangleq 1 - 1/(\rho(M(A))^2 \geq 0 \). For any \( \gamma \in (\gamma_1(A), 1] \), every positive semi-definite solution (if it exists) to MARI (2) is positive definite. If \( P \) is positive definite and \( \gamma \in (\gamma_1, 1) \), then \( MARE(1) \) has a unique positive semi-definite solution; however, \( MARE(1) \) has the limit of any sequence \( \{P_k\} \) defined by (5) for any initial \( P_0 \geq 0 \).

**Proof:** If \( P \succeq 0 \) solves (2), then by [9, Lemma 1 (o)], \( P \succeq g_\gamma(P) \geq (1 - \gamma) A^T P A + Q \succeq 0 \). Since \( A \) is Schur, there exists a \( W \succeq 0 \) such that \( W = A^T W A + Q \succeq 0 \). Thus, \( W \succeqs 3 \) and the condition of [9, Lemma 4] is satisfied. Denote \( Z_t = g_\gamma(Z_0), Z_{k+1} = g_\gamma(Z_k) \). By [9, Lemmas 1 (b), c, and 4], \( Z_t \succeq 0 \), \( Z_t \succeq Z_{t-1} \). Denote \( P \succeq 0 \), solve MARE (2) with \( P \succeq 0 \), furthermore, this solution \( P \) is the limit of any sequence \( \{P_k\} \) defined by (5) for any initial \( P_0 \geq 0 \).

**Remark 1:** The stabilizing property with respect to \( \tau = \alpha, \beta \) plays an important role in the modified optimal filtering problem [7]; the case of \( \tau = 1 \) can be verified following [29, Theorem 13.5.2]. If \( \gamma_1 \geq 1, \alpha = \gamma = 1 = \beta \), the MARI/MARE stabilizing property is the well-known one for ARI/ARE.

For MARI (2) or MARE (1) with general real symmetric \( Q \) and \( R \), if \( R \) is nonsingular and a solution \( P \) satisfies that \( (B^T P B + R)^{-1} B^T X A \) is nonsingular and \( \rho(A) \leq 1 \) with \( A = A \), \( \beta \) defined in (6) having \( \tau = \gamma \), then this solution \( P \) is said to be almost stabilizing. If \( \rho(A) \leq 1 \), it is trivial to see that \( P = 0 \) is an almost stabilizing and positive semi-definite solution to MARI (2) and MARE (1) with \( Q = 0 \). Furthermore, we will demonstrate the uniqueness of an almost stabilizing and positive semi-definite solution to the following MARE:

\[ X = A^T X A - \gamma A^T X B (B^T X B + R)^{-1} B^T X A \]

**Theorem 2:** Assume that \( R \succeq 0, (A, B) \) is stabilizable, and \( \rho(A) \leq 1 \). With the above lemma, \( X \) is the unique almost stabilizing and positive semi-definite solution to MARE (9).

**Proof:** On the contrary, suppose that there exists a non-zero almost stabilizing solution \( Y \succeq 0 \). Denote \( A_Y, X \triangleq A - \gamma B (B^T X B + R)^{-1} B^T X A \). Then, \( \tau = 0 \) and \( Y = 0 \) are both positive semi-definite solutions to the linear matrix equation (LME) (30):

\[ X = A^T X A_Y - \gamma A_Y = (A_Y)^T X A \]

Applying [29, Theorem 5.2.3], there exists an eigenvalue \( \lambda_1 \) of \( A \) and an eigenvalue \( \lambda_2 \) of \( A_Y \) such that \( \lambda_1, \lambda_2 > 1 \). Since \( |\lambda_1| = |\lambda_2| \leq 1 \), one has \( |\lambda_1| = |\lambda_2| = 1 \), and both \( A \) and \( A_Y \) are not Schur. Thus, if \( \rho(A) < 1 \), a contradiction is obtained already. Now, assume that \( \rho(A) = 1 \).

