<table>
<thead>
<tr>
<th><strong>Title</strong></th>
<th>On Pricing and Hedging Basket Credit Derivatives with Dependent Structure</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Author(s)</strong></td>
<td>Zhu, D; Xie, Y; Ching, WK; Zheng, H</td>
</tr>
<tr>
<td><strong>Citation</strong></td>
<td>The IEEE Computational Intelligence for Financial Engineering and Economics (CIFEr), London, UK, 27-28 March 2014. In the Proceedings of the IEEE/IAFE Computational Intelligence for Financial Engineering and Economics (CIFEr), 2014, p. 435--440</td>
</tr>
<tr>
<td><strong>Issued Date</strong></td>
<td>2014</td>
</tr>
<tr>
<td><strong>URL</strong></td>
<td><a href="http://hdl.handle.net/10722/207341">http://hdl.handle.net/10722/207341</a></td>
</tr>
<tr>
<td><strong>Rights</strong></td>
<td>Proceedings of the IEEE/IAFE Computational Intelligence for Financial Engineering and Economics (CIFEr). Copyright © IEEE. ©2014 IEEE. Personal use of this material is permitted. However, permission to reprint/republish this material for advertising or promotional purposes or for creating new collective works for resale or redistribution to servers or lists, or to reuse any copyrighted component of this work in other works must be obtained from the IEEE.; This work is licensed under a Creative Commons Attribution-NonCommercial-NoDerivatives 4.0 International License.</td>
</tr>
</tbody>
</table>
Abstract—In this paper, we study the problem of hedging a basket credit derivatives, in particular, we are interested in basket default swaps. For the pricing of credit derivatives, we consider a factor Copula approach. Single-name credit default swaps will be chosen as the hedging instruments. The hedging mechanism is tested using simulated data with a given measure. Numerical results reveal the efficiency of our proposed hedging method.

I. INTRODUCTION

The ongoing credit crisis has highlighted the need for a sound methodology for the pricing and risk management of various credit products. Credit derivatives, whose payoff is correlated with default events of the underlying portfolio, play a significant role in credit markets. In general, there are two types of derivative products: single-name credit derivatives and basket credit derivatives.

For basket credit derivatives, an important issue is how to capture the dependence structure of the defaults. There are three main approaches to model the default dependence within the broad range of reduced-form models: (i) Models with direct interaction between default intensities, where the default probability of one name would be affected by other names’ defaults and vice versa, see for instance, [1], [2] and [3]. (ii) Models with dependent intensities but with conditionally independent default times, where the default intensities of names are affected by common systematic factors, see [4] and [5]. (iii) The factor copula models [6] and [7]. Among these three approaches, the factor Copula models are the market standard for pricing of credit derivatives.

As a prominent risk-management issue, the hedging of loss derivatives has attracted increasing attention these years. The main goal of our paper is to consider the hedging of basket default swaps using single-name Credit Default Swaps (CDS) as hedging instruments. We will discuss the pricing of basket default swaps conditional on survivorship information and Gaussian Copula model will be employed to capture the joint distribution of all default times. Following that, we focus on the hedging of basket CDS by using delta hedging movements in single-name CDSs with respect to the shifting in the implied correlation, which is one of the parameters of Gaussian Copula model. The key advantage of our hedging method is that we use smaller numbers of single-name CDSs as a hedging instrument. This will make the hedging problem much easier and more efficient. A hedging efficiency measure would be provided for testing and making comparison with the simulated hedging results.

The paper is structured as follows. Section 2 describes the model for the joint distribution of default times. Section 3 considers the pricing of basket default swaps and Section 4 discusses the valuation of single-name credit default swaps. The delta hedging of the basket CDS and the hedging results will be indicated in Section 5. Section 6 then concludes the paper.
II. ModelinG DefauLt TiMeS

