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Synchronization of Dynamical Networks with Distributed Event-Based Communication

Tao Liu, David J. Hill and Bin Liu

Abstract—In this paper, we study synchronization of a dynamical network whose nodes are linear time-invariant systems and are interconnected through a shared communication network. Firstly, synchronization of a dynamical network with physical links and undirected topology is reinvestigated from a set stability point of view. An explicit Lyapunov function with respect to its synchronization manifold is constructed for such a network by using properties of undirected networks. Based on this Lyapunov function, a distributed event-triggered sampling scheme is designed which decides when a node should send its sampled state to its neighbors across the communication network in order to achieve asymptotic synchronization of a dynamical network with communication links. The proposed triggering rule only depends on the state of the node itself and the sampled ones that are received from its neighbors.

I. INTRODUCTION

Synchronization of dynamical networks has received a great deal of attention due to its extensive applications in various fields – see [1] and references therein. Problems like synchronization of networks with different topologies [2], [3], [4] as well as relationships between synchronizability and topological properties [5] have been extensively studied. All of these have advanced the understanding of the synchronization phenomena in real-world networks.

Roughly speaking, networks in the real world can be classified into two categories based on the way how their nodes are interconnected, i.e., networks with physical links and networks with communication links. For the former type, nodes in a network are connected by physical links where they share information with their neighbors continuously in time, e.g., generators in a power system are interconnected by transmission lines [6]. For the latter type, nodes communicate with each other over a shared communication media such as over a wireless digital network where data are transmitted at discrete time instants (see [7] for an example).

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In the literature, most existing work focused on networks with physical links, and remain problems of networks with communication links open. Apparently, methods and results for networks with physical links cannot apply to networks with communication links directly due to the discontinuous communications between nodes. One fundamental problem for such networks is when and how frequently nodes should communicate with each other to ensure a desired level of performance of a given network, in particular, when the corresponding communication network has a limited bandwidth.

In order to use the limited network bandwidth effectively, event-triggered control has been proposed for networked control systems, where the plant samples its outputs based on the occurrence of an event which is generated by some well-designed event-triggering conditions (see [8], [9] and references therein). With this control mechanism, a system can adaptively adjust its sample rates according to what is currently happening within the system. Therefore, unnecessary communications might be avoided. Along this line, centralized as well as decentralized event-triggered control methods were proposed in a framework of input-to-state stability of nonlinear systems in [8] and [10]; output-based decentralized event-triggered control was studied for linear systems via stability theory of impulsive systems in [9]; and distributed event-triggered control for interconnected subsystems was developed in [11] and [12] with the help of small gain theory of large-scale systems.

More recently, the problem of consensus of multi-agent systems, which is closely related to the topic of synchronization of dynamical networks, has also been investigated under the circumstance that agents communicate with each other over a communication network. A simply decentralized event-triggered control method was exploited for a multi-agent system in [12] under which practical (bounded) consensus of the system was obtained. In [7], asymptotic consensus was achieved by a distributed event-triggered control mechanism which was further extended to guarantee $L_2$ gain stability of the system with respect to additive disturbances in [13]. However, these works only focused on networks with very simple node dynamics (first order integrators), and the obtained results do not appear to extend to networks with more general node dynamics. In this paper, we will consider the synchronization problem of a network with general linear dynamics and communication links. In particular, we will focus on the problem of how to design a distributed event-triggered sampling rule for each node under which synchronization of a given network can be achieved. Note that the synchronization problem of a network with
physical links was usually investigated within a framework of stability analysis of an error dynamical system where an additional signal—a synchronous state of the network is needed. This signal, which is also required to be exactly the same for each node, is often not available for a design purpose. Therefore, in order to give a more practical event-triggering rule for a network with communication links, we will first study synchronization of a corresponding network with physical links from a set stability point of view. This will allow us to avoid the usage of such a signal during the design procedure. Then, based on the proposed results, event-triggering mechanisms will be explored to achieve synchronization of the network with communication links.

