A new test for the proportionality of two large-dimensional covariance matrices

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Abstract

Let $\mathbf{x}_1, \ldots, \mathbf{x}_{n_1+1} \overset{\text{iid}}{\sim} N_p(\boldsymbol{\mu}_1, \boldsymbol{\Sigma}_1)$ and $\mathbf{y}_1, \ldots, \mathbf{y}_{n_2+1} \overset{\text{iid}}{\sim} N_p(\boldsymbol{\mu}_2, \boldsymbol{\Sigma}_2)$ be two independent random samples, where $n_1 \leqslant p < n_2$. In this article, we propose a new test for the proportionality of two large $p \times p$ covariance matrices $\boldsymbol{\Sigma}_1$ and $\boldsymbol{\Sigma}_2$. By applying modern random matrix theory, we establish the asymptotic normality property for the proposed test statistic as $(p, n_1, n_2) \to \infty$ together with the ratios $p/n_1 \to y_1 \in (0, \infty)$ and $p/n_2 \to y_2 \in (0, 1)$ under suitable conditions. We further showed that these conclusions are still valid if normal populations are replaced by general populations with finite fourth moments.

Keywords: Covariance matrix, Hypothesis testing, Large-dimensional data, Limiting spectral distribution, Proportionality, Random F-matrices.

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1. Introduction

With the rapid development and wide applications of computer techniques, huge data can be collected and stored. This is called as high-dimensional data or large-dimensional data, see Bai and Silverstein [2]. Many traditional estimation and test tools are no more valid or perform badly for such large-dimensional data, since these traditional methods are often based on the classical central asymptotic theorems assuming a large sample size n and fixed dimension p.

In this article, we consider testing proportionality of large-dimensional covariance matrices from two different populations. The proportionality of covariance

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matrices is the simplest form of heteroscedasticity between populations, which has extensive applications in economics, discriminations, etc.

As an instance, consider a quantitative genetic experiment, called *paternal half-sib design*. This experiment is conducted under the hypothesis of equal heritabilities in the two populations, and it corresponds to the hypothesis of proportionality between population covariance matrices which we will discuss in this paper. The goal of this experiment is to model measurements of some quantitative traits in two independent populations of animal offsprings. More detailed description of this experiment can be found in Jensen and Madsen [10]. Other related examples are: Dargahi-Noubary [4] considered the applications of discrimination between two normal populations when covariance matrices are proportional. Nel and Groenewald [14] studied the multivariate Behrens-Fisher problem under the assumption of proportional covariance matrices. Later, Villa and Pérignon [20] investigated the sources of time variation in the covariance matrix of interest rates. In their work, they discussed the similarities among covariance matrices of bond yields including the cases of equality and proportionality.

Let $\mathbf{x}_1, \ldots, \mathbf{x}_{n_1+1} \overset{\text{iid}}{\sim} N_p(\boldsymbol{\mu}_1, \boldsymbol{\Sigma}_1)$ and $\mathbf{y}_1, \ldots, \mathbf{y}_{n_2+1} \overset{\text{iid}}{\sim} N_p(\boldsymbol{\mu}_2, \boldsymbol{\Sigma}_2)$ be two independent random samples, where $n_1 \leqslant p < n_2$. To describe the proposed new test, we define $N_k = n_k + 1$ ($k = 1, 2$) and consider the joint sufficient statistics:

$$\bar{\mathbf{x}} = \frac{1}{N_1} \sum_{i=1}^{N_1} \mathbf{x}_i, \qquad \hat{\mathbf{V}}_1 = \sum_{i=1}^{N_1} (\mathbf{x}_i - \bar{\mathbf{x}})(\mathbf{x}_i - \bar{\mathbf{x}})^\top,$$

and

$$\bar{\mathbf{y}} = \frac{1}{N_2} \sum_{j=1}^{N_2} \mathbf{y}_j, \qquad \hat{\mathbf{V}}_2 = \sum_{j=1}^{N_2} (\mathbf{y}_j - \bar{\mathbf{y}})(\mathbf{y}_j - \bar{\mathbf{y}})^\top, \tag{1}$$

for $(\boldsymbol{\mu}_1, \boldsymbol{\Sigma}_1)$ and $(\boldsymbol{\mu}_2, \boldsymbol{\Sigma}_2)$, respectively, where $\mathbf{A}^\top$ denotes the transpose of $\mathbf{A}$. Thus, $\hat{\boldsymbol{\Sigma}}_k = \hat{\mathbf{V}}_k / n_k$ is the maximum likelihood estimator of $\boldsymbol{\Sigma}_i$ for $k = 1, 2$.

The null hypothesis of interest is

$$H_0\colon \boldsymbol{\Sigma}_1 = \sigma^2 \boldsymbol{\Sigma}_2, \tag{2}$$

where $\sigma^2 > 0$ is an unknown constant. When $\sigma^2 = 1$, Wilks [21] suggested a likelihood ratio (LR) test statistic

$$\Lambda_N = \frac{N_1}{2} \log |\hat{\mathbf{V}}_1| + \frac{N_2}{2} \log |\hat{\mathbf{V}}_2| - \frac{N}{2} \log |\hat{\mathbf{V}}_1 + \hat{\mathbf{V}}_2| \ ,$$

where $N = N_1 + N_2$. However, in practice, it is often to use a modified LR test statistic

$$\widetilde{\Lambda}_N = \frac{n_1}{2}\log|\hat{\mathbf{V}}_1| + \frac{n_2}{2}\log|\hat{\mathbf{V}}_2| - \frac{n_1+n_2}{2}\log|\hat{\mathbf{V}}_1 + \hat{\mathbf{V}}_2|,$$

due to its unbiasedness and monotonicity of the power function (see, Srivastava, Khatri and Carter [18]; Sugiura and Nagao [19]).

As the sample sizes $N_1, N_2 \to \infty$, the classical central limiting theorem states that under H_0 (see [12], Sec. 8.2)

$$-2\Lambda_N \xrightarrow{\text{D}} \chi^2_{\frac{1}{2}p(p+1)} \quad \text{and} \quad -2\widetilde{\Lambda}_N \xrightarrow{\text{D}} \chi^2_{\frac{1}{2}p(p+1)}. \tag{3}$$

Note that the limiting results in (3) are true only when the variable dimension p is assumed to be fixed and $p \ll \min(N_1, N_2)$. With computer simulations, Bai et al. [1] showed that, employing the χ^2 approximation (3) for dimensions like 30 or 40, increases dramatically the size of the test. For dimension and sample sizes $(p, N_1, N_2) = (40, 800, 400)$, the test size equals 21.2% instead of the nominal 5% level. The result is even worse for the case of $(p, N_1, N_2) = (80, 1600, 800)$ which leads to a 49.5% test size.

For the general case of hypothesis (2) with unknown constant σ^2, to our knowledge, the first research was done by Federer [6] who developed a maximum likelihood method for two groups of normal populations with dimension $p \leqslant 3$. Kim [11] extensively studied the problem of proportionality between covariance matrices, and showed that the solution of the likelihood equations was unique. This result was later published by Guttman et al. [9]. Independently, Rao [15] considered the likelihood ratio test for proportionality of covariance matrices from two normal populations. As an extension, under normality assumptions, Eriksen [5] and Flury [8], independently, discussed maximum likelihood estimation of proportional covariance matrices and provided likelihood ratio tests for testing the hypothesis of proportionality.

Furthermore, Schott [16] considered the test problem of proportional covariance matrices from k different groups and obtained a Wald statistic under general conditions without normal population assumptions. However, all of the above work is based on the classical approximating theory which assumes the dimension of data, p to be held fixed. As it will be shown, this classical approximation leads to a test size much higher than the nominal test level in the case of large-dimensional data. All such problems and limitations bring up the need for further study on proportionality test of covariance matrices.

Based on the modern *random matrix theory* (RMT), we propose a new test and further prove that the distribution of the test statistic proposed approximates to a normal distribution as $(p, n_1, n_2) \to \infty$ together with the ratios $p/n_1 \to y_1 \in (0, \infty)$ and $p/n_2 \to y_2 \in (0, 1)$ under suitable conditions. The proposed new test is valid for both Gaussian data and non-Gaussian data with finite fourth moments.

The rest of the article is organized as follows. Section 2 briefly reviews some preliminary and useful RMT results. In Section 3, we propose a new test and study its asymptotic normality as $(p, n_1, n_2) \to \infty$. In Section 4, the efficiency of the proposed test is illustrated by simulation studies. Section 5 presents conclusions and some discussions. Some technical derivations are put into the Appendix.

2. Review of some useful results of random matrix theory

We first review several results from RMT, which will be used for the proposed test procedure. For any $p \times p$ matrix $\mathbf{M}$ with real eigenvalues $\{\lambda_i^{\mathbf{M}}\}_{i=1}^p$, the *empirical spectral distribution* (ESD) of $\mathbf{M}$, denoted by $F_n^{\mathbf{M}}$, is defined by

$$F_n^{\mathbf{M}}(x) = \frac{1}{p} \sum_{i=1}^{p} \mathbb{1}_{[\lambda_i^{\mathbf{M}} \leqslant x]}, \quad x \in \mathbb{R}.$$

We will consider a random matrix $\mathbf{M}$ whose ESD $F_n^{\mathbf{M}}$ converges (in a sense to be precisely) to a *limiting spectral distribution* (LSD) $F^{\mathbf{M}}$. To make statistical inference about a parameter $\theta = \int f(x) \mathrm{d}F^{\mathbf{M}}(x)$, it is natural to use an estimator

$$\hat{\theta} = \int f(x) \mathrm{d}F_n^{\mathbf{M}}(x) = \frac{1}{p} \sum_{i=1}^{p} f(\lambda_i^{\mathbf{M}}) \ ,$$

which is a so-called *linear spectral statistic* (LSS) of the random matrix $\mathbf{M}$.