In this section, we consider $N$ underlying defaultable claims, denoted as $1, 2, \ldots, N$. For each name $i$, the associated default time is denoted by $\tau_i, i = 1, \ldots, N$, which are all defined on a probability space $(\Omega, \mathcal{G}, \mathbb{P})$. Our market model is supposed to be arbitrage-free and $\mathbb{P}$ is the risk-neutral probability measure. The marginal distribution function of default time $\tau_i$ is supposed to be

$$F_i(t) = \mathbb{P}(\tau_i \leq t) = 1 - e^{-\lambda_i t}.$$ 

Let

$$F(t_1, \ldots, t_N) = \mathbb{P}(\tau_1 \leq t_1, \ldots, \tau_N \leq t_N)$$

be the joint probability distribution function of default times. By Sklar’s Theorem, there exists a Copula function $C(u_1, u_2, \ldots, u_N)$ such that

$$F(t_1, t_2, \ldots, t_N) = C(F_1(t_1), F_2(t_2), \ldots, F_N(t_N)).$$

Since $F_i$ is continuous and strictly increasing, we simply get

$$C(u_1, \ldots, u_N) = F(F_1^{-1}(u_1), \ldots, F_N^{-1}(u_N)). \quad (1)$$

Now we consider a Gaussian vector $(X_1, X_2, \ldots, X_N)$, where

$$X_i = \rho Z + \sqrt{1 - \rho^2}Z_i$$

and $Z, Z_i (1 \leq i \leq N)$ are independent and identically distributed standard normal variables. Then

$$\text{cov}(X_i, X_j) = \begin{cases} 1 & i = j \\ \rho^2 & i \neq j \end{cases}$$

Consider the Gaussian Copula

$$C(u_1, \ldots, u_N) = \Phi_{\Sigma}(\Phi^{-1}(u_1), \ldots, \Phi^{-1}(u_N)),$$

where $\Phi$ denotes the cumulative distribution function of the standard normal $N(0, 1)$ and $\Sigma$ represents the covariance matrix of $X_i (1 \leq i \leq N)$.

By iterated expectations theorem, this term can be written as follows:

$$E[\mathbb{P}(X_1 < \Phi^{-1}(u_1), \ldots, X_N < \Phi^{-1}(u_N))|Z]$$

then we have

$$C(u_1, \ldots, u_N) = \int \left( \prod_{i=1}^{N} \Phi \left( \frac{\Phi^{-1}(u_i) - \rho x}{\sqrt{1 - \rho^2}} \right) \right) \phi(x) dx.$$

Similarly, the joint cumulative distribution function can be written as follows:

$$F(t_1, \ldots, t_N) = \int \left( \prod_{i=1}^{N} \Phi \left( \frac{\Phi^{-1}(F_i(t_i)) - \rho x}{\sqrt{1 - \rho^2}} \right) \right) \phi(x) dx$$

For each $i = 1, 2, \ldots, N$, we denote the default indicator process $H_i^t = I_{\{\tau_i \leq t\}}$ and the corresponding filtration $\mathbb{H}^t_i = (H_i^s : s \leq t)$. In the succeeding part, we let

$$\mathbb{H}_t = \mathbb{H}^1_i \vee \mathbb{H}^2_i \vee \ldots \vee \mathbb{H}^N_i$$

and we assume that $\mathbb{P}(\tau_i = \tau_j) = 0$ for any $1 \leq i \neq j \leq N$. Then the collection of default times can be put in order $\tau^1 < \cdots < \tau^t < \cdots < \tau^N$, where $\tau^t$ stands for the time of the $i$-th default.

III. Pricing of Basket Credit Derivatives

In what follows, we consider the pricing of $K^{th}$-to-default swap on a set of $N$ defaultable claims. We denote by 1, the nominal of a given reference credit and denote by $r > 0$, the constant market interest rate.

