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Realization of Three-Port Spring Networks With Inerter for Effective Mechanical Control

Michael Z. Q. Chen, Kai Wang, Yun Zou, and Guanrong Chen

Abstract—This note is concerned with the passive network synthesis problem of one-port networks consisting of one inerter, one damper, and at most three springs. To solve the problem, a necessary and sufficient condition is derived for the realization of a three-port resistive network containing at most three elements, utilizing graph theory and several existing results of n-port resistive networks. By extracting the damper and the inerter, a necessary and sufficient condition is obtained for the realization of one-port networks containing one damper, one inerter, and at most three springs under an assumption that the admittance of three-port networks containing only springs is well-defined. The covering networks are also presented. Based on properties of circuit topology, a realizability condition is derived for the special case when the earlier assumption does not hold. Combining the two conditions when the assumption holds or not, the final realizability condition is obtained.

Index Terms—Inerter, mechanical network, passive network synthesis, positive-real function, three-port resistive network.

I. INTRODUCTION

Passive network synthesis has been an important branch of systems theory, which experienced a “golden era” from the 1930s to the 1970s with a vast volume of literature available today [1], [11], [16]. However, there are still many problems remaining unsolved. For instance, the general realization methods such as the Bott-Duffin procedure [1] appear to be highly non-minimal. Up to now, the minimal realization problem is still far from being solved.

Recently, a new mechanical element named “inerter” [18] was introduced, where the force applied at its two terminals is proportional to the relative acceleration between them. Naturally, the inerter completes the force-current analogy between mechanical systems and electrical ones. Therefore, the theory of passive electrical network synthesis can be directly translated to the mechanical setting, making the design of passive mechanisms more systematic. Applications of the inerter to the control of mechanical systems [3], [17], [21], [22], [23] have been investigated, and performance advantages over conventional passive mechanisms more systematic. Applications of the inerter to the control of mechanical systems [3], [17], [21], [22], [23] have been investigated, and performance advantages over conventional passive mechanisms more systematic. Applications of the inerter to the control of mechanical systems [3], [17], [21], [22], [23] have been investigated, and performance advantages over conventional passive mechanisms more systematic. Applications of the inerter to the control of mechanical systems [3], [17], [21], [22], [23] have been investigated, and performance advantages over conventional passive mechanisms more systematic. Applications of the inerter to the control of mechanical systems [3], [17], [21], [22], [23] have been investigated, and performance advantages over conventional passive mechanisms more systematic. Applications of the inerter to the control of mechanical systems [3], [17], [21], [22], [23] have been investigated, and performance advantages over conventional passive mechanisms more systematic. Applications of the inerter to the control of mechanical systems [3], [17], [21], [22], [23] have been investigated, and performance advantages over conventional passive mechanisms more systematic.
\( \tilde{F}_2 = -c \tilde{v}_2 \) and \( \tilde{F}_3 = -b s \tilde{v}_3 \), one obtains the driving-point admittance of \( Q \) as

\[
Y(s) = \frac{\tilde{F}_1}{\tilde{E}_1} = \frac{\alpha_3 s^3 + \alpha_2 s^2 + \alpha_1 s + \alpha_0}{s^4 + \beta_3 s^3 + \beta_2 s^2 + \beta_1 s}
\]

(1)

where \( \alpha_3 = K_{11}, \alpha_2 = (1/c)(K_{11}K_{22} - K_{12}^2), \alpha_1 = (1/b)(K_{11}K_{33} - K_{13}^2), \alpha_0 = (1/(bc)) \text{det}(K), \beta_3 = (1/c)K_{22}, \beta_2 = (1/b)K_{33}, \beta_1 = (1/(bc))(K_{22}K_{33} - K_{23}^2) \), and \( b, c > 0 \). Furthermore, according to the analogy to one-element-kind networks, it is obvious that \( (1/s)K \) is realizable as the admittance of a network consisting of at most three springs if and only if \( K \) is the admittance of a three-port resistive network containing at most three elements, whose realizability condition will be derived below.