Let $\{\xi_{ki} \in \mathcal{C}\colon i, k = 1, 2, \ldots\}$ and $\{\eta_{kj} \in \mathcal{C}\colon j, k = 1, 2, \ldots\}$ be two independent double arrays of $i.i.d.$ complex variables with mean 0 and variance 1. Writing $\boldsymbol{\xi}_i = (\xi_{1i}, \xi_{2i}, \ldots, \xi_{pi})^\top$ and $\boldsymbol{\eta}_j = (\eta_{1j}, \eta_{2j}, \ldots, \eta_{pj})^\top$. For any given positive integers n_1 and n_2, the vectors $\{\boldsymbol{\xi}_1, \ldots, \boldsymbol{\xi}_{n_1}\}$ and $\{\boldsymbol{\eta}_1, \ldots, \boldsymbol{\eta}_{n_2}\}$ can be thought as independent samples of size n_1 and n_2, respectively, from some p-dimensional distribution. Let $\mathbf{S}_1$ and $\mathbf{S}_2$ be the associated sample covariance matrices, i.e.,

$$\mathbf{S}_1 = \frac{1}{n_1} \sum_{i=1}^{n_1} \boldsymbol{\xi}_i \boldsymbol{\xi}_i^\top \quad \text{and} \quad \mathbf{S}_2 = \frac{1}{n_2} \sum_{j=1}^{n_2} \boldsymbol{\eta}_j \boldsymbol{\eta}_j^\top.$$

Then, the following so-called F-matrix generalizes the classical Fisher-statistic for the present p-dimensional case,

$$\mathbf{V_n} = \mathbf{S}_1\mathbf{S}_2^{-1}, \tag{4}$$

where $\mathbf{n} = (n_1, n_2)$ and $p < n_2$. Define

$$y_{n_1} = \frac{p}{n_1} \to y_1 \in (0, +\infty), \qquad y_{n_2} = \frac{p}{n_2} \to y_2 \in (0, 1), \tag{5}$$

$\mathbf{y_n} = (y_{n_1}, y_{n_2})$ and $\mathbf{y} = (y_1, y_2)$. Under suitable moment conditions, the ESD, $F_{\mathbf{n}}^{\mathbf{V_n}}$, of $\mathbf{V_n}$ has a LSD, $F_{\mathbf{y}}(x)$ which is given by (see [2], P. 79 or [22])

$$F_{\mathbf{y}}(\mathrm{d}x) = g_{\mathbf{y}}(x)\mathbb{1}_{[a,b]}(x)\mathrm{d}x + (1 - 1/y_1)\mathbb{1}_{[y_1>1]}\delta_0(\mathrm{d}x), \tag{6}$$

where $\delta_c(\cdot)$ denotes the Dirac point measure at c, and

$$h = \sqrt{y_1 + y_2 - y_1y_2}\ , \quad a = \frac{(1-h)^2}{(1-y_2)^2}\ , \quad b = \frac{(1+h)^2}{(1-y_2)^2}\ ,$$

$$g_{\mathbf{y}}(x) = \frac{(1-y_2)}{2\pi x(y_1 + y_2x)}\sqrt{(b-x)(x-a)}\ , \ \ a < x < b\ .$$

Define the empirical process $\widetilde{G}_{\mathbf{n}} = p[F_{\mathbf{n}}^{V_{\mathbf{n}}} - F_{\mathbf{y_n}}]$ and let $\widetilde{\mathcal{A}}$ be the set of analytic functions in an open region in the complex plane containing the interval $[a, b]$ which is the support of continuous part of the LSD $F_{\mathbf{y}}$ defined in (6). The following central limit theorem (CLT) of LSS for a high-dimensional F-matrix was established in Zheng [22], which will be applied in our work.

Theorem 2.1 *For each p, assume that* $(\xi_{ij_1}\colon i = 1, \ldots, p; j_1 = 1, \ldots, n_1)$ *and* $(\eta_{ij_2}\colon i = 1, \ldots, p; j_2 = 1, \ldots, n_2)$ *are i.i.d. and satisfy* $E(\xi_{11}) = E(\eta_{11}) = 0$, $E|\xi_{11}|^2 = E|\eta_{11}|^2 = 1$, $E|\xi_{11}|^4, E|\eta_{11}|^4 < \infty, y_{n_1} = p/n_1 \to y_1 \in (0, +\infty)$, *and* $y_{n_2} = p/n_2 \to y_2 \in (0, 1)$. *Furthermore, let* $f_1, \ldots, f_k \in \widetilde{\mathcal{A}}$. *Then, the random vector* $(\widetilde{G}_n(f_1), \ldots, \widetilde{G}_n(f_k))$ *weakly converges to a* k*-dimensional Gaussian* *vector with mean vector*

$$
\begin{aligned}
m(f_j) &= \lim_{r\downarrow 1} \frac{1}{4\pi \mathrm{i}} \oint_{|\zeta|=1} f_j(z(\zeta)) \left[\frac{1}{\zeta - r^{-1}} + \frac{1}{\zeta + r^{-1}} - \frac{2}{\zeta + y_2/hr} \right] \mathrm{d}\zeta \\
&\quad + \frac{\beta_x \cdot y_1 (1-y_2)^2}{2\pi \mathrm{i} \cdot h^2} \oint_{|\zeta|=1} f_j(z(\zeta)) \frac{1}{(\zeta + y_2/h)^3} \mathrm{d}\zeta \\
&\quad + \frac{\beta_y \cdot y_2 (1-y_2)}{2\pi \mathrm{i} \cdot h} \oint_{|\zeta|=1} f_j(z(\zeta)) \frac{\zeta + 1/h}{(\zeta + y_2/h)^3} \mathrm{d}\zeta, \qquad (7)
\end{aligned}
$$

and covariance function

$$
\begin{aligned}
cov(f_j, f_\ell) &= -\lim_{r\downarrow 1} \frac{1}{2\pi^2} \oint_{|\zeta_1|=1} \oint_{|\zeta_2|=1} \frac{f_j(z(\zeta_1)) f_\ell(z(\zeta_2))}{(\zeta_1 - r\zeta_2)^2} \mathrm{d}\zeta_1 \mathrm{d}\zeta_2 \\
&\quad - \frac{(\beta_x y_1 + \beta_y y_2)(1-y_2)^2}{4\pi^2 h^2} \oint_{|\zeta_1|=1} \frac{f_j\left(z(\zeta_1)\right)}{(\zeta_1 + y_2/h)^2} \mathrm{d}\zeta_1 \\
&\quad \times \oint_{|\zeta_2|=1} \frac{f_\ell\left(z(\zeta_2)\right)}{(\zeta_2 + y_2/h)^2} \mathrm{d}\zeta_2, \qquad (8)
\end{aligned}
$$

where $z(\zeta) = (1-y_2)^{-2}[1+h^2+2h\mathrm{Re}(\zeta)]$, $h = \sqrt{y_1 + y_2 - y_1 y_2}$, $\beta_x = E|\xi_{11}|^4 - 3$, *and* $\beta_y = E|\eta_{11}|^4 - 3$.

This CLT allows for both $\mathbf{n} = (n_1, n_2)$ and p approaching infinity. In the next section, based on the above CLT, we will develop a test statistic $T_{\mathbf{n}}$ and provide the limiting distribution of $T_{\mathbf{n}}$.

3. Formulation of the new test

Let $\mathbf{\Sigma}_1^{1/2}$ and $\mathbf{\Sigma}_2^{1/2}$ be two $p \times p$ positive definite matrices such that $\mathbf{\Sigma}_1 = (\mathbf{\Sigma}_1^{1/2})^2$ and $\mathbf{\Sigma}_2 = (\mathbf{\Sigma}_2^{1/2})^2$, respectively. Define $\boldsymbol{\xi}_i = \mathbf{\Sigma}_1^{-1/2}(\mathbf{x}_i - \boldsymbol{\mu}_1)$ and $\boldsymbol{\eta}_j = \mathbf{\Sigma}_2^{-1/2}(\mathbf{y}_j - \boldsymbol{\mu}_2)$. Note that all the random vectors $\{\mathbf{x}_i\}, \{\mathbf{y}_j\}$ and the covariance matrices $\mathbf{\Sigma}_1$ and $\mathbf{\Sigma}_2$ depend on the dimension p. However, we do not signify this dependence in notation for ease of statements. After standardizing, the arrays $\{\boldsymbol{\xi}_i\}_{i=1}^{n_1+1}$ and $\{\boldsymbol{\eta}_j\}_{j=1}^{n_2+1}$ contain i.i.d. variables with mean 0 and variance 1, for which we can apply Theorem 2.1.

Testing the null hypothesis specified by (2) is equivalent to testing

$$
H_0\colon \mathbf{\Sigma}_2^{-1/2} \mathbf{\Sigma}_1 \mathbf{\Sigma}_2^{-1/2} = \sigma^2 \mathbf{I}_p. \qquad (9)
$$

This is, in fact, a sphericity test. Srivastava [17] discussed this sphericity test for a single population and proposed a measure of this sphericity. As stated in Srivastava [17] (also see [7]), the above test is invariant for an orthogonal transformation $\mathbf{x} \to \mathbf{Gx}$. This test is also invariant under a scalar transformation $x \to cx$. Thus, without loss of generality, we assume that $\boldsymbol{\Sigma}_2^{-1/2}\boldsymbol{\Sigma}_1\boldsymbol{\Sigma}_2^{-1/2} = \mathrm{diag}(\lambda_1, \ldots, \lambda_p)$. It follows from the Cauchy–Schwarz inequality that

$$\left(\sum_{i=1}^{p} \lambda_i\right)^2 \leqslant p \left(\sum_{i=1}^{p} \lambda_i^2\right) ,$$

with equality holding if and only if $\lambda_1 = \cdots = \lambda_p = \lambda$, for all $i = 1, \ldots, p$ and some constant λ. Thus, following the idea of Srivastava [17], we suggest testing H_0: $\Psi = 1$ against H_1: $\Psi > 1$ with

$$\Psi = \frac{\left(\sum_{i=1}^{p} \lambda_i^2/p\right)}{\left(\sum_{i=1}^{p} \lambda_i/p\right)^2}.$$