Since the positive-semidefinite matrix \( Y \) is orthogonally diagonalizable, there exists an orthogonal matrix \( U \) such that \( Y = U^T A \) with \( A \) being diagonal, and \( A = \text{diag}(0, \overline{Y}) \), \( \overline{Y} \succeq 0 \) is a diagonal matrix with diagonal elements being the positive eigenvalues of \( Y \). Denote \( \overline{A} = U^T \overline{A} U, \overline{B} = U B, \overline{A}_Y = \overline{A} - \gamma \overline{B} (\overline{B}^T \overline{A} B + R)^{-1} \overline{B}^T \overline{A} \).

\[ \overline{A} \triangleq \overline{A}^T \overline{A} - \gamma \overline{A}^T \overline{B} (\overline{B}^T \overline{A} B + R)^{-1} \overline{B}^T \overline{A} \]

where \( \overline{A}, \overline{B} \) is stabilizable, and the eigenvalues of \( \overline{A} \) and \( \overline{A}_Y \) are all located within the closed unit disc. Now, partition \( \overline{A} \) and \( \overline{B} \)
into block matrices with compatible dimensions as $\hat{B} = \begin{bmatrix} B_1 & B_2 \end{bmatrix}$, such that (10) reduces to the following through straight-forward manipulation:

\[
\begin{align*}
\Upsilon &= \begin{bmatrix} A_2^T & A_2 \end{bmatrix} Y_{A_2} - \gamma A_2^T Y_{B_2} (B_2^T Y_{B_2} + R)^{-1} B_2^T Y_{A_2} \\
A_2^T \bar{Y}_{A_3} &= \gamma A_2^T \bar{Y}_{B_2} (B_2^T \bar{Y}_{B_2} + R)^{-1} B_2^T \bar{Y}_{A_3} \\
A_2^T \bar{Y}_{A_3} &= \gamma A_2^T \bar{Y}_{B_2} (B_2^T \bar{Y}_{B_2} + R)^{-1} B_2^T \bar{Y}_{A_3} \\
(11) \end{align*}
\]

By (11) and noting that $\Upsilon > 0$, one can easily verify that $\hat{A}_2$ is nonsingular. By (12), $\hat{A}_2 = \gamma B_2 (B_2^T Y_{B_2} + R)^{-1} B_2^T \bar{Y}_{A_3}$, Applying the Matrix Inversion Formulas [31], 

\[
B_2 (B_2^T Y_{B_2} + R)^{-1} B_2^T \bar{Y}_{A_3} = I - (I + B_2 R^{-1} B_2^T \bar{Y}^{-1})^{-1} A_3.
\]

And thus one obtains that $(11)\hat{A}_2 = \gamma (I + B_2 R^{-1} B_2^T \bar{Y}^{-1})^{-1} A_3$. Therefore,

\[
A_3 A_2^T = \gamma (1 - B_2^T R^{-1} B_2) Y_{A_3} A_2^T.
\]

For any eigenvalue $\lambda$ of $A_3 A_2^T$, if $\lambda A_2^T \geq 0$ with a real eigenvector $x \neq 0$, left-multiplying $x^T \Upsilon$ and right-multiplying $x^T \bar{Y}$, one obtains $\lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda 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Proof: It follows from Proposition 2 and Lemma 1.

Remark 3: A special case of MARI (16) is the following:

\[ P > A^T PA - \gamma A^T PB (B^T PB)^{-1} B^T PA \]  

(17)

which is presented in [8, eq. (9)], and plays an important role in [14–17]. Any possible solution \( P > 0 \) to MARI (17) also solves MARI (16) for some sufficiently small \( \epsilon, \delta > 0 \). In Theorem 3, the continuity of ARE solution with respect to parameter matrices (see [34]) is extended to MARE. The results in Theorem 3 can be applied to the discrete-time event-triggered synchronization problem [18].

D. Key Result on Parameter Dependence of MARE Solution

Assumption 1 below is standard for semi-global stabilization of control systems subject to actuator saturation [19–23].

Assumption 1: The pair \((A,B)\) is stabilizable and \( \rho(A) = 1 \).