The rest of the paper is organized as follows. In Section II, network models to be considered are given, and some concepts of set stability of nonlinear systems are briefly reviewed. Section III gives the main results of the paper, where an explicit Lyapunov function for a network with linear dynamics and physical links is constructed with respect to its synchronization manifold. This is followed by the design of a distributed event-triggering mechanism for a related network with communication links. Examples are provided in Section IV to show the effectiveness of the proposed results. Some conclusions are given in Section V.

II. NETWORK MODEL AND PRELIMINARIES

Consider a dynamical network which consists of $N$ linearly and diffusively coupled identical nodes with communication links. Each node is a linear time-invariant system. The state equations of the entire network are given below

$$
\dot{x}_i(t) = Hx_i(t) + c \sum_{j=1}^{N} a_{ij} \Gamma x_j(t), \quad t \in [t_{k_i}^i, t_{k_i+1}^i), \quad i = 1, 2, \ldots, N, \tag{1}
$$

where $x_i = (x_{i1}, x_{i2}, \ldots, x_{in})^\top \in \mathbb{R}^n$ is the state variable of node $i$; $H \in \mathbb{R}^{n \times n}$ is a constant matrix representing the identical node dynamics; $\Gamma \in \mathbb{R}^{n \times n}$ is the inner coupling matrix; $c > 0$ is the coupling strength; $A \in \mathbb{R}^{N \times N}$ is the outer coupling matrix representing topological structure of the network. In this paper, we are interesting in undirected networks, i.e., if there is a connection between nodes $i$ and $j$ ($i \neq j$), then $a_{ij} = a_{ji} = 1$; otherwise $a_{ij} = a_{ji} = 0$, and the diagonal elements of $A$ satisfy

$$
a_{ii} = -\sum_{j=1}^{N} a_{ij} = -\sum_{j=1, j \neq i}^{N} a_{ji}, \quad i = 1, 2, \ldots, N. \tag{2}
$$

For each $i$, $\{t_{k_i}^i\}$ is a time sequence with $t_{k_i+1}^i > t_{k_i}^i$, $k_i \in \mathbb{Z}$ and $t_{k_i}^i$ representing the $k_i$th sample time instant and also the time when node $i$ sends its sampled value $x_i(t_{k_i}^i)$ to its neighbors. $N_i = \{j \in \{1, 2, \ldots, N\} \mid a_{ij} > 0\}$ denotes the index set of neighbors of node $i$. For each $t \in [t_{k_i}^i, t_{k_i+1}^i)$,

$$
\dot{x}_j(t) = x_j(t_{k_i}^i(t)), \quad \text{and} \quad t_{k_i}^i(t) \text{ is the last sample time of node } j \in N_i, i.e.,
$$

$$
k_i^j(t) = \arg \min_{t \in \mathbb{Z}, t \geq t_{k_i}^i} \{t - t_{k_i}^i\}
$$

with $\mathbb{Z}$ being the set of non-negative integers.

We suppose that neighbors can receive information from each other and update their states simultaneously, i.e. there is no time delay for a signal traveling through the communication network. We also assume that there are no data dropouts. Therefore, for each node $i$ in the network (1), its state depends on the lastest sampled state values of its neighbors, and its interconnection term will change its values at its own sample times $t_{k_i}^i$, $t_{k_i+1}^i$, ..., as well as at its neighbors’ sample times $t_{k_i}^j$, $t_{k_i+1}^j$, ..., $j \in N_i$.

We assume that $A$ is irreducible, which means the network is connected. Thus we have

$$
0 = \lambda_1(A) > \lambda_2(A) \geq \lambda_3(A) \geq \cdots \geq \lambda_N(A), \tag{3}
$$

where $\lambda_i(\cdot)$ represents the $i$th real eigenvalues of a symmetric matrix, and we will use $\lambda_i$ for simplicity when no confusion occurs.

Let $x(t;x_0) = (x_1(t;x_0)^\top, x_2(t;x_0)^\top, \ldots, x_N(t;x_0)^\top)^\top \in \mathbb{R}^{nN}$ be a solution of the network (1) with initial condition $x_0 = (x_1(t_0)^\top, x_2(t_0)^\top, \ldots, x_N(t_0)^\top)^\top$. We assume that $x(t;x_0)$ exists uniquely for all $t > t_0$.