**B. Three-Port Resistive Networks With At Most Three Elements**

It is well known that any third-order real symmetric matrix

\[
Y_N = \begin{bmatrix} y_{11} & y_{12} & y_{13} \\ y_{12} & y_{22} & y_{23} \\ y_{13} & y_{23} & y_{33} \end{bmatrix}
\]

(2)

is realizable by a three-port resistive network if and only if \( Y_N \) is paramount [6], [19]. However, necessary and sufficient conditions for the realization of three-port resistive networks with at most three elements have not yet been found.

Now, the main result of this subsection is established.

**Theorem 1:** A third-order real symmetric matrix \( Y_N \) in the form of (2) can be realized as the admittance of a three-port resistive network with at most three elements if and only if one of the following two conditions holds:

1. \( y_{12}y_{13}y_{23} \leq 0, y_{11} - |y_{12}| - |y_{13}| \geq 0, y_{22} - |y_{12}| - |y_{23}| \geq 0, y_{33} - |y_{13}| - |y_{23}| \geq 0, \) and at least three of \( y_{12}, y_{13}, y_{23}, (y_{11} - |y_{12}| - |y_{13}|), (y_{22} - |y_{12}| - |y_{23}|), \) and \( (y_{33} - |y_{13}| - |y_{23}|) \) are zero.

2. \( y_{12}y_{13}y_{23} \geq 0 \) and at least one of the following three conditions holds with at least three of the six inequality signs being equality:

   a) \( -|y_{11}| \leq 0, y_{12} \leq |y_{12}| \leq y_{11}, y_{13} \leq |y_{13}| \leq y_{12}, |y_{13}| \leq y_{12} \leq |y_{13}| \leq y_{11}, y_{12} \leq |y_{12}| \leq |y_{13}| \leq y_{12}, y_{13} \leq |y_{13}| \leq y_{11} \).

**Proof:** See [10] for details.

**C. Realizability Conditions**

To reduce the number of parameters to six, the following transformation will be used:

\[
G := \begin{bmatrix} G_1 & G_4 & G_5 \\ G_4 & G_6 & G_2 \\ G_5 & G_6 & G_3 \end{bmatrix} = T \begin{bmatrix} K_{11} & K_{12} & K_{13} \\ K_{12} & K_{22} & K_{23} \\ K_{13} & K_{23} & K_{33} \end{bmatrix} T
\]

(3)

where \( T = \text{diag}(1, 1/\sqrt{c}, 1/\sqrt{b}) \). Then, \( Y(s) \) is equivalent to

\[
Y(s) = \frac{G_{11}s^3 + (G_{12} - G_{22})s^2 + (G_{13} - G_{33})s + \text{det}(G)}{s(s^2 + G_{22}s + G_{33} + (G_{23} + G_{32}))}
\]

(4)

Using Lemmas 1–3 (see Appendix A), the following theorem is obtained.

**Theorem 2:** A positive-real function \( Y(s) \) is realizable as the driving-point admittance of a one-port network, consisting of one damper, one inverter, and at most three springs, and satisfying Assumption 1, if and only if \( Y(s) \) can be written in the form of (4), where \( G \) as defined in (3) is non-negative definite and satisfies the conditions of Lemma 2 or Lemma 3.

**D. Realization Configurations**

**Theorem 3:** Consider a positive-real function \( Y(s) \) in the form of (4), where \( G \) as defined in (3) is a non-negative definite matrix. If any first-order minor or second-order minor of \( G \) is zero, then \( Y(s) \) can be realized as a series-parallel network consisting of at most three springs, one inverter, and one damper, through the Foster Preamble [2, p. 19].