Suppose the premium payments date of the basket CDS is $0 < t^1 < t^2 < \ldots < t^M = T$, where $T$ is the expiry date of the basket default swap. The periodic premium paid at each time slot is supposed to be deterministic $Y$, which is chosen such that the value of the CDS contract is equal to 0 at the issue time. For simplicity, we do not take into consideration of the accrued premium payments due to defaults between premium payments dates. Assume the default payments are made immediately after the default and the deterministic recovery rate is denoted as $\delta$. Then based on the given survivorship information, the value of this $K^{th}$-to-default CDS at time $t$ will be

$$V_K(t, \rho) = \mathbb{E} \left[ e^{-r(\tau^K_t - t)} I_{\{t < \tau^K_{\leq t} \}} (1 - \delta) | \mathbb{H}_t \right]$$

$$- \mathbb{E} \left[ \sum_{t^k > t} I_{\{\tau^K_{> t} \}} Y e^{-r(t^k - t)} | \mathbb{H}_t \right]$$

where $I_A$ is the indicator function.

Suppose at time $0 < s \leq T$, $k_s$ names out of a total of $N$ names have already defaulted. To introduce a convenient notation, the surviving names are denoted as $\{j_1, j_2, \ldots, j_{N-k_s}\}$, and the $k_s$ defaulted time of names $i_1, \ldots, i_{k_s}$ are ordered $u_1 < \cdots < u_{k_s}$. For the sake of brevity, we will write $D_{k_s} = \{x_{i_1} = u_1, \ldots, x_{i_{k_s}} = u_{k_s} \}$. 

436
For arbitrary $s < t$, we denote $(\tau^K > s)$

\[ G^K(t, s) = P(\tau^K > t | \mathcal{H}_s). \]

By Bayes’ theorem

\[ P(\tau^K > t | \mathcal{H}_s) = P(\tau^K > t | \tau_i = u_1, \ldots, \tau_{ik_s} = u_{k_s}, \tau_{j_1} > s, \ldots, \tau_{j_{N-k_s}} > s) \]

whose numerator can be written as

\[ P(\tau^K > t, \tau_{i} \in du_1, \ldots, \tau_{ik_s} \in du_{k_s}, \tau_{j_1} > s, \ldots, \tau_{j_{N-k_s}} > s) \]

which is equal to

\[ P(\tau_{i} \in du_1, \ldots, \tau_{j_{N-k_s}} > s, \sum_{l=1}^{N-k_s} I_{s<\tau_{l} \leq t} \leq K - k_s - 1) \]

and the denominator is given by

\[ P(\tau_{i} \in du_1, \ldots, \tau_{ik_s} \in du_{k_s}, \tau_{j_1} > s, \ldots, \tau_{j_{N-k_s}} > s). \]

Two cases will be discussed in the following. If all $\lambda_i$ are identical, then

\[ P(\tau^K > t | \mathcal{H}_s) = \frac{\sum_{l=0}^{K-k_s-1} (N-k_s)}{f[s, +\infty] \cdots [s, +\infty]} I_{s<\tau_i \leq t} \leq K - k_s - 1 \]

where

\[ A := [s, t] \times \cdots \times [s, t] [t, +\infty] \times \cdots \times [t, +\infty], \]

\[ f(t_1, \ldots, t_N) = \int \left( \prod_{i=1}^{N} \frac{\partial}{\partial t_i} \Phi \left( \frac{\Phi^{-1}(F_i(t_i)) - \rho x}{\sqrt{1 - \rho^2}} \right) \right) \phi(x) dx \]

and $f(x_1, \ldots, x_N) | D_k_s$ denotes that $f(x_1, \ldots, x_N) = u_{ij_1}, x_{j_1}$

where $1 \leq j_1 \leq k_s$ and $1 \leq i_1 \leq N - k_s$. In the succeeding part, denote the $l$ names that default between time $s$ and $t$ as names $\{q_1, \ldots, q_l\}$ and the $N - k_s - l$ names that default after time $t$ will be assumed to be $\{m_1, \ldots, m_{N-k_s-l}\}$.