Definition 1: Let

$$
\mathcal{A}_s = \{x \in \mathbb{R}^{nN} \mid x_1 = x_2 = \cdots = x_N\} \tag{4}
$$

with $x = (x_1^\top, x_2^\top, \ldots, x_N^\top)^\top$. If there exists a $\delta > 0$ such that

$$
\lim_{t \to \infty} |x(t;x_0)|_{\mathcal{A}_s} = 0 \tag{5}
$$

whenever $|x_0|_{\mathcal{A}_s} < \delta$, then the network (1) is said to achieve local asymptotic synchronization. Moreover, if $\delta = \infty$, then global asymptotic synchronization is achieved.

The set $\mathcal{A}_s$ is called the synchronization manifold of the network (1), and $|x|_{\mathcal{A}_s}$ denotes the Euclidean point-to-set distance, namely

$$
|x|_{\mathcal{A}_s} = d(x, \mathcal{A}_s) = \inf_{y \in \mathcal{A}_s} \|x - y\| \tag{6}
$$

with $\| \cdot \|$ representing the Euclidean norm of a vector.

Obviously, synchronization of the network (1) depends on the time instants that each node in the network samples and sends its state to its neighbors. Moreover, as each node can only detect its own state and receive sampled states of its neighbors at certain time instants, it is more reasonable and practical to design an event-triggering rule for a node which only depends on local information that it can obtain. More precisely, if a local error signal of a node exceeds a given threshold (we say an event occurs), which can be detected by hardware detectors, then the node will sample its state and send the sampled value to its neighbors over a communication network. Then the purpose of this paper is to design a proper distributed event-triggered sampling mechanism, i.e., time sequence $\{t_{k_i}^i\}$ for each node $i$ in order to achieve synchronization.

Before discussing synchronization of the network (1), let us consider the following corresponding network with
physical links.

$$\dot{x}_i(t) = H x_i(t) + c \sum_{j=1}^{N} a_{ij} \Gamma x_j(t), \ i = 1, 2, \ldots, N. \ (7)$$

Generally, the study of synchronization of the network (7) was carried out by investigating stability of the equilibrium point of an error dynamical system with a state variable $$\eta_i = x_i - s(t)$$, where $$s(t)$$ is usually chosen as an average state of all nodes [14] or a solution of an isolated node [15]. However, to get exact information of such an $$s(t)$$ is not trivial in practice, and sometimes even impossible, especially when using it for control purpose or to make a decision for each individual node. Therefore, these results cannot extend to the design of an event-triggering rule of the network (1). To overcome such an issue, we will first study synchronization of the network (7) from set stability point of view.

Now, we will briefly review some concepts and properties of stability of a nonlinear system with respect to a closed invariant set, and then give a lemma which infers synchronization of the network (7) from stability of its synchronization manifold $$\mathcal{A}_s$$. This lemma will provide theoretical foundation for the design problem of the paper.

Consider a nonlinear system:

$$\dot{\xi}(t) = g(\xi(t)), \ (8)$$

where $$\xi \in \mathbb{R}^n$$ is the state variable, and $$g : \mathbb{R}^n \to \mathbb{R}^n$$ is continuous and locally Lipschitz on $$\xi$$. We assume that the system has unique solutions for each initial condition $$\xi_0 = \xi(t_0) \in \mathbb{D} \subseteq \mathbb{R}^n$$ and all $$t \geq t_0$$, and denote its solution at time $$t$$ with $$\xi(t)$$ by $$\xi(t; \xi_0)$$.