Theorem 4: Let Assumption 1 hold and \( \gamma \in (0,1], \delta > 0 \). Then, for each \( \epsilon \in (0,\tilde{\epsilon}] \), there exists a unique positive definite matrix \( P(\epsilon) \) that solves the MARE

\[ P = A^T PA - \gamma A^T PB (B^T PB + I)^{-1} B^T PA + \epsilon E \]  

(18)

Moreover, \( \lim_{\epsilon \to 0^+} P(\epsilon) = 0 \) monotonically.

Proof: Combining Lemma 2, Proposition 3, and Theorems 1–3, one obtains the results.

Remark 4: To prove \( \lim_{\epsilon \to 0^+} P(\epsilon) = 0 \), it is necessary to establish the uniqueness of an almost stabilizing and positive semi-definite solution. The result in [23, Lemma 3.1] is merely a special case of Theorem 4. If \( \gamma \neq 1 \), no scaling manipulation can let \( \gamma \) be absorbed into \( P, B, A, R \), and \( Q \).

III. SEMI-GLOBAL DISCRETE-TIME SYNCHRONIZATION VIA LOW-GAIN FEEDBACK

In this section, the MARE results are applied to the semi-global synchronization problem for discrete-time linear MAS subject to input saturation. Consider a group of \( N \) agents, labeled as \( 1,2,\ldots, N \). The motion of each agent is as follows:

\[ x_i(t+1) = Ax_i(t) + B \cdot \text{sat}(u_i(t)), i = 1,2,\ldots, N \]  

(19)

where \( x_i \in \mathbb{R}^n \) is the state of agent \( i; u_i \in \mathbb{R}^m \) is the control input for agent \( i; \) and sat: \( \mathbb{R}^m \to \mathbb{R}^m \) is a saturation operator defined as \( \text{sat}(u_{ij}) \equiv \text{sat}(u_{i1}), \text{sat}(u_{i2}), \ldots, \text{sat}(u_{im}) \) with the saturation function \( \text{sat}(u_{ij}) \equiv \text{sgn}(u_{ij})\min(|u_{ij}|, \tilde{\epsilon}) \) for an \( a \)-priori given input-saturation threshold \( \tilde{\epsilon} > 0 \). Denote \( u \equiv [u_{1}^T, u_{2}^T, \ldots, u_{N}^T]^T \) and \( \text{sat} \equiv [\text{sat}(u_{11})^T, \text{sat}(u_{21})^T, \ldots, \text{sat}(u_{N1})^T]^T \). The dynamics of the leader, labeled as \( N + 1 \), are described by

\[ x_{N+1}(t+1) = Ax_{N+1}(t) \]  

(20)

The problem of semi-global leader-following synchronization for the agents and leader described above is as follows: Design some linear feedback law \( u_i \) for each agent \( i \), which uses only local information from the other agents, such that for any \( a \)-priori given bounded set \( X \subset \mathbb{R}^n \), the synchronization limit set \( \lim_{t \to \infty} \|x_i(t) - x_{N+1}(t)\| = 0 \), \( i = 1,2,\ldots, N \), is exponentially achieved as long as \( x_i(0) \in X \), \( i = 1,2,\ldots, N \).

A communication network consisting of \( N \) agents is described by an undirected graph \( G(t) = \{V,E(t)\} \) [20, 21]. Let \( \bar{G}(t) = \bar{G}(\delta(t)) \) be an extended graph generated by the leader and \( G(t) \), and \( H(t) \equiv \text{diag}(h_1(t), h_2(t), \ldots, h_N(t)) = H(\delta(t)) \) be defined as \( h_i(t) = 1 \) if agent \( i \) is a neighbor of the leader at time \( t \), and \( h_i(t) = 0 \) otherwise, and \( L(t) = L_{\delta(t)} \) be the Laplacian matrix ([20], [21]) of \( G(t) \); denote \( \mathcal{L}(t) = L(t) + H(t) = \mathcal{L}_{\delta(t)} \). Here, \( \delta : N \to \Gamma \) is a switching signal whose value at time \( t \) equals the index of \( \mathcal{G}(t) \), and the index set \( \Gamma \) contains indexes of all extended graphs. \( \Gamma_{\text{tree}} \) denotes the set of indexes of extended graphs that contain a spanning tree with the leader as the root vertex. If \( \delta(t) = s \in \Gamma_{\text{tree}} \) and \( \mathcal{Z}_s = L_s + H_s \), then \( \mathcal{Z}_s > 0 \) [20, 21].