For convenience, we set

\[ g_j(t_i, \rho, x) = \Phi \left( \frac{\Phi^{-1}(F_j(t_i)) - \rho x}{\sqrt{1 - \rho^2}} \right). \]

For $G^K(t, s, \int_A f(x_1, \ldots, x_N) | D_k_s dx_{j_1} \cdots dx_{j_{N-k_s}}$ in the numerator of (3) can be calculated as follows:

\[ \int \prod_{j=1}^{k_s} \frac{\partial g_j(t_i, \rho, x)}{\partial t_i} \bigg|_{t_j = u_j} \prod_{j=1}^{l} (1 - g_j(t)) \]

and the denominator can be calculated as

\[ \int \prod_{j=1}^{k_s} \frac{\partial g_j(t, \rho, x)}{\partial t_j} \bigg|_{t_j = u_j} \prod_{j=1}^{N-k_s} (1 - g_j(t)) \phi(x) dx. \]

Otherwise, if $\lambda_i$ are distinct we have

\[ \int \prod_{j=1}^{k_s} \frac{\partial g_j(t, \rho, x)}{\partial t_j} \bigg|_{t_j = u_j} \prod_{j=1}^{N-k_s} (1 - g_j(t)) \phi(x) dx. \]

\[ \sum_{i=1}^{K-k_s-1} \int \prod_{j=1}^{N-k_s} (1 - g_j(t, \rho, x)) \phi(x) dx. \]

$I^l$ denotes the set of all possible $l$ elements chosen from $N - k_s$ surviving names and $f(x_1, x_2, \ldots, x_N) | D_k_s$ remains the same. The numerator and denominator can be calculated similarly as before.

The price of the premium payment leg of the $K$-th-to-default swap is given by

\[ V_K^{\text{prem}}(t, \rho) = E \left[ \sum_{t^l > t} I_{t^l > t^l} Y e^{-r(t^l-t)} | \mathcal{H}_t \right] = \sum_{t^l > t} Y e^{-r(t^l-t)} G^K(t^l, t), \]

where $r$ denotes the given constant interest rate.

The value of the default payment leg of the $K$-th-to-default swap at time $t$ would be written as

\[ V_K^{\text{def}}(t, \rho) = E[ (1 - \delta) I_{t^l, t^l} \tau^K e^{-r(T-t)} | \mathcal{H}_t] = (1 - \delta) G^K(t, t) - e^{-r(T-t)} G^K(T, t) \]

where $G^K(t, s)$ denotes the survival function of the $K$-th to default time conditional on the information structure up to time $s$, which has been given before.
IV. Pricing of a Single-name CDS

In our model, single-name CDSs will be chosen as the hedging instruments. For these single-name CDSs, the premiums payment date and the expiry date are supposed to be the same as in the basket default swap contract, which are \(t^i (1 \leq i \leq M)\) and \(T\) respectively. Now we would like to price a single-name CDS whose underlying asset is chosen from the set of the original \(N\) defaultable names. Without loss of generality, we assume the chosen name is \(i\), then the price at time \(t\) of the corresponding CDS is

\[
V^i(t, \rho) = \mathbb{E}[e^{-r(t-t)}I_{(t<\tau_{i}\leq T)}(1-\delta^i)|\mathcal{H}_t] - \mathbb{E}\left[\sum_{t'>t} I_{(\tau_{i}>t')} Y^i e^{-r(t'-t)}|\mathcal{H}_t\right] \tag{7}
\]

where \(\delta^i\) is the recovery rate and \(Y^i\) is the periodic paid premium.

For arbitrary \(s < t\), on the event \(\{\tau_{i} > s\}\) we denote \(G_i(t, s) = \mathbb{P}(\tau_{i} > t|\mathcal{H}_s)\). Then we have

\[
\mathbb{P}(\tau_{i} > t|\mathcal{H}_s) = \mathbb{P}(\tau_{i} > t|\tau_{i_1} = u_{1}, \ldots, \tau_{i_{k_s}} = u_{k_s}, \tau_{j_1} > s, \ldots, \tau_{j_{N-s}} > s) = \int_{A_2} f(x_1, \ldots, x_N)|_{D_{k_s}} dx_{j_1} \ldots dx_{j_{N-s}} \quad \int_{B_2} f(x_1, \ldots, x_N)|_{D_{k_s}} dx_{j_1} \ldots dx_{j_{N-s}}
\]

where

\[
A_2 = [t, +\infty] \frac{[s, +\infty] \ldots [s, +\infty]}{N-k_s-2}, \quad B_2 = [s, +\infty] \ldots [s, +\infty]
\]

\[
f(t_1, \ldots, t_N) = \int \left( \prod_{i=1}^{N} \frac{\partial}{\partial t_i} \phi \left( \frac{\Phi^{-1}(F_i(t_i)) - \rho x}{\sqrt{1 - \rho^2}} \right) \right) \phi(x) dx.
\]