Note that $\mathrm{tr}(\boldsymbol{\Sigma}_1\boldsymbol{\Sigma}_2^{-1}) = \sum_{i=1}^{p} \lambda_i$ and $\mathrm{tr}(\boldsymbol{\Sigma}_1\boldsymbol{\Sigma}_2^{-1}\boldsymbol{\Sigma}_1\boldsymbol{\Sigma}_2^{-1}) = \sum_{i=1}^{p} \lambda_i^2$; hence, we propose the following test statistic

$$T_{\mathbf{n}} = \frac{p^2 \mathrm{tr}(\hat{\boldsymbol{\Sigma}}_1\hat{\boldsymbol{\Sigma}}_2^{-1}\hat{\boldsymbol{\Sigma}}_1\hat{\boldsymbol{\Sigma}}_2^{-1})}{[\mathrm{tr}(\hat{\boldsymbol{\Sigma}}_1\hat{\boldsymbol{\Sigma}}_2^{-1})]^2} - p, \tag{10}$$

where $\hat{\boldsymbol{\Sigma}}_1$ and $\hat{\boldsymbol{\Sigma}}_2$ are unbiased sample covariance matrices. On the other hand, a scaled distance measure between $\sigma^{-2}\boldsymbol{\Sigma}_2^{-1/2}\boldsymbol{\Sigma}_1\boldsymbol{\Sigma}_2^{-1/2}$ and $\mathbf{I}_p$ can be defined by

$$\mathrm{tr}\left[\left(\frac{\sigma^{-2}\boldsymbol{\Sigma}_2^{-1/2}\boldsymbol{\Sigma}_1\boldsymbol{\Sigma}_2^{-1/2}}{(1/p)\mathrm{tr}(\sigma^{-2}\boldsymbol{\Sigma}_2^{-1/2}\boldsymbol{\Sigma}_1\boldsymbol{\Sigma}_2^{-1/2})} - \mathbf{I}_p\right)^2\right] = \frac{p^2 \mathrm{tr}(\boldsymbol{\Sigma}_1\boldsymbol{\Sigma}_2^{-1}\boldsymbol{\Sigma}_1\boldsymbol{\Sigma}_2^{-1})}{[\mathrm{tr}(\boldsymbol{\Sigma}_1\boldsymbol{\Sigma}_2^{-1})]^2} - p.$$

This scaled distance measure was also discussed by Nagao [13] for the sphericity test of a single population. Thus, it is reasonable to consider the test statistic (10).

Define $\mathbf{A} = \boldsymbol{\Sigma}_1^{-1/2}\mathbf{C}\boldsymbol{\Sigma}_1^{-1/2}$ and $\mathbf{B} = \boldsymbol{\Sigma}_2^{-1/2}\mathbf{D}\boldsymbol{\Sigma}_2^{-1/2}$, where

$$\mathbf{C} = \frac{1}{n_1}\sum_{i=1}^{N_1}(\mathbf{x}_i - \boldsymbol{\mu}_1)(\mathbf{x}_i - \boldsymbol{\mu}_1)^* \quad \text{and} \quad \mathbf{D} = \frac{1}{n_2}\sum_{j=1}^{N_2}(\mathbf{y}_j - \boldsymbol{\mu}_2)(\mathbf{y}_j - \boldsymbol{\mu}_2)^*.$$

Note that $\mathbf{V_n} = \mathbf{AB}^{-1}$ indeed forms a random F-matrix. Then, we define

$$\widetilde{T}_{\mathbf{n}} = \frac{p^2 \mathrm{tr}(\mathbf{CD}^{-1}\mathbf{CD}^{-1})}{[\mathrm{tr}(\mathbf{CD}^{-1})]^2} - p. \tag{11}$$

As $T_{\mathbf{n}}$ and $\widetilde{T}_{\mathbf{n}}$ have the same asymptotic distribution and CLT under H_0, in the next, we will consider the CLT of $\widetilde{T}_{\mathbf{n}}$.

Theorem 3.1 *Suppose that the conditions in Theorem 2.1 hold under H_0 and $T_{\mathbf{n}}$ is defined in (10), then, under H_0 and as $(n_1, n_2) \to \infty$,*

$$v_{T_{\mathbf{n}}}^{-1/2}[T_{\mathbf{n}} - \mu_{T_{\mathbf{n}}} - ph^2(1-y_2)^{-1}] \xrightarrow{\mathrm{D}} N(0,1) \ , \tag{12}$$

with

$$\mu_{T_{\mathbf{n}}} = (h^2 + y_2^2)(1-y_2)^{-2} + \beta_x y_1 + \beta_y y_2, \tag{13}$$

$$v_{T_{\mathbf{n}}} = 4h^2(h^2 + 2y_2^2)(1-y_2)^{-4} \ , \tag{14}$$

where $h = \sqrt{y_1 + y_2 - y_1 y_2}$.

PROOF. For convenience, we define $g_1(x) = x^2$ and $g_2(x) = x$. Let $\{\lambda_i\}_{i=1}^p$ be the eigenvalues of the F-matrix $\mathbf{V_n}$. Following Theorem 2.1, we have, as $(n_1, n_2) \to \infty$,

$$p \begin{pmatrix} (1/p)\sum_{j=1}^p (\lambda_j/\sigma^2)^2 - \int g_1(x) F_{\mathbf{y}}(\mathrm{d}x) \\ (1/p)\sum_{j=1}^p \lambda_j/\sigma^2 - \int g_2(x) F_{\mathbf{y}}(\mathrm{d}x) \end{pmatrix} \xrightarrow{\mathrm{D}} N(\boldsymbol{\mu}^{\mathrm{CLT}}, \boldsymbol{\Sigma}^{\mathrm{CLT}}),$$

where

$$\boldsymbol{\mu}^{\mathrm{CLT}} = \begin{pmatrix} m(g_1) \\ m(g_2) \end{pmatrix}, \quad \boldsymbol{\Sigma}^{\mathrm{CLT}} = \begin{pmatrix} v(g_1, g_1) & v(g_1, g_2) \\ v(g_1, g_2) & v(g_2, g_2) \end{pmatrix},$$

and $F_{\mathbf{y}}(\mathrm{d}x)$ is defined in (6).

Denote $\delta(\xi) = (1-y_2)^{-2}(1 + h^2 + 2h\mathrm{Re}\xi)$, then the components of $\boldsymbol{\mu}^{\mathrm{CLT}}$ are given by

$$\begin{aligned} m(g_j) &= \lim_{r\downarrow 1} \frac{1}{4\pi i} \oint_{|\xi|=1} g_j(\delta(\xi)) \left(\frac{1}{\xi - r^{-1}} + \frac{1}{\xi + r^{-1}} - \frac{2}{\xi + y_2/h} \right) \mathrm{d}\xi \\ &\quad + \frac{\beta_x \cdot y_1(1-y_2)^2}{2\pi i \cdot h^2} \oint_{|\xi|=1} g_j(\delta(\xi)) \frac{1}{(\xi + y_2/h)^3} \mathrm{d}\xi \\ &\quad + \frac{\beta_y \cdot y_2(1-y_2)}{2\pi i h} \oint_{|\xi|=1} g_j(\delta(\xi)) \frac{\xi + 1/h}{(\xi + y_2/h)^3} \mathrm{d}\xi, \quad j = 1, 2, \end{aligned}$$

and the elements of $\mathbf{\Sigma}^{\mathrm{CLT}}$ are given by

$$\begin{aligned} v(g_k, g_l) &= -\lim_{r\downarrow 1} \frac{1}{2\pi^2} \oint_{|\xi_1|=1} \oint_{|\xi_2|=1} \frac{g_k(\delta(\xi_1)) g_l(\delta(\xi_2))}{(\xi_1 - r\xi_2)^2} \mathrm{d}\xi_1 \mathrm{d}\xi_2 \\ &\quad - \frac{(\beta_x y_1 + \beta_y y_2)(1-y_2)^2}{4\pi^2 h^2} \oint_{|\xi_1|=1} \frac{g_k(\delta(\xi_1))}{(\xi_1 + y_2/h)^2} \mathrm{d}\xi_1 \\ &\quad \times \oint_{|\xi_2|=1} \frac{g_l(\delta(\xi_2))}{(\xi_2 + y_2/h)^2} \mathrm{d}\xi_2 \,, \qquad k, l \in \{1, 2\}, \end{aligned}$$

with $\beta_x = E|\xi_{11}|^4 - 3$ and $\beta_y = E|\eta_{11}|^4 - 3$.

By the Residue theorem, we have

$$m(g_1) = \frac{h^2 - y_2^2 + 2y_2(1+h^2)}{(1-y_2)^4} + \frac{\beta_x y_1 + 3\beta_y y_2}{(1-y_2)^2} + \frac{2h^2 \beta_y y_2}{(1-y_2)^3} \,, \tag{15}$$

$$m(g_2) = \frac{y_2}{(1-y_2)^2} + \frac{\beta_y y_2}{(1-y_2)} \,, \tag{16}$$

$$v(g_1, g_1) = \frac{4h^2(2h^4 + 5h^2 + 2)}{(1-y_2)^8} + \frac{4(\beta_x y_1 + \beta_y y_2)(1 + h^2 - y_2)^2}{(1-y_2)^6} \,, \tag{17}$$

$$v(g_1, g_2) = \frac{4h^2(1+h^2)}{(1-y_2)^6} + \frac{2(\beta_x y_1 + \beta_y y_2)(1 + h^2 - y_2)}{(1-y_2)^4} \,, \tag{18}$$

$$v(g_2, g_2) = \frac{2h^2}{(1-y_2)^4} + \frac{\beta_x y_1 + \beta_y y_2}{(1-y_2)^2} \,. \tag{19}$$

Moreover, let $b_0 = \int_0^\infty x^2 F_{\mathbf{y}}(\mathrm{d}x)$ and $b_1 = \int_0^\infty x F_{\mathbf{y}}(\mathrm{d}x)$. Then, we have

$$b_0 = \frac{1 + h^2 - y_2}{(1-y_2)^3} \,, \qquad b_1 = \frac{1}{1 - y_2} \,. \tag{20}$$

The details of the above derivations are postponed to the Appendix.