**Definition 2 ([16]):** Let $$\mathcal{A} \subseteq \mathbb{R}^n$$ be a nonempty, closed set with

$$\sup_{\xi \in \mathbb{R}^n} \{||\xi||_{\mathcal{A}}\} = \infty. \ (9)$$

Then it is called an invariant set for the system (8), if

$$\xi(t; \xi_0) \in \mathcal{A}, \ \text{for all } \xi_0 \in \mathcal{A}, \ \text{and all } t > t_0. \ (10)$$

**Definition 3 ([16]):** The system (8) is said to be **globally asymptotically stable** with respect to a nonempty, closed, invariant set $$\mathcal{A}$$ if the following two properties hold:

i). **Stability.** There exists a $$\mathcal{K}_\infty$$-function$$^1$$ $$\delta(\cdot)$$ such that for any $$\epsilon > 0$$,

$$||\xi(t; \xi_0)||_{\mathcal{A}} < \epsilon, \ \text{whenever } ||\xi_0||_{\mathcal{A}} < \delta(\epsilon) \text{ and } t \geq t_0. \ (11)$$

ii). **Attraction.** For any $$r, \ \epsilon > 0$$, there is a $$T > 0$$, such that

$$||\xi(t; \xi_0)||_{\mathcal{A}} < \epsilon \ (12)$$

whenever $$||x_0||_{\mathcal{A}} < r$$ and $$t \geq T$$.

**Definition 4 ([16]):** A Lyapunov function for the system (8) with respect to a nonempty, closed, invariant set $$\mathcal{A} \in \mathbb{R}^n$$ is a function $$V : \mathbb{R}^n \to \mathbb{R}$$ such that $$V$$ is smooth on $$\mathbb{R}^n/\mathcal{A}$$ and satisfies

i). there exists two $$\mathcal{K}_\infty$$-functions $$\alpha_1$$ and $$\alpha_2$$ such that for any $$\xi \in \mathbb{R}^n$$,

$$\alpha_1(||\xi||_{\mathcal{A}}) \leq V(\xi) \leq \alpha_2(||\xi||_{\mathcal{A}}); \ (13)$$

ii). there exists a $$\mathcal{K}$$-function $$\alpha_3$$ such that for any $$\xi \in \mathbb{R}^n/\mathcal{A},$$

$$\frac{\partial V}{\partial \xi} f(\xi) \leq -\alpha_3(||\xi||_{\mathcal{A}}). \ (14)$$

**Lemma 1 ([16]):** The system (8) is globally asymptotically stable with respect to $$\mathcal{A}$$ if and only if there exists a Lyapunov function $$V$$ with respect to the set $$\mathcal{A}$$.

**Remark 1:** When $$\mathcal{A}$$ only contains an equilibrium point of the system (8), then the Lyapunov function defined in Definition 4 coincides with the one for an equilibrium point, and Lemma 1 reduces to conditions under which global asymptotic stability of an equilibrium point of the system can be guaranteed (see [17] for an example).

Note that diffusively coupling terms as well as identical node dynamics of the network (7) guarantee that the state $$x(t; x_0)$$ of the network (7) will remain on the synchronization manifold $$\mathcal{A}_s$$ if it starts from $$\mathcal{A}_s$$, i.e., for all $$x_0 \in \mathcal{A}_s$$, one has $$x(t; x_0) \in \mathcal{A}_s, \ \forall t > t_0$$. Therefore, the synchronization manifold $$\mathcal{A}_s$$ is an invariant set of the network (7). This leads to the following lemma.

**Lemma 2:** The network (7) is globally asymptotically synchronized if there exists a Lyapunov function satisfied conditions in Definition 4 with respect to the synchronization manifold $$\mathcal{A}_s$$.

**Proof:** It is straight forward from the Definition 1, 3 and Lemma 1. \nobreak \hfill \blacksquare

**Remark 2:** Apparently, Lemma 2 and the results proposed subsequently for the network (7) are equivalent to their counterparts by investigating stability of the error dynamical system. The advantage of the obtained results in this paper is that they do not need the information of $$s(t)$$ and therefore can apply to the design of a distributed event-triggering rule for the network (1) directly.

III. MAIN RESULTS

In this section, we will construct a Lyapunov function with respect to $$\mathcal{A}_s$$ for the network (7). This will be followed by a distributed event-triggering mechanism design method for the network (1) by using such a Lyapunov function.

A. A Lyapunov Function for Synchronization Manifold

For the network (7), Theorem 1 gives an explicit Lyapunov function with respect to its synchronization manifold $$\mathcal{A}_s$$.