**Proof:** Case 1: \( G_{4} = 0 \). One obtains

\[
Y(s) = \frac{k_1}{s} + \left( \frac{s}{k_2} + \left( \frac{s}{k_3} + \frac{1}{c} \right)^{-1} \right)^{-1}
\]

where

\[
k_1 = \text{det}(G)/(G_{23}G_{32} - G_{22}^2), \quad k_2 = G_{23}^2/(G_{23}G_{32} - G_{22}^2), \quad k_3 = G_{23}G_{32}^2/(G_{23}G_{32} - G_{22}^2)^2, \quad c = G_{23}^2G_{32}^2/(G_{23}G_{32} - G_{22}^2)^2.
\]

If \( G_{23}G_{32} - G_{22}^2 \neq 0 \), then \( Y(s) \) is realizable as in Fig. 1(a) with \( k_1, k_2, k_3, b, c \geq 0 \). Specifically, if \( G_{23}G_{32} - G_{22}^2 = 0 \), then \( Y(s) = G_1/s \), which is realizable as in Fig. 1(a) with \( k_1 = G_1 \geq 0 \) and \( k_2 = \infty \).

Case 2: \( G_{5} = 0 \). One obtains

\[
Y(s) = \frac{k_1}{s} + \left( \frac{s}{k_2} + c + \left( \frac{s}{k_3} + \frac{1}{c} \right)^{-1} \right)^{-1}
\]

where

\[
k_1 = \text{det}(G)/(G_{23}G_{32} - G_{22}^2), \quad k_2 = G_{23}G_{32}^2/(G_{23}G_{32} - G_{22}^2)^2, \quad k_3 = G_{23}G_{32}/(G_{23}G_{32} - G_{22}^2)^2, \quad c = G_{23}^2G_{32}^2/(G_{23}G_{32} - G_{22}^2)^2.
\]

If \( G_{23}G_{32} - G_{22}^2 \neq 0 \), then \( Y(s) \) is realizable as in Fig. 1(b) with \( k_1, k_2, k_3, b, c \geq 0 \). Specifically, if \( G_{23}G_{32} - G_{22}^2 = 0 \), then \( Y(s) = G_1/s \), which is realizable as in Fig. 1(b) with \( k_1 = G_1 \geq 0 \) and \( k_2 = \infty \).

Case 3: \( G_{6} = 0 \). One obtains

\[
Y(s) = \frac{k_1}{s} + \left( \frac{s}{k_2} + \frac{1}{bs} \right)^{-1} + \left( \frac{s}{k_3} + \frac{1}{c} \right)^{-1}
\]

where

\[
k_1 = \text{det}(G)/(G_{23}G_{32}), \quad k_2 = G_{23}G_{32}^2/(G_{23}G_{32} - G_{22}^2)^2, \quad k_3 = G_{23}G_{32}^2/(G_{23}G_{32} - G_{22}^2)^2, \quad c = G_{23}^2G_{32}^2/(G_{23}G_{32} - G_{22}^2)^2.
\]

If \( G_{23}G_{32} \neq 0 \), then \( Y(s) \) is realizable as in Fig. 1(c) with \( k_1, k_2, k_3, b, c \geq 0 \). Specially, if \( G_{23} = 0 \) and \( G_{32} \neq 0 \), then \( Y(s) \) is realizable as in Fig. 1(d) with \( k_1, k_2, k_3, b, c \geq 0 \).
obtains (4), where the first-order minors and all the second-order minors being non-zero.

If Condition 1 holds, then \( Y(s) \) is realizable as Fig. 2(a), with \( k_1 = G_1 G_2 (G_1 - G_2 G_6/G_6) \), \( k_2 = G_2^2 (G_1 - G_2 G_5/G_6) \), \( k_3 = G_2^2 (G_2 G_6 - G_2 G_5/G_6) \), \( b = G_2^2 (G_2 G_6 - G_2 G_5/G_6) \), and \( c = G_2^2 (G_2 G_6 - G_2 G_5/G_6) \). Then, \( Y(s) \) is realizable by the network as in Fig. 1(c) with \( k_1, k_2, k_3, b \geq 0 \) and \( c \geq 0 \).