Lemma 3: Let \( s \in \Gamma_{\text{tree}} \) and \( \mathcal{Z}_s = L_s + H_s \) with eigenvalues \( 0 < \lambda_1 \leq \cdots \leq \lambda_N \). Then, \( \min_{\epsilon \in E(s)} \max_{i=1,\ldots,N} |1 - \omega \lambda_i| = (\lambda_N - \lambda_1) / (\lambda_N + \lambda_1) < 1 \), \( \min_{\epsilon \in E(s)} \max_{i=1,\ldots,N} |1 - \omega \lambda_i| = 2 / (\lambda_1 + \lambda_N) \).

Proof: It can be proved similarly to [15, eq. (14)].

Remark 5: For \( s \in \Gamma_{\text{tree}} \), because the number of possible values of \( \lambda_1/N (\mathcal{Z}_s) \equiv \lambda_1/N \) is finite, one can find the minimum \( \lambda_1/N (\mathcal{Z}_s) \), denoted as \( \min \{\lambda_1/N \} \), using an exhaustive search method; and there are a finite number of possible values of \( (2/(\lambda_1 + \lambda_N)) \), the set of which is denoted by \( \Omega \).

Assumption 2: Extended graph \( \mathcal{G}(t) \) consisting of the \( N \) agents and the leader contains a spanning tree rooted at the leader all the time, that is, \( \delta(t) \in \Gamma_{\text{tree}}, \forall t \in N \).

A low-gain feedback design for leader-following multi-agent systems (19) and (20) is carried out in three steps.

Step (i). Find \( P = P(\epsilon) > 0 \) to solve the MARE

\[ P = A^T PA - \theta A^T PB (B^T PB + I)^{-1} B^T PA + \epsilon E \]  

(21)

where \( \epsilon \in (0,1] \) is the low-gain parameter to be designed, and the MARE characteristic parameter \( \theta \) is given by

\[ \theta \equiv \frac{2}{\lambda_1(t) + \lambda_N(t)} \in \Omega \]  

(22)

with \( \min \{\lambda_1/N \} \) defined in Remark 5.

Step (ii). Set the controller parameter \( \omega(t) \) as

\[ \omega(t) \equiv \frac{2}{\lambda_1(t) + \lambda_N(t)} \in \Omega \]  

(23)

where \( \lambda_1(t) \) and \( \lambda_N(t) \) denote the smallest and the largest eigenvalues, respectively, of \( \mathcal{L}(t) = L(t) + H(t) \); and \( \Omega \) is defined in Remark 5.

Step (iii). Construct a linear feedback law as

\[ u_i = K \left( h_i(t)(x_i - x_{N+1}) + \sum_{j=1}^{N} a_{ij}(t)(x_i - x_j) \right) \]  

(24)

for agents \( i = 1,2,\ldots, N \), where

\[ K = -\omega(B^T P(\epsilon)B + I)^{-1}B^T P(\epsilon)A \]  

(25)

with \( \omega = \omega(t) \) at time step \( t \).

The numerical solution for \( P(\epsilon) \) is referred to [9, Theorem 6]. The key MARE result \( \lim_{\epsilon \to 0^+} P(\epsilon) = 0 \) in Theorem 4 guarantees the effectiveness of the bounded input technique.

Lemma 4: The unique positive definite solution \( P(\epsilon) \) to MARE (21) and the controller matrix \( K \) in (25) satisfy that

\[ I_N \otimes P(\epsilon) \leq (\mathcal{L}(\mathcal{t}) \otimes K)^T (\mathcal{L}(\mathcal{t}) \otimes K) \]  

(26)

where \( \otimes \) denotes the Kronecker product [30].