Define $g(x_1, x_2) = x_1/x_2^2$ and $\mathbf{a}_0 = (1/b_1^2, -2b_0/b_1^3)^\top$. Applying the Delta method (see Casella and Berger [3]), we obtain, as $(n_1, n_2) \to \infty$,

$$p\left[\frac{(1/p)\sum_{j=1}^{p}(\lambda_j/\sigma^2)^2}{(1/p^2)(\sum_{j=1}^{p}\lambda_j/\sigma^2)^2} - \frac{b_0}{b_1^2}\right] \xrightarrow{\mathrm{D}} N\left(\mathbf{a}_0^\top \boldsymbol{\mu}^{\mathrm{CLT}},\ \mathbf{a}_0^\top \boldsymbol{\Sigma}^{\mathrm{CLT}} \mathbf{a}_0\right),$$

that is

$$T_{\mathbf{n}} - p(b_0/b_1^2 - 1) \xrightarrow{\mathrm{D}} N\left(\mathbf{a}_0^\top \boldsymbol{\mu}^{\mathrm{CLT}},\ \mathbf{a}_0^\top \boldsymbol{\Sigma}^{\mathrm{CLT}} \mathbf{a}_0\right). \tag{21}$$

Furthermore, substituting the equations (15)–(20) into (21) and after some manipulations, we obtain

$$T_{\mathbf{n}} - ph^2(1-y_2)^{-1} \xrightarrow{\mathrm{D}} N(\mu_{T_{\mathbf{n}}}, \upsilon_{T_{\mathbf{n}}})\ ,$$

where $\mu_{T_{\mathbf{n}}}$ and $\upsilon_{T_{\mathbf{n}}}$ are given in (13) and (14). This is equivalent to the statement of (12). The proof is completed.

Theorem 3.2 *Under the assumptions of Theorem 3.1 and when* $\|\boldsymbol{\Sigma}_1\|$ *and* $\|\boldsymbol{\Sigma}_2\|$ *are bounded,* β_x *and* β_y *have consistent estimators* $\hat{\beta}_x$ *and* $\hat{\beta}_y$ *given by*

$$\hat{\beta}_x = \frac{\frac{1}{p(N_1-1)}\sum_{i=1}^{N_1}\left(\mathbf{x}_i^\top\mathbf{x}_i - \frac{1}{N_1}\sum_{i=1}^{N_1}\mathbf{x}_i^\top\mathbf{x}_i\right)^2 - \frac{1}{pN_1(N_1-1)}\left[\sum_{i=1}^{N_1}\left(\mathbf{x}_i^\top\mathbf{x}_i - \frac{1}{N_1}\sum_{j=1}^{N_1}\mathbf{x}_j^\top\mathbf{x}_j\right)\right]^2}{\frac{1}{p}\sum_{k=1}^{p}\left[\frac{1}{N_1-1}\sum_{i=1}^{N_1}\left(x_{ik} - \frac{1}{N_1}\sum_{j=1}^{N_1}x_{jk}\right)^2\right]^2} - 2$$

and

$$\hat{\beta}_y = \frac{\frac{1}{p(N_2-1)}\sum_{i=1}^{N_2}\left(\mathbf{y}_i^\top\mathbf{y}_i - \frac{1}{N_2}\sum_{i=1}^{N_2}\mathbf{y}_i^\top\mathbf{y}_i\right)^2 - \frac{1}{pN_2(N_2-1)}\left[\sum_{i=1}^{N_2}\left(\mathbf{y}_i^\top\mathbf{y}_i - \frac{1}{N_2}\sum_{j=1}^{N_2}\mathbf{y}_j^\top\mathbf{y}_j\right)\right]^2}{\frac{1}{p}\sum_{k=1}^{p}\left[\frac{1}{N_2-1}\sum_{i=1}^{N_2}\left(y_{ik} - \frac{1}{N_2}\sum_{j=1}^{N_2}y_{jk}\right)^2\right]^2} - 2$$

where $\mathbf{x}_i = (x_{i1}, \ldots, x_{ip})^\top$ *and* $\mathbf{y}_i = (y_{i1}, \ldots, y_{ip})^\top$.

PROOF. We have $\mathbf{x} = (X_1, \dots, X_p)^\top = \mathbf{\Sigma}_1^{\frac{1}{2}} \boldsymbol{\xi}$, where $\mathbf{\Sigma}_1^{\frac{1}{2}} = (a_{ij})$, $\mathbf{\Sigma} = (\sigma_{kk})$ and $\boldsymbol{\xi} = (\xi_1, \dots, \xi_p)^\top$. Hence,

$$\begin{aligned}
E(X_k^4) &= E(a_{k1}\xi_1 + \cdots + a_{kp}\xi_p)^4 \\
&= \sum_{j=1}^p a_{kj}^4 \cdot (E\xi_1^4 - 1) + \left(\sum_{j=1}^p a_{kj}^2 \right)^2, \\
E(X_k^2 X_l^2) &= \sum_{j=1}^p a_{kj}^2 a_{lj}^2 \cdot (E\xi_1^4 - 1) + \left(\sum_{j=1}^p a_{kj}^2 \right) \left(\sum_{j=1}^p a_{lj}^2 \right), \\
E(X_1^2 + \cdots + X_p^2)^2 &= \sum_{i=1}^p E(X_i^4) + 2 \sum_{k \neq l} E(X_k^2 X_l^2) \\
&= \sum_{j=1}^p \left(\sum_{k=1}^p a_{kj}^2 \right)^2 \cdot (E\xi_1^4 - 1) + \left(\sum_{j=1}^p \sum_{k=1}^p a_{kj}^2 \right)^2,
\end{aligned}$$

where

$$E(X_1^2 + \cdots + X_p^2)^2 = E(\mathbf{x}^\top \mathbf{x})^2, \quad \sigma_{kk} = \sum_{k=1}^p a_{kj}^2, \quad \sum_{j=1}^p \sum_{k=1}^p a_{kj}^2 = \sum_{k=1}^p \sigma_{kk}.$$

That is,

$$\frac{1}{p} E(\mathbf{x}^\top \mathbf{x})^2 = \frac{1}{p} \sum_{j=1}^p \sigma_{kk}^2 \cdot (E\xi_1^4 - 1) + \frac{1}{p} \left(\sum_{j=1}^p \sigma_{kk} \right)^2.$$

When $\|\mathbf{\Sigma}\|$ is bounded, then we have

$$\begin{aligned}
E|\hat{\sigma}_{kk}^2 - \sigma_{kk}^2| &= E|\hat{\sigma}_{kk} + \sigma_{kk}| \cdot |\hat{\sigma}_{kk} - \sigma_{kk}| \\
&\leqslant \sqrt{E(\hat{\sigma}_{kk} + \sigma_{kk})^2 \cdot E(\hat{\sigma}_{kk} - \sigma_{kk})^2} = o(1)
\end{aligned}$$

uniformly for k. Then we have

$$E \left| \frac{1}{p} \sum_{k=1}^p \hat{\sigma}_{kk}^2 - \frac{1}{p} \sum_{k=1}^p \sigma_{kk}^2 \right| \leqslant \frac{1}{p} \sum_{k=1}^p E|\hat{\sigma}_{kk}^2 - \sigma_{kk}^2| = o(1).$$

That is

$$\frac{1}{p}\sum_{k=1}^{p}\hat{\sigma}_{kk}^2 - \frac{1}{p}\sum_{k=1}^{p}\sigma_{kk}^2 \xrightarrow{\text{P}} 0.$$

Let the estimator of $\frac{1}{p}E(\mathbf{x}^\top\mathbf{x})^2 - \frac{1}{p}\left(\sum_{j=1}^{p}\sigma_{kk}\right)^2$ be

$$\begin{aligned}
& \frac{1}{N_1 p}\sum_{i=1}^{N_1}(\mathbf{x}_i^\top\mathbf{x}_i)^2 - \frac{1}{p}\frac{\sum_{i<j}\mathbf{x}_i^\top\mathbf{x}_i \cdot \mathbf{x}_j^\top\mathbf{x}_j}{\binom{N_1}{2}} \\
= \;& \frac{1}{p(N_1-1)}\sum_i(\mathbf{x}_i^\top\mathbf{x}_i)^2 - \frac{1}{pN_1(N_1-1)}\left(\sum_i \mathbf{x}_i^\top\mathbf{x}_i\right)^2,
\end{aligned}$$

where

$$\begin{aligned}
\frac{1}{p(N_1-1)}\sum_i(\mathbf{x}_i^\top\mathbf{x}_i)^2 &= \frac{1}{p(N_1-1)}\sum_i(\mathbf{x}_i^\top\mathbf{x}_i - \text{tr}\boldsymbol{\Sigma} + \text{tr}\boldsymbol{\Sigma})^2 \\
&= \frac{N_1}{p(N_1-1)}(\text{tr}\boldsymbol{\Sigma})^2 + \frac{2}{p}\text{tr}\boldsymbol{\Sigma}\cdot\frac{1}{N_1-1}\sum_i(\mathbf{x}_i^\top\mathbf{x}_i - \text{tr}\boldsymbol{\Sigma}) \\
&\quad + \frac{1}{p(N_1-1)}\sum_i(\mathbf{x}_i^\top\mathbf{x}_i - \text{tr}\boldsymbol{\Sigma})^2, \\
\frac{1}{pN_1(N_1-1)}\left(\sum_i \mathbf{x}_i^\top\mathbf{x}_i\right)^2 &= \frac{1}{pN_1(N_1-1)}\left(\sum_i \mathbf{x}_i^\top\mathbf{x}_i - n\text{tr}\boldsymbol{\Sigma} + n\text{tr}\boldsymbol{\Sigma}\right)^2 \\
&= \frac{N_1}{p(N_1-1)}(\text{tr}\boldsymbol{\Sigma})^2 + \frac{2}{p}\text{tr}\boldsymbol{\Sigma}\cdot\frac{1}{N_1-1}\sum_i(\mathbf{x}_i^\top\mathbf{x}_i - \text{tr}\boldsymbol{\Sigma}) \\
&\quad + \frac{[\sum_{i=1}^{N_1}(\mathbf{x}_i^\top\mathbf{x}_i - \text{tr}\boldsymbol{\Sigma})]^2}{pN_1(N_1-1)}.
\end{aligned}$$