If Condition 2 holds, then \( Y(s) \) is realizable as Fig. 2(b), with \( k_1 = G_1 G_2 (G_1 - G_2 G_6/G_6) \), \( k_2 = G_2^2 (G_1 - G_2 G_5/G_6) \), \( k_3 = G_2^2 (G_2 G_6 - G_2 G_5/G_6) \), \( b = G_2^2 (G_2 G_6 - G_2 G_5/G_6) \), and \( c = G_2^2 (G_2 G_6 - G_2 G_5/G_6) \). Then, \( Y(s) \) is realizable by the network as in Fig. 1(c) with \( k_1, k_2, k_3, b \geq 0 \) and \( c \geq 0 \).

If Condition 3 holds, then \( Y(s) \) is realizable as Fig. 2(c), with \( k_1 = G_1 G_2 (G_1 - G_2 G_6/G_6) \), \( k_2 = G_2^2 (G_1 - G_2 G_5/G_6) \), \( k_3 = G_2^2 (G_2 G_6 - G_2 G_5/G_6) \), \( b = G_2^2 (G_2 G_6 - G_2 G_5/G_6) \), and \( c = G_2^2 (G_2 G_6 - G_2 G_5/G_6) \). Then, \( Y(s) \) is realizable by the network as in Fig. 1(c) with \( k_1, k_2, k_3, b \geq 0 \) and \( c \geq 0 \).

If Condition 4 holds, then \( Y(s) \) is realizable as Fig. 2(d), with \( k_1 = G_1 G_2 (G_1 - G_2 G_6/G_6) \), \( k_2 = G_2^2 (G_1 - G_2 G_5/G_6) \), \( k_3 = G_2^2 (G_2 G_6 - G_2 G_5/G_6) \), \( b = G_2^2 (G_2 G_6 - G_2 G_5/G_6) \), and \( c = G_2^2 (G_2 G_6 - G_2 G_5/G_6) \).

Proof: Condition 1: Since \( G_1 G_2 G_6 < 0 \), \( \text{det}(G) = 0 \), and \( G \) with first-order minors and second-order minors being non-zero is non-negative definite, one implies that \( k_1, k_2, k_3, b \geq 0 \), and the admittance of the network in Fig. 1(a) is equivalent to (4).

Condition 2: Since \( G_1 G_2 G_6 < 0 \), \( G_1 G_2 G_6 + G_1 G_2 G_6 - G_1 G_2 G_6 < 0 \), and \( G \) with first- and second-order minors being non-zero is non-negative definite, one implies that \( G_1 - G_2 G_6/G_6 > 0 \), \( G_2 - G_2 G_6/G_6 < 0 \), and \( G_3 - G_2 G_6/G_6 > 0 \) by the discussions in the proof of Lemma 3. Hence, it is implied that \( k_1, k_2, k_3, b \geq 0 \), and the admittance of the network in Fig. 1(b) is equivalent to (4).

Condition 3: Since \( G_1 G_2 G_6 < 0 \), \( G_1 G_2 G_6 + G_1 G_2 G_6 - G_1 G_2 G_6 < 0 \), and \( G \) with first- and second-order minors being non-zero is non-negative definite, one implies that \( G_1 - G_2 G_6/G_6 > 0 \), \( G_2 - G_2 G_6/G_6 < 0 \), \( G_3 - G_2 G_6/G_6 > 0 \) by the discussions in the proof of Lemma 3. Hence, it is implied that \( k_1, k_2, k_3, b \geq 0 \), and the admittance of the network in Fig. 1(c) is equivalent to (4).