Proof. Let \( \delta = \arg \min_{s \in \Gamma_{\text{tree}}} \lambda_1(\mathcal{Z}_s) \). Then, \( \theta = \beta \equiv 4\lambda_1(\mathcal{Z}_s)\lambda_N(\mathcal{Z}_s)/(\lambda_1(\mathcal{Z}_s) + \lambda_N(\mathcal{Z}_s))^2 \). Denoting \( \beta \equiv 1 + \sqrt{1 - \theta} = 2\lambda_N(\mathcal{Z}_s)/(\lambda_1(\mathcal{Z}_s) + \lambda_N(\mathcal{Z}_s)) \), and by (4) with \( (\mathcal{Q},R,\gamma,\tau) = (\epsilon I, I, 0, \beta) \), one has \( P(\epsilon) \equiv \beta \).
\(\lambda_N(t)^2K^TK + \varepsilon I \succ \lambda_N(t)^2K^TK\). Obviously, \(\lambda_N(t)^2I_N \succeq \mathcal{L}^2(t)\). Then, one has \(I_N \otimes P(\varepsilon) \succeq \lambda_N(t)^2I_N \otimes K^TK \succeq \mathcal{L}^2(t) \otimes K^TK = (\mathcal{L}(t) \otimes K)(\mathcal{L}(t) \otimes K)\).

\textbf{Theorem 5:} Consider a multi-agent system consisting of \(N\) agents with linear dynamics (19) subject to an \textit{a priori} given input-saturation threshold \(\varpi > 0\), and a leader with dynamics (20). Let Assumptions 1 and 2 hold. Then, the controller given by (24) and (25) achieves \(V = \mathcal{L}(t) \otimes K\).

\(\|x_i(0)\|_\infty < R_x\) with \(R_x > 0\), there exists an \(\varepsilon^* \in (0, 1]\) such that for the low-gain parameter \(\varepsilon = \varepsilon^*, \lim_{t \to \infty} \|x_i(t) - x_{N+1}(t)\| = 0\), \(i = 1, 2, \ldots, N\), as long as \(x_i(0) \in \mathcal{X}\) for all \(i = 1, 2, \ldots, N, N + 1\). Moreover, the convergence speed for the synchronization is exponential. Furthermore, an \(\varepsilon^*\) can be chosen such that

\[
p(\mathcal{P}(\varepsilon^*)) \leq \frac{\varpi^2}{(4NnR_x^2)}. \quad (27)
\]

\textit{Proof:} Denote \(\tilde{x}_i = x_i - x_{N+1}\) and \(\tilde{x} = [\tilde{x}_1^T, \tilde{x}_2^T, \ldots, \tilde{x}_N^T]^T\).

Applying the Kronecker product [30], \(u_i = K_i(t)(\tilde{x}_i) + \sum_{j=1}^{N} K_i(a_{ij}(t)(\tilde{x}_i - \tilde{x}_j))\); \(u = (\mathcal{L}(t) \otimes K)(\tilde{x})\). Denote \(\bar{A} \triangleq I_N \otimes A, \bar{B} \triangleq I_N \otimes B\). One has

\[
\dot{\tilde{x}}(t + 1) = \bar{A}\tilde{x}(t) + \bar{B} \cdot \text{sat} ((\mathcal{L}(t) \otimes K)\tilde{x}(t))
\]

for which the common quadratic Liapunov function \(V(\tilde{x}) \triangleq \frac{1}{2}\sum_{i=1}^{N} \tilde{x}_i^T P(\varepsilon)\tilde{x}_i = \tilde{x}^T(\mathcal{L} + I_N \otimes P(\varepsilon))\tilde{x}\) is used.