Then, we have

$$\begin{aligned}
& \frac{1}{p(N_1-1)}\sum_i(\mathbf{x}_i^\top\mathbf{x}_i)^2 - \frac{1}{pN_1(N_1-1)}\left(\sum_i \mathbf{x}_i^\top\mathbf{x}_i\right)^2 - \left(\frac{E(\mathbf{x}^\top\mathbf{x})}{p} - \frac{(\mathrm{tr}\boldsymbol{\Sigma})^2}{p}\right) \\
= & \frac{1}{pN_1}\sum_i(\mathbf{x}_i^\top\mathbf{x}_i - \mathrm{tr}\boldsymbol{\Sigma})^2 - \frac{\sum_{i\neq j}(\mathbf{x}_i^\top\mathbf{x}_i - \mathrm{tr}\boldsymbol{\Sigma})\cdot(\mathbf{x}_j^\top\mathbf{x}_j - \mathrm{tr}\boldsymbol{\Sigma})]}{pN_1(N_1-1)} - \frac{E(\mathbf{x}^\top\mathbf{x} - \mathrm{tr}\boldsymbol{\Sigma})^2}{p} \\
= & \frac{1}{pN_1}\sum_i(\mathbf{x}_i^\top\mathbf{x}_i - \mathrm{tr}\boldsymbol{\Sigma})^2 - \frac{E(\mathbf{x}^\top\mathbf{x} - \mathrm{tr}\boldsymbol{\Sigma})^2}{p} - \frac{\sum_{i\neq j}(\mathbf{x}_i^\top\mathbf{x}_i - \mathrm{tr}\boldsymbol{\Sigma})\cdot(\mathbf{x}_j^\top\mathbf{x}_j - \mathrm{tr}\boldsymbol{\Sigma})]}{pN_1(N_1-1)} \\
\stackrel{\mathrm{P}}{\to} & \ 0.
\end{aligned}$$

Then we obtain the consistent estimator of $E(\xi_1^4)$ as follows

$$\frac{\frac{1}{p(N_1-1)}\sum_{i=1}^{N_1}(\mathbf{x}_i^\top\mathbf{x}_i)^2 - \frac{1}{pN_1(N_1-1)}\left(\sum_{i=1}^{N_1}\mathbf{x}_i^\top\mathbf{x}_i\right)^2}{\frac{1}{p}\sum_{k=1}^{p}(\frac{1}{N_1}\sum_{i=1}^{N_1}x_{ik}^2)^2} + 1,$$

where $\mathbf{x}_i = (x_{i1}, \ldots, x_{ip})^\top$.

When $E(\mathbf{x})$ is unknown, we also obtain the consistent estimators of $E(\xi_1^2)$ as follows:

$$\frac{\frac{1}{p(N_1-1)}\sum_{i=1}^{N_1}\left(\mathbf{x}_i^\top\mathbf{x}_i - \frac{1}{N_1}\sum_{i=1}^{N_1}\mathbf{x}_i^\top\mathbf{x}_i\right)^2 - \frac{1}{pN_1(N_1-1)}\left[\sum_{i=1}^{N_1}\left(\mathbf{x}_i^\top\mathbf{x}_i - \frac{1}{N_1}\sum_{j=1}^{N_1}\mathbf{x}_j^\top\mathbf{x}_j\right)\right]^2}{\frac{1}{p}\sum_{k=1}^{p}\left[\frac{1}{N_1-1}\sum_{i=1}^{N_1}\left(x_{ik} - \frac{1}{N_1}\sum_{j=1}^{N_1}x_{jk}\right)^2\right]^2} + 1.$$

Then, the consistent estimator of $\beta_x = E(\xi_1^4) - 3$ is as follows

$$\hat{\beta}_x = \frac{\frac{1}{p(N_1-1)} \sum_{i=1}^{N_1} \left(\mathbf{x}_i^\top \mathbf{x}_i - \frac{1}{N_1} \sum_{i=1}^{N_1} \mathbf{x}_i^\top \mathbf{x}_i \right)^2 - \frac{1}{pN_1(N_1-1)} \left[\sum_{i=1}^{N_1} \left(\mathbf{x}_i^\top \mathbf{x}_i - \frac{1}{N_1} \sum_{j=1}^{N_1} \mathbf{x}_j^\top \mathbf{x}_j \right) \right]^2}{\frac{1}{p} \sum_{k=1}^{p} \left[\frac{1}{N_1-1} \sum_{i=1}^{N_1} \left(x_{ik} - \frac{1}{N_1} \sum_{j=1}^{N_1} x_{jk} \right)^2 \right]^2} - 2.$$

The derivation of $\hat{\beta}_y$ is similar. The proof is completed.

4. Simulation studies

4.1. Large-dimensional Gaussian data

In this section, we perform some simulation studies to evaluate the efficiency of the new test (LZ test) proposed in Section 3 for testing hypothesis (2) with large-dimensional Gaussian data. For comparison, we also conduct the Bartlett test (Bartlett adjusted likelihood ratio test, see Eriksen [5] or Flury [8]) and the Wald test (see Schott [16]). We generate $\mathbf{x}_1, \ldots, \mathbf{x}_{n_1+1}$ from $N_p(\mathbf{0}, \sigma^2 \mathbf{I}_p)$ and $\mathbf{y}_1, \ldots, \mathbf{y}_{n_2+1}$ from $N_p(\mathbf{0}, \mathbf{\Sigma}_2)$ with $\mathbf{\Sigma}_2 = (\rho^{|i-j|})$ for $\sigma^2 = 0.5, 1.0, 2.0$ and $\rho = 0.0, 0.4, 0.8$, respectively. We intend to test the null hypothesis: H_0: $\mathbf{\Sigma}_1 = \sigma^2 \mathbf{\Sigma}_2$. Notice that $\rho = 0$ corresponds to the realized size (Type-I error) of the test. The nominal test level is set as $\alpha = 0.05$. We simulate 20,000 independent replicates for different values of (p, n_1, n_2) under the assumption of $p < n_2$ and compare the above three test. The simulation results are summarized in Table 1. It needs to point out that the proposed new test allows for $p \geqslant n_1$, which means that the observations from one of two groups can be fewer than the dimension. Table 4 displays the summary of Gaussian data as well as non-gaussian data for the case of $p > n_1$.

From Table 1, we can see that, as the dimension p increases, both Wald test and Bartlett test lead to an increasing test size while the proposed LZ test is so stable. Especially, when the dimension $p > 600$, both Wald test and Bartlett test are so worse which almost always reject the null hypothesis; in contrast, our proposed test remains even accurate.

Figure 1 shows the QQ-plots for the 1,000 observed values of the LZ test $T_\mathbf{n}$ with $\rho = 0.0$. The values of (p, n_1, n_2) are chosen as $(320, 640, 640)$ and $(500, 50, 1000)$. In both cases, the normality result appears to be satisfied by the

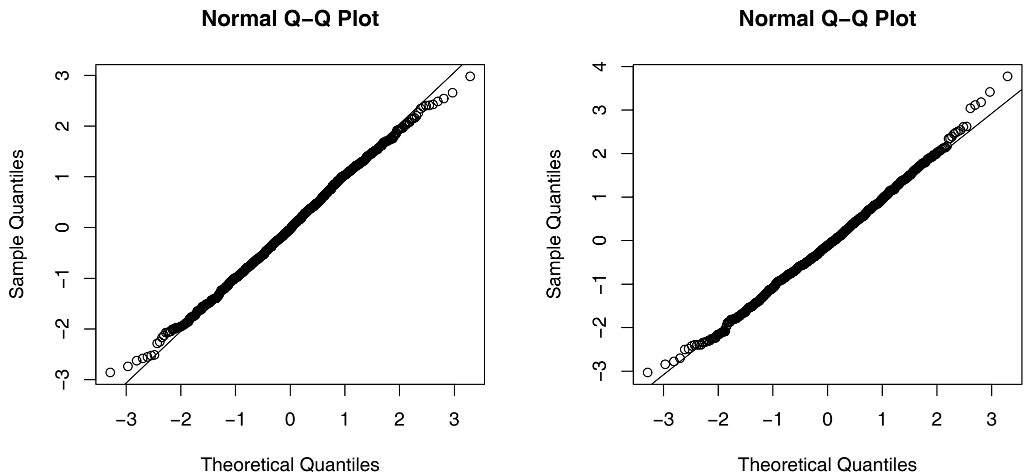

Figure 1: Normal QQ-Plot for T_n as in (10) under H_0 based on 1000 replicates. The left is for the case of $p < n_1$; the right is for the case of $p \geqslant n_1$.

QQ-plots for large (p, n_1, n_2) which validates our theoretical asymptotic normality results. We also check the QQ-plots of $T_{\mathbf{n}}$ under the alternative hypothesis (i.e. $\rho > 0$). The similar normality property appears to be satisfied, but for the restriction of space, we ignore the QQ-plots at here.

4.2. Large-dimensional non-Gaussian data

In fact, Theorem 3.1 is valid for general population distributions with finite fourth moments. To see this, two distributions are considered. One is the gamma distribution. We generate $\mathbf{x}_i = \sigma \mathbf{z}_i^{(1)}$, $i = 1, \ldots, n_1 + 1$ and $\mathbf{y}_j = \mathbf{\Gamma} \mathbf{z}_j^{(2)}$, $j = 1, \ldots, n_2 + 1$ with $\mathbf{z}_i^{(1)} = (Z_{i1}^{(1)}, ..., Z_{ip}^{(1)})^\top$ and $\mathbf{z}_j^{(2)} = (Z_{j1}^{(2)}, \ldots, Z_{jp}^{(2)})^\top$ consisting of identical and independent standardized gamma(4, 0.5) random variables so that they have mean 0 and variance 1, and $\sigma = (\sigma^2)^{1/2}$ and $\mathbf{\Gamma}_{p \times p} = (\rho^{|i-j|})^{1/2}$ for $\sigma^2 = 0.5, 1.0, 2.0$ and $\rho = 0.0, 0.4, 0.8$, respectively. The nominal test level is set as $\alpha = 0.05$. We simulate 20,000 independent replications for different values of (p, n_1, n_2) and compute the estimated significance level of the LZ test. The simulation Results are summarized in Table 2.