**Theorem 1:** If there exist a positive definite matrix $$P \in \mathbb{R}^{n(N-1) \times n(N-1)}$$ and a constant $$\alpha > 0$$ such that

$$\bar{H}^T \bar{P} + \bar{P} \bar{H} \leq -\alpha \bar{\Phi} \bar{\Phi}^T \ (15)$$

with $$\bar{H} = I_N \otimes H + cA \otimes \Gamma$$ and $$\bar{P} = \bar{\Phi} P \bar{\Phi}^T$$, then

$$V(x) = x^T \bar{P} x. \ (16)$$
is a Lyapunov function for the network (7) with respect to its synchronization manifold $A_s$, and the network is asymptotically synchronized.

Here, $I_N$ is an $N \times N$ identity matrix. $\otimes$ denotes the Kronecker product of two matrices. $\Phi = \Phi \otimes I_n$. $\Phi = (\phi_1, \phi_2, \ldots, \phi_N) \in \mathbb{R}^{N \times N-1}$ with $\phi_i = (\phi_{i1}, \phi_{i2}, \ldots, \phi_{iN})^T \in \mathbb{R}^N$ being the orthonormal eigenvector of $A$ corresponding to its nonzero eigenvalue $\lambda_i$, $i = 2, 3, \ldots, N$ and satisfying $\sum_{j=1}^{N} \phi_{ij} = 0$.

**Proof:** Since $A$ is irreducible, symmetric, and has the zero row sum property (2), there always exists a unitary matrix $\Psi = (\psi_1, \psi_2, \ldots, \psi_N) \in \mathbb{R}^{N \times N}$ with $\psi_i = (\psi_{i1}, \psi_{i2}, \ldots, \psi_{iN})^T \in \mathbb{R}^N$ such that

$$\Psi^T A \Psi = \Lambda = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_N).$$

Furthermore, we can choose $\psi_i = \frac{1}{\sqrt{N}}(1, 1, \ldots, 1)^T$ which corresponds to $\lambda_1 = 0$. This leads to

$$\sum_{i=1}^{N} \psi_{ij} = 0 \text{ for all } i = 2, 3, \ldots, N.$$

Let $\phi_i = \psi_i$, $i = 2, 3, \ldots, N$. Then by the definition of $A_s$ in (4), we have

$$\|\Phi^T x\|^2 = x^T \Phi^T x = x^T (U \otimes I_n)x = x^T (U \otimes I_n)(U \otimes I_n)x = \sum_{i=1}^{N} \|x_i - \bar{x}\|^2 = |x|^2_{A_s},$$

where $\bar{x} = \frac{1}{N} \sum_{i=1}^{N} x_i$. The third equality in (17) is followed by the fact $\Phi^T \Phi = I_{N-1}$ which leads to $U^2 = U$ with

$$U = \Phi \Phi^T = \frac{1}{N} \begin{pmatrix} N - 1 & -1 & \cdots & -1 \\ -1 & N - 1 & \cdots & -1 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & \cdots & N - 1 \end{pmatrix}$$

Thus, (17) together with (15) and (16) guarantees (13) and (14) in Definition 4 being satisfied, where $\alpha_1(|x|_{A_s}) = \lambda_{\min}(P)|x|_{A_s}^2$, $\alpha_2(|x|_{A_s}) = \lambda_{\max}(P)|x|_{A_s}^2$, and $\alpha_3(|x|_{A_s}) = \alpha_i|_{A_s}$ with $\lambda_{\min}(\cdot)$ and $\lambda_{\max}(\cdot)$ being the minimum and maximum eigenvalues of a symmetric matrix. These prove that (16) is a Lyapunov function for the network (7) with respect to $A_s$. Moreover, the network is globally asymptotically synchronized based on Lemma 2.

In Theorem 1, a positive definite matrix $\bar{P}$ can be obtained by solving the linear matrix inequality (LMI) (15). But this condition might be hard to check for a network with a large number of nodes due to its high dimension. Hence, we give an alternative method to construct such a Lyapunov function by solving $N - 1$ lower dimensional LMIs, which is stated in Theorem 2.