After some simple calculations, the numerator of \(G_i(t, s)\) can be written as

\[
\int \prod_{j=1}^{k_s} \frac{\partial g_i_j(\rho, x)}{\partial t_j}|_{t_j = u_j} \prod_{q=1}^{N-k_s-1} (1 - g_i_j(s, \rho, x))(1 - g_i(t, \rho, x)) \phi(x) dx
\]

where \(g_i(t, \rho, x)\) denotes the same formula as in (4).

Consequently, the premium leg of \(i\) CDS at time \(t\) would be given by

\[
V^i_{\text{prem}}(t) = \mathbb{E} \left[ \sum_{t'>t} I_{(\tau_{i}>t')} Y^i e^{-r(t'-t)}|\mathcal{H}_t\right] = \sum_{t'>t} Y^i e^{-r(t'-t)}G_i(t', t).
\]

The default leg of this single-name CDS at time \(t\) would be

\[
V^i_{\text{def}}(t) = \mathbb{E}[I_{(t,T)}(1-\delta^i)|\mathcal{H}_t] = (1-\delta^i)[G_i(t, t) - e^{-r(T-t)}G_i(T, t)] - r \int_t^T G_i(x, t)e^{-r(x-t)}dx.
\]

The “fair” spread can be obtained by setting the above two legs equal to each other at the initial time.

V. Hedging Shifts in the Implied Correlation

In this section, we shall discuss the hedging of \(K\)-to-default basket CDS using single-name CDSs. First we consider a single-name CDS based on \(i (1 \leq i \leq N)\) name which is randomly chosen from the set of \(N\) defaultable claims.

In practice, it is important to manage the risk of the fluctuation in the implied correlation, one of the parameters of Gaussian Copula model. The approach that we use for hedging basket CDS is delta hedging movements in the single CDS \(i\) with respect to shifting in the correlation \(\rho\), while other parameters remain unchanged. The hedging position in CDS \(i\) is equal to

\[
\phi^i(t) = \frac{\partial}{\partial \rho} V_K(t, \rho).
\]

If \(\tau_{i} \geq T^K\), then the succeeding trading strategy \(\phi^i\) can be applied to hedge the \(K\)-to-default basket CDS, otherwise, another asset would be chosen from the survival assets. With similar method, the corresponding trading strategy can be obtained.

In what follows, we will consider a hedging efficiency measure and show the effectiveness of our hedging method. As in [8], we will measure the efficiency of delta hedge by measuring the difference between the accumulated change in the values of the basket CDS and the accumulated change in value of the chosen single-name CDSs when correlation \(\rho\) changes in time.

Suppose we consider the following portfolio:

\[
V_k = V_K(t, \rho) - \phi^i(t)V^i(t, \rho)
\]
If there is no arbitrage opportunity and this portfolio is perfectly hedged, then it is expected that
\[ d\mathcal{V}_t = r\mathcal{V}_t dt, \]
where \( r \) denotes the constant interest rate. For such portfolio, we are interested in its dynamics until some time point \( \tau^K \land T \), where \( T \) is the expiry date. We let
\[
\begin{align*}
\dot{V}_t &= V_t e^{r(\tau^K \land T - t)}; \\
\dot{V}_K(t, \rho) &= V_K(t, \rho) e^{r(\tau^K \land T - t)}; \\
\dot{V}_{ij}(t, \rho) &= V_{ij}(t, \rho) e^{r(\tau^K \land T - t)}
\end{align*}
\]
then from Eq. (8), we have
\[ d\dot{V}_i = 0 \]
which implies that
\[ \int_t^{\tau^K \land T} d\dot{V}_K(t, \rho) = \int_t^{\tau^K \land T} \phi^i(t) d\dot{V}_{ij}(t, \rho). \tag{9} \]
We divide the interval \([t, \tau^K \land T]\) into \( J \) subintervals such that \( t \leq t_1 \leq \cdots \leq t_{j-1} \leq t_j = \tau^K \land T \). The discrete form of the equation (9) by using these subintervals becomes:
\[ V_K(t_j, \rho) - e^{r(t_j-t)} V_K(t, \rho) - \sum_{j=1}^J \phi^i(t_{j-1})(V_{ij}(t_j, \rho) - V_{ij}(t_{j-1}, \rho)) = 0 \tag{10} \]
This equation would be expected to hold if the basket CDS is perfectly hedged.