Another multivariate distribution considered is a mixture of multivariate normal distributions. We generate $\mathbf{z}_i^{(1)}$, $i = 1, \ldots, n_1 + 1$ and $\mathbf{z}_j^{(2)}$, $j = 1, \ldots, n_2 + 1$ with each component is i.i.d. from $0.9N(0, 1) + 0.1N(0, 3^2)$. Let $\mathbf{x}_i = \sigma \mathbf{z}_i^{(1)}$ and

$\mathbf{y}_j = \mathbf{\Gamma z}_j^{(2)}$ with the same values of σ and $\mathbf{\Gamma}$ as in the case of gamma distribution. The simulation results are summarized in Table 3.

From Tables 2 and 3, we can see that, our proposed test is still valid for non-Gaussian data. We also studied the QQ-plots of $T_{\mathbf{n}}$ based on 1,000 replicates under the null and alternative hypotheses for the gamma data and mixture of multivariate normal data. The normality property appears to be satisfied by the QQ-plots for large (p, n_1, n_2).

4.3. More simulations on the LZ test

In this section, we continue the simulations of Section 4.1 and 4.2 but with another different set of values for (p, n_1, n_2). We set $p = 10, 25, 50, 100$, $n_1 = 2p$, and $n_2 = p + 5, p + 10, \ldots$. The results are displayed in Table 5 and Table 6. Obviously, when p is too small or when p/n_2 is close to 1, the LZ test is not so efficient. Therefore, we recommend the practitioners to be careful to use the LZ test when p is small or when p/n_2 is close to 1.

5. Conclusions and future work

In this article, we developed a new test for comparing the proportionality of large-dimensional covariance matrices from two different population with finite fourth-moments. Its asymptotic normality is established based on modern random matrix theory as the dimension increases proportionally with the sample sizes. Simulation studies show that our proposed method is valid for both Gaussian data and non-Gaussian data. Our proposed method requires that $p < n_2$ but allows for $p \geqslant n_1$. Generalizing our proposed method to the case of $p \geqslant \max(n_1, n_2)$ will be the future work.

6. Appendix

Derivations of (14)

Notice that $0 < y_2, \dfrac{y_2}{h} < 1$. Applying Cauchy integration formula, for any given $r > 1$, we have

$$\oint_{|\xi|=1} \left(\frac{1}{\xi - r^{-1}} + \frac{1}{\xi + r^{-1}} - \frac{2}{\xi + y_2/h} \right) \mathrm{d}\xi = 0 \,. \tag{22}$$

We know that $Re\xi = (\xi + \bar{\xi})/2$ and $\bar{\xi} = \xi^{-1}$ for any ξ with $|\xi| = 1$. Then for $g_1(x) = x^2$ and $\delta(\xi) = (1-y_2)^{-2}(1 + h^2 + 2hRe\xi)$, in the formula

$$\begin{aligned} m(g_1) = &\lim_{r\downarrow 1} \frac{1}{4\pi \mathrm{i}} \oint_{|\xi|=1} g_1(\delta(\xi)) \left(\frac{1}{\xi - r^{-1}} + \frac{1}{\xi + r^{-1}} - \frac{2}{\xi + y_2/h} \right) \mathrm{d}\xi \\ &+ \frac{\beta_x \cdot y_1(1-y_2)^2}{2\pi \mathrm{i} \cdot h^2} \oint_{|\xi|=1} g_1(\delta(\xi)) \frac{1}{(\xi + y_2/h)^3} \mathrm{d}\xi \\ &+ \frac{\beta_y \cdot y_2(1-y_2)}{2\pi \mathrm{i} \cdot h} \oint_{|\xi|=1} g_1(\delta(\xi)) \frac{\xi + 1/h}{(\xi + y_2/h)^3} \mathrm{d}\xi \end{aligned}$$

we have

$$\begin{aligned} &\lim_{r\downarrow 1} \oint_{|\xi|=1} g_1(\delta(\xi)) \left(\frac{1}{\xi - r^{-1}} + \frac{1}{\xi + r^{-1}} - \frac{2}{\xi + y_2/h} \right) \mathrm{d}\xi \\ &= \lim_{r\downarrow 1} \frac{1}{(1-y_2)^4} \left\{ \oint_{|\xi|=1} [2h^2 + (1 + h^2 + h\xi)^2] \left(\frac{1}{\xi - r^{-1}} + \frac{1}{\xi + r^{-1}} - \frac{2}{\xi + y_2/h} \right) \mathrm{d}\xi \right. \\ &\quad \left. + \oint_{|\xi|=1} \left[\frac{h}{\xi^2} + \frac{2h(1+h^2)}{\xi} \right] \left(\frac{1}{\xi - r^{-1}} + \frac{1}{\xi + r^{-1}} - \frac{2}{\xi + y_2/h} \right) \mathrm{d}\xi \right\} \\ &= \frac{2\pi \mathrm{i}}{(1-y_2)^4} \cdot \left[2(h^2 - y_2^2 + 2y_2(1+h^2)) + 0 + 0 \right] \\ &= \frac{4\pi \mathrm{i}}{(1-y_2)^4} \cdot [h^2 - y_2^2 + 2y_2(1+h^2)] \ , \end{aligned}$$

$$\begin{aligned} \oint_{|\xi|=1} g_1(\delta(\xi)) \frac{1}{(\xi + y_2/h)^3} \mathrm{d}\xi &= \frac{1}{(1-y_2)^4} \left\{ \oint_{|\xi|=1} \frac{2h^2 + (1 + h^2 + h\xi)^2}{(\xi + y_2/h)^3} \mathrm{d}\xi \right. \\ &\quad \left. + \oint_{|\xi|=1} \left[\frac{h}{\xi^2} + \frac{2h(1+h^2)}{\xi} \right] \frac{1}{(\xi + y_2/h)^3} \mathrm{d}\xi \right\} \\ &= 2\pi \mathrm{i} \cdot \left[\frac{h^2}{(1-y_2)^4} + 0 \right] \\ &= 2\pi \mathrm{i} \cdot \frac{h^2}{(1-y_2)^4} \ , \end{aligned}$$

and

$$\begin{aligned}
\oint_{|\xi|=1} g_1(\delta(\xi))\frac{\xi+1/h}{(\xi+y_2/h)^3}\mathrm{d}\xi &= \frac{1}{(1-y_2)^4}\left\{\oint_{|\xi|=1}[2h^2+(1+h^2+h\xi)^2]\frac{(\xi+1/h)}{(\xi+y_2/h)^3}\mathrm{d}\xi\right. \\
&\quad \left. +\oint_{|\xi|=1}\left[\frac{h}{\xi^2}+\frac{2h(1+h^2)}{\xi}\right]\frac{(\xi+1/h)}{(\xi+y_2/h)^3}\mathrm{d}\xi\right\} \\
&= 2\pi\mathrm{i}\cdot\left[\frac{h(3+2h^2-3y_2)}{(1-y_2)^4}+0\right] \\
&= 2\pi\mathrm{i}\cdot\frac{h(3+2h^2-3y_2)}{(1-y_2)^4}\ .
\end{aligned}$$

Thus, we obtain

$$m(g_1)=\frac{h^2-y_2^2+2y_2(1+h^2)}{(1-y_2)^4}+\frac{\beta_x y_1+3\beta_y y_2}{(1-y_2)^2}+\frac{2h^2\beta_y y_2}{(1-y_2)^3}\ .$$

Derivations of (15)

For $g_2(x)=x$ and $\delta(\xi)=(1-y_2)^{-2}(1+h^2+2hRe\xi)$, in the formula

$$\begin{aligned}
m(g_2) &= \lim_{r\downarrow 1}\frac{1}{4\pi\mathrm{i}}\oint_{|\xi|=1} g_2(\delta(\xi))\left(\frac{1}{\xi-r^{-1}}+\frac{1}{\xi+r^{-1}}-\frac{2}{\xi+y_2/h}\right)\mathrm{d}\xi \\
&\quad +\frac{\beta_x\cdot y_1(1-y_2)^2}{2\pi\mathrm{i}\cdot h^2}\oint_{|\xi|=1} g_2(\delta(\xi))\frac{1}{(\xi+y_2/h)^3}\mathrm{d}\xi \\
&\quad +\frac{\beta_y\cdot y_2(1-y_2)}{2\pi\mathrm{i}\cdot h}\oint_{|\xi|=1} g_2(\delta(\xi))\frac{\xi+1/h}{(\xi+y_2/h)^3}\mathrm{d}\xi \\
&= \lim_{r\downarrow 1}\frac{1}{4\pi\mathrm{i}\cdot(1-y_2)^2}\oint_{|\xi|=1}(1+h^2+h\xi+h\xi^{-1})\left(\frac{1}{\xi-r^{-1}}+\frac{1}{\xi+r^{-1}}-\frac{2}{\xi+y_2/h}\right)\mathrm{d}\xi \\
&\quad +\frac{\beta_x\cdot y_1}{2\pi\mathrm{i}\cdot h^2}\oint_{|\xi|=1}(1+h^2+h\xi+h\xi^{-1})\frac{1}{(\xi+y_2/h)^3}\mathrm{d}\xi \\
&\quad +\frac{\beta_y\cdot y_2}{2\pi\mathrm{i}\cdot h(1-y_2)}\oint_{|\xi|=1}(1+h^2+h\xi+h\xi^{-1})\frac{\xi+1/h}{(\xi+y_2/h)^3}\mathrm{d}\xi \\
&= \frac{1}{4\pi\mathrm{i}\cdot(1-y_2)^2}\cdot 2\pi\mathrm{i}\cdot(2y_2+0)+\frac{\beta_x y_1}{h^2}\cdot 0+\frac{\beta_y y_2}{2\pi\mathrm{i}\cdot h(1-y_2)}\cdot 2\pi\mathrm{i}\cdot(h+0) \\
&= \frac{y_2}{(1-y_2)^2}+\frac{y_2\beta_y}{(1-y_2)}\ .
\end{aligned}$$