**Theorem 2:** If there exist positive matrices $P_i \in \mathbb{R}^{n \times n}$ such that

$$H_i^T P_i + P_i H_i < 0, \ i = 2, 3, \ldots, N$$

with $H_i = H + \lambda_i I$, then

$$V(x) = x^T \bar{P} \Phi \Phi^T x$$

is a Lyapunov function for the network (7) with $P = \text{diag}(P_2, P_3, \ldots, P_N)$ and $\lambda_i$, $\Phi$ being defined the same as in Theorem 1.

**Proof:** If inequalities in (19) are satisfied, then selecting a Lyapunov function candidate as (20) gives

$$\dot{V} = x^T P_x + x^T \bar{P} \dot{x}$$

$$= 2x^T \bar{P} \Phi \Phi^T \dot{x}$$

$$= 2x^T \bar{P} \Phi \Phi^T \dot{H}(I_n \otimes I_n - \Phi \Phi^T + \Phi \Phi^T) x$$

$$= 2x^T \bar{P} \Phi \Phi^T \dot{H} \Phi \Phi^T x + 2x^T \Phi \Phi^T \dot{H}(I_n \otimes I_n - \Phi \Phi^T) x.$$

Since $\Phi \Phi^T = U$, $UA = AU$ and $\Phi \Phi^T 1_{N \times N} = 0_{(N-1) \times N}$, where $1_{m \times N}$ and $0_{m \times N}$ are $m \times n$ matrices with all elements being 1 and 0, respectively, we have

$$\Phi \Phi^T H(I_n \otimes I_n - \Phi \Phi^T)$$

$$= \Phi \Phi^T H(I_n \otimes I_n - I_n)$$

$$= \Phi \Phi^T (I_n \otimes I_n - I_n) H$$

$$= \frac{1}{N} \Phi \Phi^T (1_{N \times N} \otimes I_n) H$$

$$= 0_{N \times N \times N}.$$

Here we use a property of Kronecker product, i.e., $(A \otimes B)(C \otimes D) = AC \otimes BD$ with $A, B, C$ and $D$ having compatible dimensions. In addition, we have

$$\Phi^T A \Phi = \bar{A} = \text{diag}(\lambda_2, \lambda_3, \ldots, \lambda_N).$$

Combining (21) with (22) gives

$$\dot{V} = x^T \Phi \Phi^T \dot{H} \Phi \Phi^T P + P \Phi \Phi^T \dot{H} \Phi \Phi^T x$$

$$= x^T \Phi \left( (H \otimes I_n - P \otimes H) \Phi \Phi^T \right) x$$

$$= -x^T \Phi \bar{Q} \Phi^T x$$

$$\leq - \lambda_{\min}(Q)|x|_{A_s}^2,$$

where $\bar{K} = \text{diag}(H_2, H_3, \ldots, H_N)$, and $Q = \text{diag}(Q_2, Q_3, \ldots, Q_N)$ with $Q_i = -H_i^T P_i - P_i H_i$, $Q_i > 0$, so is $Q$, therefore $\lambda_{\min}(Q) > 0$.

Hence, (20) is a Lyapunov function for the network (7) with respect to $A_s$. This completes the proof.

**B. Distributed Event-Triggering Mechanism Design**

Now, consider the network model (1), and the purpose of this subsection is to design a sampling time sequence $\{t_i\}$ for each node $i$ under which synchronization of the network can be achieved. Moreover, the sampling time sequence of a given node $i$ will depend on the occurrence of an event which is defined by a violation of a local error signal to a given threshold, i.e., if the error signal exceeds the given threshold, the sampler of the node will sample and send its states to its neighbors. We suppose that at $t = t_0$, all the nodes sample its state and share its value with its neighbors, i.e., $x(t_0) = \bar{x}(t_0)$.

In the case of all state information of the entire network can be accessed for the design purpose, we can get a
centralized event-triggering rule which bases on an error between the current state and its latest sampled value.

