Generalized Gauge for Multi-scale Inhomogeneous Media

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Abstract—The vector potential $A$ has no direct physical meaning in classical electromagnetics. However, it manifests itself in quantum physics in terms of the Aharonov-Bohm effect. The vector potential $A$ is similar to momentum. By itself, it is hard to detect classically, but its time variation generates a force in terms of electric field. Hence, the electric field is of the form

$$E = -\partial_t A - \nabla \Phi$$

where the electric field, which exerts a force on a charge, is generated by a time varying $A$ and the gradient of the scalar potential $\Phi$. The magnetic flux is given by $B = \nabla \times A$

By using Lorentz gauge

$$\nabla \cdot A = -\mu \varepsilon \partial_t \Phi$$

Maxwell’s equations in vacuum reduce to

$$\nabla^2 \Phi - \mu \varepsilon \partial_t^2 \Phi = -\rho / \varepsilon,$$

$$\nabla^2 A - \mu \varepsilon \partial_t^2 A = -\mu J$$

For inhomogeneous medium, we pick the generalized gauge

$$\varepsilon^{-1} \nabla \cdot \varepsilon A = -\mu \varepsilon \partial_t \Phi.$$  

Then it can be shown that Maxwell’s equations reduce to

$$\varepsilon^{-1} \nabla \cdot \varepsilon \nabla \Phi - \mu \varepsilon \partial_t^2 \Phi = -\rho / \varepsilon,$$

$$-\mu \nabla \times \mu^{-1} \nabla \times A - \mu \varepsilon \partial_t^2 A + \mu \varepsilon \nabla \frac{1}{\mu \varepsilon} \varepsilon^{-1} \nabla \cdot \varepsilon A = -\mu J.$$  

For homogeneous medium, (6) and (7) reduce to (3) and (4).

The above equations have no low-frequency breakdown when solved numerically irrespective of how small the meshes are. Moreover, since $A$ and $\Phi$ are needed in writing the Hamiltonian of an atom-field system, it is particularly suited for solving Maxwell-Schrodinger system of equations.

The discretization of the above equations can be inspired by differential forms from differential geometry. The vector potential $A$ can be regarded as a one form which is curl-conforming. But the permittivity function can be regarded as a Hodge operator that converts a one form to a two form. Hence, $\varepsilon A$ becomes a two form which has to be divergence conforming. The Hodge operator can also be implemented numerically.