Derivations of (16)

For $g_1(x) = x^2$ and $\delta(\xi) = (1-y_2)^{-2}(1+h^2+2hRe\xi)$, in the formula,

$$\begin{aligned} v(g_1,g_1) &= -\lim_{r\downarrow 1}\frac{1}{2\pi^2}\oint_{|\xi_1|=1}\oint_{|\xi_2|=1}\frac{g_1(\delta(\xi_1))g_1(\delta(\xi_2))}{(\xi_1-r\xi_2)^2}\mathrm{d}\xi_1\mathrm{d}\xi_2 \\ &\quad -\frac{(\beta_x y_1+\beta_y y_2)(1-y_2)^2}{4\pi^2h^2}\left[\oint_{|\xi_1|=1}\frac{g_1(\delta(\xi_1))}{(\xi_1+y_2/h)^2}\mathrm{d}\xi_1\right]^2 . \end{aligned}$$

we have

$$\begin{aligned} &\lim_{r\downarrow 1}\oint_{|\xi_1|=1}\oint_{|\xi_2|=1}\frac{g_1(\delta(\xi_1))g_1(\delta(\xi_2))}{(\xi_1-r\xi_2)^2}\mathrm{d}\xi_1\mathrm{d}\xi_2 \\ &\quad=\lim_{r\downarrow 1}\frac{1}{(1-y_2)^8}\oint_{|\xi_1|=1}(1+h^2+h\xi_1+h\xi_1^{-1})^2\left[\oint_{|\xi_2|=1}\frac{(1+h^2+h\xi_2+h\xi_2^{-1})^2}{(\xi_1-r\xi_2)^2}\mathrm{d}\xi_2\right]\mathrm{d}\xi_1 \\ &\quad=\lim_{r\downarrow 1}\frac{1}{(1-y_2)^8}\oint_{|\xi_1|=1}(1+h^2+h\xi_1+h\xi_1^{-1})^2\left[\oint_{|\xi_2|=1}\frac{(1+h^2+h\xi_2+h\xi_2^{-1})^2}{r^2(\xi_2-\xi_1/r)^2}\mathrm{d}\xi_2\right]\mathrm{d}\xi_1 \\ &\quad=\lim_{r\downarrow 1}\frac{1}{(1-y_2)^8}\oint_{|\xi_1|=1}(1+h^2+h\xi_1+h\xi_1^{-1})^2\cdot 2\pi\mathrm{i}\cdot 2h(1+h^2+h\xi_1/r)r^2\mathrm{d}\xi_1 \\ &\quad=\frac{2\pi\mathrm{i}\cdot 2h}{(1-y_2)^8}\oint_{|\xi_1|=1}\left[(1+h^2+h\xi_1)^3+\frac{h^2(1+h^2+h\xi_1)}{\xi_1^2}+\frac{2h(1+h^2+h\xi_1)^2}{\xi_1}\right]\mathrm{d}\xi_1 \\ &\quad=-\frac{4\pi^2\cdot 2h^2(2h^4+5h^2+2)}{(1-y_2)^8} , \end{aligned}$$

and

$$\begin{aligned} &\oint_{|\xi_1|=1}\frac{g_1(\delta(\xi_1))}{(\xi_1+y_2/h)^2}\mathrm{d}\xi_1 \\ &\quad=\frac{1}{(1-y_2)^4}\oint_{|\xi_1|=1}\frac{(1+h^2+h\xi_1+h\xi_1^{-1})^2}{(\xi_1+y_2/h)^2}\mathrm{d}\xi_1 \\ &\quad=\frac{1}{(1-y_2)^4}\left[\oint_{|\xi_1|=1}\frac{2h^2+(1+h^2+h\xi_1)^2}{(\xi_1+y_2/h)^2}\mathrm{d}\xi_1+\oint_{|\xi_1|=1}\frac{2h(1+h^2)\xi_1^{-1}+h^2\xi_1^{-2}}{(\xi_1+y_2/h)^2}\mathrm{d}\xi_1\right] \\ &\quad=\frac{2\pi\mathrm{i}}{(1-y_2)^4}[2h(1+h^2-y_2)+0] \\ &\quad=\frac{2\pi\mathrm{i}}{(1-y_2)^4}\cdot 2h(1+h^2-y_2) , \end{aligned}$$

So, we derived that

$$v(g_1, g_1) = \frac{4h^2(2h^4 + 5h^2 + 2)}{(1 - y_2)^8} + \frac{4(\beta_x y_1 + \beta_y y_2)(1 + h^2 - y_2)^2}{(1 - y_2)^6} \ .$$

Derivations of (17)

For $g_1(x) = x^2$ and $g_2(x) = x$, we have

$$\begin{aligned} v(g_1, g_2) \quad = \quad & -\lim_{r\downarrow 1} \frac{1}{2\pi^2} \oint_{|\xi_1|=1} g_1(\delta(\xi_1)) \left[\oint_{|\xi_2|=1} \frac{g_2(\delta(\xi_2))}{r^2(\xi_2 - \xi_1/r)^2} \mathrm{d}\xi_2 \right] \mathrm{d}\xi_1 \\ & - \frac{(\beta_x y_1 + \beta_y y_2)(1 - y_2)^2}{4\pi^2 h^2} \oint_{|\xi_1|=1} \frac{g_1(\delta(\xi_1))}{(\xi_1 + y_2/h)^2} \mathrm{d}\xi_1 \oint_{|\xi_1|=1} \frac{g_2(\delta(\xi_2))}{(\xi_2 + y_2/h)^2} \mathrm{d}\xi_2 \ . \end{aligned}$$

Now, we can derive that

$$\begin{aligned} &\lim_{r\downarrow 1} \oint_{|\xi_1|=1} g_1(\delta(\xi_1)) \left[\oint_{|\xi_2|=1} \frac{g_2(\delta(\xi_2))}{r^2(\xi_2 - \xi_1/r)^2} \mathrm{d}\xi_2 \right] \mathrm{d}\xi_1 \\ &\quad = \lim_{r\downarrow 1} \frac{1}{(1 - y_2)^6} \oint_{|\xi_1|=1} (1 + h^2 + h\xi_1 + h\xi_1^{-1})^2 \left[\oint_{|\xi_2|=1} \frac{1 + h^2 + h\xi_2 + h\xi_2^{-1}}{r^2(\xi_2 - \xi_1/r)^2} \mathrm{d}\xi_2 \right] \mathrm{d}\xi_1 \\ &\quad = \lim_{r\downarrow 1} \frac{2\pi \mathrm{i}}{(1 - y_2)^6} \oint_{|\xi_1|=1} (1 + h^2 + h\xi_1 + h\xi_1^{-1})^2 \cdot (h + 0) \mathrm{d}\xi_1 \\ &\quad = \lim_{r\downarrow 1} \frac{2\pi \mathrm{i} \cdot h}{(1 - y_2)^6} \oint_{|\xi_1|=1} \left[\frac{h^2}{\xi_1^2} + \frac{2h(1 + h^2)}{\xi_1} \right] \mathrm{d}\xi_1 \\ &\quad = -\frac{4\pi^2 \cdot 2h^2(1 + h^2)}{(1 - y_2)^6} \ , \end{aligned}$$

and

$$\oint_{|\xi_2|=1} \frac{g_2(\delta(\xi_2))}{(\xi_2 + y_2/h)^2} \mathrm{d}\xi_2 = \frac{1}{(1 - y_2)^2} \oint_{|\xi_2|=1} \frac{1 + h^2 + h\xi_2 + h\xi_2^{-1}}{(\xi_2 + y_2/h)^2} \mathrm{d}\xi_2 = \frac{2\pi \mathrm{i} \cdot h}{(1 - y_2)^2} \ .$$

So, we derived that

$$v(g_1, g_2) = \frac{4h^2(1 + h^2)}{(1 - y_2)^6} + \frac{2(\beta_x y_1 + \beta_y y_2)(1 + h^2 - y_2)}{(1 - y_2)^4} \ .$$

Derivations of (18)

For $g_2(x) = x$, we have

$$v(g_2, g_2) = -\lim_{r\downarrow 1} \frac{1}{2\pi^2} \oint_{|\xi_1|=1} g_2(\delta(\xi_1)) \left[\oint_{|\xi_2|=1} \frac{g_2(\delta(\xi_2))}{r^2(\xi_2 - \xi_1/r)^2} \mathrm{d}\xi_2 \right] \mathrm{d}\xi_1 - \frac{(\beta_x y_1 + \beta_y y_2)(1-y_2)^2}{4\pi^2 h^2} \left[\oint_{|\xi_1|=1} \frac{g_2(\delta(\xi_1))}{(\xi_1 + y_2/h)^2} \mathrm{d}\xi_1 \right]^2 .$$

Since that

$$\begin{aligned} &\lim_{r\downarrow 1} \oint_{|\xi_1|=1} g_2(\delta(\xi_1)) \left[\oint_{|\xi_2|=1} \frac{g_2(\delta(\xi_2))}{r^2(\xi_2 - \xi_1/r)^2} \mathrm{d}\xi_2 \right] \mathrm{d}\xi_1 \\ &= \lim_{r\downarrow 1} \frac{1}{(1-y_2)^4} \oint_{|\xi_1|=1} (1 + h^2 + h\xi_1 + h\xi_1^{-1}) \left[\oint_{|\xi_2|=1} \frac{1 + h^2 + h\xi_2 + h\xi_2^{-1}}{r^2(\xi_2 - \xi_1/r)^2} \mathrm{d}\xi_2 \right] \mathrm{d}\xi_1 \\ &= \lim_{r\downarrow 1} \frac{2\pi \mathrm{i}}{(1-y_2)^4} \oint_{|\xi_1|=1} (1 + h^2 + h\xi_1 + h\xi_1^{-1}) \cdot (h + 0) \mathrm{d}\xi_1 \\ &= \lim_{r\downarrow 1} \frac{2\pi \mathrm{i} \cdot h}{(1-y_2)^6} \oint_{|\xi_1|=1} \left[\frac{h^2}{\xi_1^2} + \frac{2h(1+h^2)}{\xi_1} \right] \mathrm{d}\xi_1 \\ &= -\frac{4\pi^2 \cdot h^2}{(1-y_2)^4} , \end{aligned}$$