**Lemma 3:** If there exist positive definite matrices $P_i \in \mathbb{R}^{n \times n}$ such that

$$H_i^T P_i + P_i H_i = -2I_n, \quad i = 2, 3, \ldots, N,$$  \hspace{1cm} (24)

then the network (1) is globally asymptotically synchronized under the following sampling time sequence based on a centralized event-triggering rule

$$t_{k+1}^{i} = \inf \left\{ t > t_k \mid \| \Phi^T e \| - \frac{\delta}{\alpha} \| \Phi^T x \| \geq 0 \right\},$$  \hspace{1cm} (25)

where $e = \hat{x} - x$, $\delta \in (0, 1)$, and $\alpha = \max_{i=2,3,\ldots,N} \{ -c \lambda_i(A) \| P_i \Gamma \| \}$, \hspace{1cm} (26)

and here $\| \cdot \|$ also denotes the induced norm of a matrix.

**Proof:** Rewrite the network (1) as follows

$$\dot{x}(t) = (I_N \otimes H)(x(t) + (cA \otimes \Gamma)(x(t) + e(t)))$$

$$= \bar{H}x(t) + \bar{A}e(t),$$  \hspace{1cm} (27)

where $\bar{A} = cA \otimes \Gamma$.

By selecting a Lyapunov function candidate given in Theorem 2, we have

$$\dot{V} = \dot{\hat{x}}^T P \dot{x} + \dot{x}^T P \dot{x}$$

$$= 2x^T \bar{P}(\bar{H}x + \bar{A}e).$$  \hspace{1cm} (28)

At $t = t_0$, we have $\| \Phi^T e \| < \| \Phi^T x \|$. Again, by (21), (22), and under the event-triggering rule (25), we get

$$\dot{V} = -2x^T \Phi \Phi^T e + 2x^T \bar{P}(cA \otimes \Gamma) \Phi^T e$$

$$\leq -2 \| \Phi^T x \|^2 + 2 \| P(cA \otimes \Gamma) \Phi^T e \| = -2 \| \Phi^T x \|^2 + 2 \| \Phi^T e \|$$

$$\leq -2(1 - \delta) \| \Phi^T x \|^2.$$  \hspace{1cm} (29)

Applying Theorem 2 proves the result.

**Remark 3:** Similar to [8], we can conclude that there exists a minimum inter-event time between two consecutive events which is lower bounded by a non-zero constant $\tau_{\text{min}} > 0$, i.e., $t_{k+1} - t_{k} \geq \tau_{\text{min}}$. This can be proved by calculating $\frac{d}{dt} \| \Phi^T e \|$, for details, please refer to [8].

In practice, a centralized event-triggering rule (25) is usually hard for implementation. For one thing, it might be costly and time consuming to gather all the state information for the design purpose. Secondly, each node in the network can only get information of sampled states from its neighbors at certain discrete time instants, rather than $x(t)$ for all $t > t_0$. Therefore, a distributed event-triggering mechanism for a given node which only relies on information that it can get is desirable. Such a triggering rule is proposed in the following theorem.

**Theorem 3:** If there exist positive definite matrix solutions $P_i \in \mathbb{R}^{n \times n}$ satisfying (24), then the network (1) is globally asymptotically synchronized under the following distributed event-triggering rule

$$t_{k+1}^{i} = \inf \left\{ t > t_k^{i} \mid \| e_i \| + \frac{\delta}{\lambda_N(\alpha + \delta)} \| \hat{z}_i \| \geq 0 \right\},$$  \hspace{1cm} (30)

where $\delta \in (0, 1)$ and $\hat{z}_i = \sum_{j=1}^{N} a_{ij} \hat{x}_j$, $\lambda_N < 0$ is the smallest eigenvalue of $A$.

**Proof:** Under the event-triggering rule (30), we have

$$\| e \| < \frac{-\delta}{\lambda_N(\alpha + \delta)} \| \hat{z} \|$$

$$= \frac{-\delta}{\lambda_N(\alpha + \delta)} \| (A \otimes I_n) \hat{z} \|$$

$$= \frac{-\delta}{\lambda_N(\alpha + \delta)} \| z + (A \otimes I_n) e \|$$

$$\leq \frac{-\delta}{\lambda_N(\alpha + \delta)} (\| z \| + \| (A \otimes I_n) \| \| e \| )$$

$$\leq \frac{-\delta}{\lambda_N(\alpha + \delta)} (\| z \| - \lambda_N \| e \| ),$$  \hspace{1cm} (31)

which is equivalent to

$$\| e \| < \frac{-\delta}{\alpha \lambda_N} \| z \|,$$  \hspace{1cm} (32)

where $z = (A \otimes I_n)x$, and $\| A \otimes I_n \| = -\lambda_N$.