Condition 4: Since \( G_1 G_2 G_6 < 0 \), \( G_1 G_2 G_6 + G_1 G_2 G_6 - G_1 G_2 G_6 < 0 \), and \( G \) with first- and second-order minors being non-zero is non-negative definite, one implies that \( G_1 - G_2 G_6/G_6 > 0 \), \( G_2 - G_2 G_6/G_6 < 0 \), \( G_3 - G_2 G_6/G_6 > 0 \) by the discussions in the proof of Lemma 3. Hence, it is implied that \( k_1, k_2, k_3, b \geq 0 \), and the admittance of the network in Fig. 1(d) is equivalent to (4).
where five conditions holds: 1) $\alpha$, $\alpha_1$, $\alpha_2$, $\alpha_3$, $\beta_1$, $\beta_2$, $\beta_3 \geq 0$, $W_1$, $W_2$, $W_3 \geq 0$, $W^2 = 4W_1W_2W_3$, and also satisfy either 1) at least one of $\alpha$, $\alpha_2$, $\alpha_3$, $\beta_1$, $\beta_2$, $\beta_3$, $W_1$, $W_2$, $W_3$, $(\beta_2 - W/(2W_1))$, $(\beta_3 - W/(2W_2))$, and $(\alpha_3 - W/(2W_3))$ is zero; or 2) one of the following holds with Condition 1 not being satisfied: a) $W < 0$ and $\alpha_0 = 0$; b) $W > 0$ and $\alpha_0 + \alpha_3\beta_1 + \alpha_2\beta_2 - \alpha_1\beta_3 = 0$; c) $W > 0$ and $\alpha_0 + \alpha_3\beta_1 + \alpha_1\beta_3 - \alpha_2\beta_2 = 0$; d) $W > 0$ and $\alpha_0 + \alpha_3\beta_1 + \alpha_2\beta_2 - \alpha_1\beta_3 = 0$.

**Proof:** See [10] for a detailed proof.

Figs. 1 and 2(a)-(d) are configurations achieving Conditions 1 and 2a-2d of Theorem 5, respectively, whose expressions of element values can be obtained from those in terms of $G_1$ to $G_6$ through (5).

### III. Final Realizability Results

To complete the present study, one considers the case when Assumption 1 does not hold.

**Theorem 6:** A positive-real function $Y(s)$ can be realized as the driving-point admittance of a one-port network, consisting of one damper, one inverter, and at most three springs, and not satisfying Assumption 1, if and only if $Y(s)$ can be written in the form of

$$Y(s) = \frac{\alpha_3s^3 + \alpha_2s^2 + \alpha_1s + \alpha_0}{\beta_3s^3 + \beta_2s^2 + \beta_1s}$$

where $\alpha_0$, $\alpha_1$, $\alpha_2$, $\alpha_3$, $\beta_1$, $\beta_2$, $\beta_3 \geq 0$, and one of the following five conditions holds: 1) $\alpha_3 = \beta_3 = \beta_0 = 0$, $\alpha_1$, $\alpha_2$, $\beta_1$, $\beta_2 > 0$; 2) $\alpha_3 > 0$, $\beta_2 > 0$, $\alpha_1\beta_3 - \alpha_0\beta_2 = \alpha_3\beta_1$, $\alpha_2\beta_2 \geq 0$, $\alpha_0\beta_2 + \beta_2\beta_1 \geq 1$, $\alpha_1\beta_3 = \alpha_3\beta_1$; 3) $\beta_2 = 0$, $\alpha_3$, $\beta_1 > 0$, $\alpha_1\beta_3 - \alpha_0\beta_2 \geq 0$, $\alpha_2\beta_2 \geq 0$, $\alpha_0\beta_2 + \beta_2\beta_1 \geq 1$, $\alpha_1\beta_3 = \alpha_3\beta_1$; 4) $\beta_3 = 0$, $\beta_1 > 0$, $\alpha_0\beta_2 - \alpha_1\beta_3 \geq 0$, $\alpha_2\beta_2 \geq 0$, $\alpha_0\beta_2 + \beta_2\beta_1 \geq 1$, $\alpha_1\beta_3 = \alpha_3\beta_1$; 5) $\alpha_3$, $\beta_1 > 0$, $\alpha_1\beta_3 - \alpha_0\beta_2 \geq 0$, $\alpha_2\beta_2 \geq 0$, $\alpha_0\beta_2 + \beta_2\beta_1 \geq 1$, $\alpha_1\beta_3 = \alpha_3\beta_1$,

Furthermore, networks in Fig. 3 with $b$, $c > 0$ and $k_1$, $k_2$, $k_3 \geq 0$ can realize each of the five conditions above, respectively.