A. Numerical Examples

In this section some simulated examples will be presented to show the efficiency of our hedging method. First, we give some hypothetical values for the model parameters. We assume that the underlying portfolio consists of \( N = 3 \) defaultable claims and we will consider second-to-default basket CDS. We also suppose that the initial time \( t_0 = 0 \), the expiry date \( T = 2 \) years, the constant interest rate \( r = 0.04 \), the premium will be paid quarterly, the recovery rate of the basket CDS \( \delta = 0.6 \), the recovery rates for single CDS on Names 1, 2 and 3 are supposed to be \( \delta^1 = 0.4, \delta^2 = 0.5, \delta^3 = 0.6 \) respectively. For convenience, we suppose all the intensities are the same \( \lambda = 0.1 \). In discrete form Eq. (10) \( \Delta t = t_j - t_{j-1} = \frac{1}{40} \) year.

We simulate dynamics of the default times of those three assets by using Gaussian vector. Let
\[ a(t) = V_K(t_j, \rho) - e^{r(t_j-t)} V_K(t, \rho); \]
and
\[ b(t) = \sum_{j=1}^J \phi^i(t_{j-1})(V_{ij}(t_j, \rho) - V_{ij}(t_{j-1}, \rho)); \]
the values of these two variables will be computed and compared. The pricing functions here are evaluated by calculating the sum over all the discretization time points, and these time points are different from the previous hedging times \( t_j \) we used. In our simulated default times, Name 1 defaults first, Name 2 follows and then Name 3. Name 1 defaults after about half a year and Name 2 defaults after about one and a half years. Since we hedge the basket CDS by using single-name CDS, different chosen orders will be tested and compared. With shifting of the implied correlation \( \rho \), the dynamics of \( a(t) \) and \( b(t) \) are shown in Figs. 1-4. In Fig. 1, Name 2 is chosen as the underlying asset of the hedging instrument. In Fig. 2, Name 3 is chosen. Fig. 3 provides the results if Name 1 is chosen first and Name 2 is subsequently chosen as the underlying assets. Similarly, Fig. 4 addresses the results for choosing Name 1 and Name 3 as hedging instruments.

The first default occurs after about half a year, and the figures show that the accumulated values change dramatically once the default happens. This is reasonable since when one default occurs, it is expected that more defaults may occur which will therefore decrease the value of basket derivatives that we concerned. From these figures, one can also see that with shifting of the implied correlation \( \rho \), the accumulated
change of the value in the basket CDS is approximately mirrored by the accumulated change of the value in the chosen hedging instruments. These imply that our method can be employed to reduce the risk caused by fluctuation of the correlation $\rho$. We also note that the hedging results will be better if finer discretization time step is adopted for evaluating those pricing functions. Additionally, among all the choices, if Name 2, which defaults at the same time as the basket CDS, is chosen as hedging instrument, the hedging result would be the best.

VI. CONCLUSIONS

In this paper we discuss the pricing and hedging of basket credit derivatives using Gaussian Copula model. Numerical results are given to show the efficiency of the hedging method. The hedging instruments that we used are a few numbers of single-name CDSs, in this way, investors can manage their portfolio more easily and it is more cost effective. Our method can also be applied to hedge other credit derivatives under the same model. The method can also be extended to discuss the hedging problems when the default dependence is modeled using other approaches. In such cases, other parameters would be considered, for instance, the CDS spread, instead of the correlation $\rho$. This will an interesting future research issue.

REFERENCES