For any \(\varepsilon = \varepsilon_0 \in (0, 1]\), since \(\mathcal{X}\) is bounded, a level set parameter \(c(\varepsilon_0) > 0\) can be defined as

\[
c(\varepsilon_0) \triangleq \sup_{\varepsilon = \varepsilon_0, x_i(0) \in \mathcal{X}, i = 1, 2, \ldots, N+1} V(x_i(0)). \quad (29)
\]

And for \(\varepsilon = \varepsilon_0\), define the level set

\[
L_V(c(\varepsilon_0)) \triangleq \{\xi \in \mathbb{R}^{Nn} | V(\xi) \leq c(\varepsilon_0)\} \quad (30)
\]

which is bounded. Choose a sufficiently small \(\varepsilon^*\) such that when \(\varepsilon_0 = \varepsilon^*, \text{conditions } \tilde{x}(t) \in L_V(c(\varepsilon^*))\) and \(\omega(t) \in \mathcal{P}\) imply that \(\|\mathcal{L}(t) \otimes K\| \leq \varpi\) for \(i = 1, 2, \ldots, N\), where \(\varpi > 0\) is the \textit{a priori} given saturation threshold. The existence of an \(\varepsilon^*\) is guaranteed by the convergence \(\lim_{\varepsilon \to 0} P(\varepsilon) = 0\) established in Theorem 4.

In fact, if \(\varepsilon^*\) is chosen such that (27) holds, since \(P(\varepsilon^*) \preceq \rho(P(\varepsilon^*))I\), by (29), one has \(c(\varepsilon^*) \leq \rho(P(\varepsilon^*)) \sum_{i=1}^{N} \|\tilde{x}_i(0)\|^2 \leq (\varpi^2/(4NnR_x^2))\sum_{i=1}^{N} \|\tilde{x}_i(0)\|^2 \leq (\varpi^2/(4R_x^2))\sum_{i=1}^{N} \|\tilde{x}_i(0)\|^2 \leq \varpi^2/4R_x^2\).

By (26) and (30), if \(\tilde{x}(t) \in L_V(c(\varepsilon^*))\) and \(\omega(t) \in \mathcal{P}\), then \(\|\mathcal{L}(t) \otimes K\| \leq \varpi\) \(\leq \sqrt{\mathcal{L}(t) \otimes K}\| \leq \varepsilon^*\), \(\|x_i(t) - x_{N+1}(t)\| \leq c(\varepsilon^*) \leq \varpi\).

Thus, \(L_V(c(\varepsilon^*))\) remain linear for any \(\delta(t) = s \in T_{\text{tree}}, \text{and can be equivalently expressed as } \tilde{x}(t+1) = (I_N \otimes A + \mathcal{L}(t) \otimes BK)\tilde{x}(t), \text{where } \mathcal{L}(t) = L(t) + H(t), \text{and } K(t) \text{ is given in (25).}

For the remainder of the proof, \(\varepsilon = \varepsilon^*, \text{and } P(\varepsilon^*) \text{ is denoted as } P \text{ for short. Now, through straightforward manipulation, one can evaluate } \Delta V(t) \triangleq V(\tilde{x}(t + 1)) - V(\tilde{x}(t)), \text{which is the variation of } V \text{ along the discrete-time trajectories of } \tilde{x} \text{ within the set } L_V(c), \text{as follows:}\)

\[
\Delta V(t) = \tilde{x}^T(I_N \otimes (A^TPA - P) + 2\mathcal{L}(t) \otimes A^TPBK)
\]

\[
+ \mathcal{L}(t)^2 \otimes K^TB^TBPK \tilde{x}. \quad (31)
\]

Since \(\mathcal{L}(t) > 0\), there exists an orthogonal matrix \(U(t) = U_{\delta(t)}\) such that \(\mathcal{L}(t) = U_{\delta(t)}\mathcal{L}(t) U_{\delta(t)}^T(\Lambda(t)U(t), \Lambda(t)^{\dagger}\text{diag}(\lambda_1(t), \lambda_2(t), \ldots, \lambda_N(t)) = \Lambda_\delta(t), \text{where } \lambda_i(t) > 0 \text{ are eigenvalues of } \mathcal{L}(t). \text{De-}

\textbf{IV. CONCLUSION}

In this note, the stabilizing property and the parameter dependence have been studied for MARE. The uniqueness of an almost stabilizing and positive semi-definite solution has been established for MARE with \(Q = 0\). The discrete-time semi-global synchronization problem has been solved for linear MAS subject to input saturation. Future studies include: MARE with general real symmetric parameter matrices \(Q\) and \(R\); the global synchronization for networks containing high-order integrator dynamics subject to actuator saturation.

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