**Proof:** See [10] for a detailed proof.

Now, the final integrated conclusion is stated as follows.

**Theorem 7:** A positive-real function $Y(s)$ can be realized as the driving-point admittance of a one-port network $Q$ consisting of one inverter, one damper, and at most three springs, if and only if $Y(s)$ can be written in the form of

$$Y(s) = \frac{\alpha_3s^3 + \alpha_2s^2 + \alpha_1s + \alpha_0}{\beta_3s^3 + \beta_2s^2 + \beta_1s}$$

where $\alpha_0$, $\alpha_1$, $\alpha_2$, $\alpha_3$, $\beta_1$, $\beta_2$, $\beta_3 \geq 0$, which satisfies the conditions of Theorem 5 when $\beta_3 = 1$, or the conditions of Theorem 6 when $\beta_3 = 0$. Moreover, $Y(s)$ is realizable as one of configurations in Fig. 1, 2, or 3.

**Proof:** It is proved by combining Theorems 5 and 6.

**Remark 1:** A necessary and sufficient condition for the realizability of any positive-real function as the admittance of a one-port network with one inverter, one damper, and a finite number of springs in terms of (6) is given in [10], from which one can assert that the number of springs may not always be possible to reduce to three. Since there are five independent parameters in the networks of this note (corresponding to five elements), there must exist an equality constraint for the conditions in terms of $G_1$ to $G_6$, which obviously cannot be equivalent to the conditions for networks without limiting the number of springs (involving only inequality constraints). After converting the coefficients, the corresponding conditions in terms of (6) for the realizability of the two classes of networks are consequently not equivalent.

**Remark 2:** The direct enumeration method can be used as an alternative way to obtain the results of this note.

**Remark 3:** Through simulation, it can be shown that both the comfort performance and the dynamic tyre loads performance for a quarter-car system (see [3], [22]) are not substantially degraded by reducing the number of springs from four to three.

### IV. Numerical Examples

**Example 1:** Given a positive-real function $Y(s) = (45s^3 + 99s^2 + 54s + 22)/(9s^4 + 36s^3 + 27s^2 + 107)$, one can check that the conditions of Theorem 7 (Condition 2d of Theorem 5) holds. Therefore, $Y(s)$ is realizable as shown in Fig. 2(d) with $b = 1$, $c = 9/10$, $k_1 = 2$, $k_2 = 1/4$, and $k_3 = 11/4$.

**Example 2:** Given a positive-real function $Y(s) = (2s^2 + 4s + 1)/(s^3 + 4s^2 + s)$, one can check that the conditions of Theorem 7 (Condition 1 of Theorem 5) holds. Therefore, $Y(s)$ is realizable as shown in Fig. 2(e) with $b = 4$, $c = 1$, $k_1 = 2$, $k_2 = 2$, and $k_3 = 0$ (two springs).

**Example 3:** Given a positive-real function $Y(s) = (20s^3 + 44s^2 + 24s + 19)/(4s^4 + 16s^3 + 12s^2 + 47s)$, one can check that the conditions of Theorem 7 do not hold. Therefore, $Y(s)$ cannot be realized with one damper, one inverter, and at most three springs. However, it can be realized by the series connection of a spring $k_4$ with Fig. 2(d), where $b = 1$, $c = 9/16$, $k_1 = 15/8$, $k_2 = 3/8$, $k_3 = 21/8$, and $k_4 = 1/8$.