Moreover, we have

$$\| z \|^2 = x^T (A^2 \otimes I_n) x \leq \lambda_N^2 x^T (U^2 \otimes I_n) x$$

$$= \lambda_N^2 \| \Phi^T x \|^2,$$  \hspace{1cm} (33)

which comes from $A^2 \leq \lambda_N^2 U^2$ and $U^2 = U = \Phi \Phi^T$. Therefore, we have

$$\| e \| < \delta \| \Phi^T x \|.$$  \hspace{1cm} (34)

Since $\| \Phi^T e \| \leq \| \Phi^T \| \| e \|$ with $\| \Phi \| = 1$, combining this with (32) and (33) gives

$$\| \Phi^T e \| \leq \frac{\delta}{\alpha} \| \Phi^T x \|.$$  \hspace{1cm} (35)

By selecting the same Lyapunov function candidate as in Lemma 3, (28) and (29) hold under the triggering rule (30). Therefore, the network (1) is globally asymptotically synchronized.

**Remark 4:** The triggering rule (30) for the network (1) is only related to the minimum eigenvalue of its outer coupling matrix $A$, thus it is scalable to a large-scale network, in particular to a network with a small value of $\| \lambda_N \|$. It is worth pointing out that the consensus problem of multi-agent systems discussed in [7] can be seen as a special case of the problem formulated in this paper. Moreover, the triggering rules given in [7], [13] depend on the state of neighboring agents $z(t)$ continuously which requires continuous communication between agents and their neighbors with respect to time. Contrary to this, our proposed triggering rules which trigger the sampler of a node, only rely on the state of the node itself and the sampled states it receives from its neighbors $\hat{z}_i(t)$. Thus, from an implementation point of view, our rules appear to offer practical advantages.

**Remark 5:** Zeno behavior which refers to infinitely fast switching in a finite time interval is undesirable in practice. In this paper, we assume that no Zeno phenomenon happens in the network (1). However, how to exclude such a phenomenon by designing a suitable distributed event-triggering rule is an important issue and deserves attention in further study.
In this section, we give an example to show the effectiveness of the proposed results. The network considered here has 10 nodes with $c = 1$, and $H$, $\Gamma$, $A$ being given as follows

$$H = \begin{pmatrix} 0 & -0.5 \\ 0.5 & 0 \end{pmatrix}, \quad \Gamma = \begin{pmatrix} 0.25 & 0 \\ -1 & 0.25 \end{pmatrix},$$

$$A = \begin{pmatrix} -4 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & -4 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & -4 & 1 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & -4 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & -4 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & -4 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & -4 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & -4 \\
\end{pmatrix}.$$  

In this example, the isolated node of the network is stable, rather than asymptotically stable with both eigenvalues $\pm 0.5i$ of $H$ on the imaginary axis, so the network cannot asymptotically synchronize to its equilibrium point, but to a stable solution.

Figure 1 and Figure 2 show simulation results of the network (1) with different triggering rules proposed in Lemma 3 and Theorem 3, respectively. The figures show that the simulation result of the network with a centralized triggering rule has better performances than that of the network with a distributed triggering method both in synchronization performance and sample frequency (non-periodic). This is reasonable as the centralized rule uses all information of the entire network. However, such a centralized method is usually impossible for implementation, in particular for a network with a large number of nodes. In this case, the distributed triggering rule will show its advantages.

### V. CONCLUSIONS

In this paper, set stability has been applied to studying asymptotic synchronization of a linear dynamical network with physical links, and synchronization criteria have been established from this point of view. Based on the proposed results, different event-triggering rules have been studied for a network with communication links, which decide when a node should sample and send its state to its neighbors in order to achieve synchronization. Simulations have been provided to show the effectiveness of designed triggering mechanisms. According to the simulation results, some comparisons have been made between different rules. They show that interesting problems related to this topic remain open and deserve more attention.

### REFERENCES