**Example 4:** For the suspension system [17, Fig. 3], $K_{ag}(s)$ is presented in [17, Eq. (27)], which can guarantee the same value of $J_3$ (see [17, Eq. (5)]) as the optimization function $K(s)$ of $J_3$ when $k_s = 50$ kN/m (intermediate static stiffness range) by the YALMIP method. As a result, the admittance of the suspension strut becomes $Y(s) = 50000/s + K_{ag}(s)$ ($289010000s^2 + 50555000s + 4876200000$)/(s$^3 + 1011s^2 + 97524s$). From [17, Fig. 12], it is known that $Y(s)$ is realizable with one damper, one inverter, and at most three springs. One can check that the conditions of Theorem 7...
(Condition 1) holds, which further illustrates the validity and benefits of the results obtained in this note.

V. CONCLUSION

This note has studied the realization problem of one-port networks containing one inerter, one damper, and at most three springs. The first main contribution of this note is the derivation of a necessary and sufficient condition for a real symmetric matrix to be realizable as the admittance of a three-port resistive network containing at most three elements. Corresponding results for one-element-type mechanical networks then follow from the force-current analogy. The second main contribution is to obtain a necessary and sufficient condition for the realizability of any positive-real function as the admittance of a one-port mechanical network consisting of one damper, one inerter, and at most three springs (Theorem 7) as well as the network configurations to cover the condition (Figs. 1–3). The element extraction approach was utilized, and the two cases when the admittance of the resulting three-port spring network is well-defined and when the admittance is non-well-defined were discussed, respectively. Numerical examples were provided for illustration. Further research endeavor will be directed to laboratory implementation of the new networks and their potential practical applications.

APPENDIX A

PREVIOUS LEMMAS OF THEOREM 2

Lemma 1: A positive-real function $Y(s)$ is realizable as the driving-point admittance of a one-port network, consisting of one damper, one inerter, and at most three springs, and satisfying Assumption 1, if and only if it can be written in the form of (4), where $G$ as defined in (3) is non-negative definite, and there exists an invertible diagonal matrix $D = \text{diag}\{x, y\}$ with $x, y > 0$ such that $DGD$ satisfies the conditions of Theorem 1.

Proof: Necessity. Let $K$ be the admittance of the three-port resistive network obtained by extracting one damper and one inerter. Let $x = \sqrt{c}, y = \sqrt{b}$, and let $D = \text{diag}\{a, b, c\}$, where $a, b, c > 0$. Then, $Y(s)$ can be in the form of (4), where $K = DGD$ as in (3) satisfies the conditions of Theorem 1.

Sufficiency. Since $DGD$ satisfies the conditions of Theorem 1, $(1/s)DGD$ is realizable as a three-port network consisting of at most three springs. Consequently, $Y(s)$ is the admittance of the network in [6, Fig. 5], where $c = x^2, b = y^2$, and the admittance of $X$ is $(1/s)DGD$.

Lemma 2: Consider a non-negative definite matrix $G$ as defined in (3). If any first-order minor or second-order minor of $G$ is zero, then there must exist an invertible diagonal matrix $D = \text{diag}\{x, y\}$ with $x, y > 0$ such that $DGD$ satisfies the conditions of Theorem 1.

Proof: Case 1: At least one of the first-order minors is zero. It will be shown that there exists $D = \text{diag}\{x, y\}$ with $x, y > 0$ such that $DGD$ satisfies Condition 1 of Theorem 1. If $G_4 = 0$ and other entries are nonzero, then Condition 1 of Theorem 1 is equivalent to $G_1 - y[G_5] \geq 0, x^2G_2 - xyG_6 \geq 0$, and $y^2G_3 - y[G_5] - xyG_6 \geq 0$ with $x, y > 0$ and at least two of the inequality signs being equality signs. If one chooses $x = G_1[|G_6|/|G_2G_3|]$ and $y = G_1[|G_5|]$, then the first and second inequality signs become equality signs, and the third item always holds because of $G_1G_6 - G_4G_5 = 0$, then $G_1 = G_4G_2/G_6$, implying $G_4G_2G_3 > 0$ and $G_1/|G_4| = |G_5|/|G_6|$. Condition 2a of Theorem 1 becomes

$$
\frac{|G_5|}{|G_6|} \leq x \leq \frac{G_1}{|G_4|}, \quad \frac{|G_6|}{G_3} \leq y \leq \frac{|G_4|}{G_5},
$$

(7)

$$
(x|G_2| - |G_4|) \left(\frac{y}{x}\right) \leq xG_2 - |G_4|,
$$

(8)

with at least three inequality signs being equality signs. Letting $x = G_1/|G_4| = |G_5|/|G_6|$, the first and second inequality signs of (7) are both equality signs, which implies that (8) holds because of $G_1G_2 - G_4G_5 \geq 0$. Since $G_3 - G_6G_4/G_2 = G_3 - G_6G_2/G_1 = G_3 - G_6G_2/G_1 \geq 0$, one can choose some $y > 0$ such that the second item of (7) holds with one equality sign. Similarly, the subsuces of $G_4G_2 - G_4G_5 = 0$ and that of $G_3G_4 - G_3G_5 = 0$ can be proved. It has been shown in [2, pg. 46] that the following expressions hold:

$$
(G_2G_3 - G_4G_5)(G_4G_2 - G_4G_5) - (G_1G_4 - G_3G_6)^2 = G_3G_4 - G_3G_5 - G_2G_4 - G_2G_5 = 0,
$$

and $G_4G_2 - G_4G_5 = 0$. Then, one can obtain

$$
G_1|G_5| + G_1|G_6| + G_2|G_3| + G_4|G_6| = \frac{G_1G_2 - G_4^2}{|G_4||G_6| + G_2|G_3|},
$$

(9)

and det$(G) = 0$, which implies Condition 1 of this lemma.

Case 2: $G_4G_2G_3 > 0$. Condition 2a of Theorem 1 becomes (7) and (8) with each of the three having one and only one equality sign, implying $G_1 > G_4G_2/G_3$ and $G_3 > G_3G_6/G_4$. Since $G_2 \neq G_4G_2G_3/G_6$, it is only possible that $|G_3||G_6| \leq |x| = |G_1|/|G_4|$ and $|G_1|/|G_6| = |x|/|G_4|$, (8) with the equality sign gives $G_1G_6 - G_4G_5 - G_2G_4 - G_2G_5 = 0$, which implies Condition 2 of this lemma. Similarly, one can prove that Condition 2b of Theorem 1 implies Condition 3 of this lemma, and Condition 2c of Theorem 1 implies Condition 4 of this lemma.

Sufficiency. If Condition 1 of this lemma holds, let $x, y > 0$ satisfy (9). Then, it can be verified that Condition 1 of Theorem 1 holds following the proof of the necessity part.

In [6], it is shown that the non-negative definiteness of $G$ and $G_4G_2G_3 > 0$ implies that at most one of $G_1 - G_4G_2G_3/G_6 > 0$ is negative. If Condition 2 of this lemma holds, let $x = G_1/|G_4|$, and $y = |G_4|/|G_3|$. Then, one can obtain $\text{det}(G) = G_1G_6(G_4G_2G_3 - G_4G_5)$, implying $G_1 > G_4G_2G_3/G_6$, $G_2 < G_4G_2G_3/G_6$, and $G_3 > G_4G_2G_3/G_6$. Then, (7) holds with each of the two items having one equality sign. Since $G_2G_4G_6 - G_1G_6 - G_3G_4 = 0$, (8) holds with equality sign. Hence, $D = \text{diag}\{x, y\}$ with $x, y > 0$ exists such that $DGD$ satisfies Condition 2 of Theorem 1. Similarly, one can also prove that Condition 2 of Theorem 1 holds if Condition 3 or Condition 4 of this lemma holds. Since Condition 3 implies $G_1 > G_4G_2G_3/G_6$, $G_2 > G_4G_2G_3/G_6$, and Condition 4 implies $G_1 < G_4G_2G_3/G_6$, $G_2 > G_4G_2G_3/G_6$, and $G_3 > G_4G_2G_3/G_6$, the four conditions given in the lemma have no overlap.
REFERENCES