Then, we have

$$v(g_2, g_2) = \frac{2h^2}{(1-y_2)^4} + \frac{\beta_x y_1 + \beta_y y_2}{(1-y_2)^2} .$$

Derivations of (19)

Following the definition of $F_{\mathbf{y}}(\mathrm{d}x)$ and using the substitution $x = (1-y_2)^{-2}(1+h^2 - 2hcos\theta)$, $0 < \theta < \pi$, we have

$$\sqrt{(b-x)(x-a)} = \frac{2hsin\theta}{(1-y_2)^2}, \qquad \mathrm{dx} = \frac{2hsin\theta}{(1-y_2)^2} \mathrm{d}\theta,$$

$$x = \frac{1 + h^2 - 2hcos\theta}{(1-y_2)^2}, \qquad y_1 + y_2 x = \frac{h^2 + y_2^2 - 2hy_2 cos\theta}{(1-y_2)^2}.$$

Therefore,

$$
\begin{aligned}
b_0 &= \int_a^b x^2 \cdot \frac{(1-y_2)\sqrt{(b-x)(x-a)}}{2\pi x(y_1+y_2x)}\,\mathrm{d}x \\
&= \frac{4h^2}{2\pi(1-y_2)^3}\int_0^{\pi} \frac{(1+h^2-2hcos\theta)sin^2\theta}{h^2+y_2^2-2hy_2cos\theta}\,\mathrm{d}\theta \\
&= \frac{4h^2}{4\pi(1-y_2)^3}\int_0^{2\pi} \frac{(1+h^2-2hcos\theta)sin^2\theta}{h^2+y_2^2-2hy_2cos\theta}\,\mathrm{d}\theta \\
&= \frac{h^2}{\pi(1-y_2)^3}\int_{|z|=1} \frac{1+h^2-h(z+z^{-1})}{h^2+y_2^2-hy_2(z+z^{-1})}\cdot\frac{-(z^2-1)^2}{4z^2}\cdot\frac{\mathrm{d}z}{iz} \text{ (letting } z=e^{i\theta}) \\
&= \frac{ih^2}{4\pi y_2(1-y_2)^3}\int_{|z|=1} \frac{(z^2-\frac{1+h^2}{h}z+1)(z^2-1)^2}{z^3(z^2-\frac{h^2+y_2^2}{hy_2}z+1)}\,\mathrm{d}z \\
&= \frac{ih^2}{4\pi y_2(1-y_2)^3}\int_{|z|=1} \frac{(z-1/h)(z-h)(z-1)^2(z+1)^2}{z^3(z-y_2/h)(z-h/y_2)}\,\mathrm{d}z
\end{aligned}
$$

Obviously, there are two poles inside the unit circle: 0 and y_2/h. Their corresponding residues are

$$
\begin{aligned}
R(0) &= -2+\frac{(h^2+y_2^2)[(h^2+y_2^2)-y_2(1+h^2)]}{h^2y_2^2} \\
R(y_2/h) &= \frac{(h^2-y_2^2)(1-y_2)(y_2-h^2)}{h^2y_2^2}.
\end{aligned}
$$

Hence,

$$
b_0 = \frac{ih^2}{4\pi y_2(1-y_2)^3}\cdot 2\pi i\cdot([R(0)+R(y_2/h)] = \frac{1+h^2-y_2}{(1-y_2)^3}.
$$

On the other hand,

$$
\begin{aligned}
b_1 &= \int_a^b x \cdot \frac{(1-y_2)\sqrt{(b-x)(x-a)}}{2\pi x(y_1+y_2x)}\,\mathrm{d}x \\
&= \frac{4h^2}{2\pi(1-y_2)} \int_0^{\pi} \frac{sin^2\theta}{h^2+y_2^2-2hy_2cos\theta}\,\mathrm{d}\theta \\
&= \frac{4h^2}{4\pi(1-y_2)} \int_0^{2\pi} \frac{sin^2\theta}{h^2+y_2^2-2hy_2cos\theta}\,\mathrm{d}\theta \\
&= \frac{h^2}{\pi(1-y_2)} \int_{|z|=1} \frac{1}{h^2+y_2^2-hy_2(z+z^{-1})} \cdot \frac{-(z^2-1)^2}{4z^2} \cdot \frac{\mathrm{d}z}{iz} \text{ (letting } z=e^{i\theta}) \\
&= \frac{h}{4\pi i \cdot y_2(1-y_2)} \int_{|z|=1} \frac{(z^2-1)^2}{z^2(z^2-\frac{h^2+y_2^2}{hy_2}z+1)}\,\mathrm{d}z \\
&= \frac{h}{4\pi i \cdot y_2(1-y_2)} \int_{|z|=1} \frac{(z^2-1)^2}{z^2(z-h/y_2)(z-y_2/h)}\,\mathrm{d}z
\end{aligned}
$$

There are two poles inside the unit circle: 0 and y_2/h. Their corresponding residues are

$$
R(0) = \frac{y_2^2+h^2}{hy_2}, \quad R(y_2/h) = \frac{y_2^2-h^2}{hy_2}.
$$

Hence,

$$
b_1 = \frac{h}{4\pi i \cdot y_2(1-y_2)} \cdot 2\pi i \cdot [R(0)+R(y_2/h)] = \frac{1}{1-y_2}.
$$

Acknowledgments

The authors would like to thank the Editor, an Associate Editor and two referees for their helpful comments and useful suggestions. GL Tian's research was fully supported by a grant (HKU 779210M) from the Research Grant Council of the Hong Kong Special Administrative Region.

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Table 1: Sizes and powers of the Wald test, Bartlett test, and proposed LZ test based on 20,000 independent replications using real Gaussian variables with $y_1 = 0.5$ and $y_2 = 0.5$. Test sizes are calculated under the null hypothesis H_0: $\mathbf{\Sigma}_1 = \sigma^2\mathbf{\Sigma}_2$ with $\rho = 0$; the powers are estimated under the alternative hypothesis H_1: $\mathbf{\Sigma}_1 \neq \sigma^2\mathbf{\Sigma}_2$ with $\rho > 0$, where $\mathbf{\Sigma}_1 = \sigma^2\mathbf{I}_p$ and $\mathbf{\Sigma}_2 = (\rho^{|i-j|})_{p\times p}$.

ρ	(p, n_1, n_2)	Wald Test			Bartlett Test			LZ Test		
		$\sigma^2 = 0.5$	1.0	2.0	0.5	1.0	2.0	0.5	1.0	2.0
0.0	$(40, 80, 80)$	0.0514	0.0095	0.0517	0.0755	0.0803	0.0764	0.0656	0.0665	0.0666
	$(80, 160, 160)$	0.1435	0.0112	0.1416	0.1103	0.1101	0.1139	0.0681	0.0649	0.0679
	$(160, 320, 320)$	0.5111	0.0087	0.5128	0.2011	0.1913	0.1935	0.0608	0.0611	0.0576
	$(320, 640, 640)$	0.9784	0.0118	0.9879	0.4827	0.4643	0.4471	0.0580	0.0574	0.0548
	$(640, 1280, 1280)$	1.0000	0.0098	1.0000	0.9369	0.9232	0.9457	0.0534	0.0540	0.0542
0.4	$(40, 80, 80)$	0.9887	0.9723	0.9992	0.9987	0.9974	0.9993	0.2901	0.2905	0.2921
	$(80, 160, 160)$	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.7315	0.7320	0.7288
	$(160, 320, 320)$	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9994
	$(320, 640, 640)$	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	$(640, 1280, 1280)$	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
0.8	$(40, 80, 80)$	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.7153	0.7133	0.7109
	$(80, 160, 160)$	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9978	0.9982	1.0000
	$(160, 320, 320)$	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	$(320, 640, 640)$	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	$(640, 1280, 1280)$	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

Table 2: Sizes and powers of the proposed LZ test based on 20,000 independent replications using real Gamma variables with $y_1 = 0.5$ and $y_2 = 0.5$. Test sizes are calculated under the null hypothesis H_0: $\mathbf{\Sigma}_1 = \sigma^2\mathbf{\Sigma}_2$ with $\rho = 0$; the powers are estimated under the alternative hypothesis H_1: $\mathbf{\Sigma}_1 \neq \sigma^2\mathbf{\Sigma}_2$ with $\rho > 0$, where $\mathbf{\Sigma}_1 = \sigma^2\mathbf{I}_p$ and $\mathbf{\Sigma}_2 = (\rho^{|i-j|})_{p\times p}$.

(p, n_1, n_2)	$\rho = 0.0$			$\rho = 0.4$			$\rho = 0.8$		
	$\sigma^2 = 0.5$	1.0	2.0	0.5	1.0	2.0	0.5	1.0	2.0
$(40, 80, 80)$	0.0722	0.0726	0.0681	0.2882	0.2899	0.2914	0.7087	0.7083	0.7037
$(80, 160, 160)$	0.0689	0.0686	0.0697	0.7268	0.7247	0.7252	1.0000	1.0000	1.0000
$(160, 320, 320)$	0.0615	0.0616	0.0605	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
$(320, 640, 640)$	0.0536	0.0545	0.0543	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
$(640, 1280, 1280)$	0.0525	0.0548	0.0529	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

Table 3: Sizes and powers of the proposed LZ test based on 20,000 independent replications using real mixture of multivariate normal distributions variables with $y_1 = 0.5$ and $y_2 = 0.5$. Test sizes are calculated under the null hypothesis H_0: $\mathbf{\Sigma}_1 = \sigma^2\mathbf{\Sigma}_2$ with $\rho = 0$; the powers are estimated under the alternative hypothesis H_1: $\mathbf{\Sigma}_1 \neq \sigma^2\mathbf{\Sigma}_2$ with $\rho > 0$, where $\mathbf{\Sigma}_1 = \sigma^2\mathbf{I}_p$ and $\mathbf{\Sigma}_2 = (\rho^{|i-j|})_{p\times p}$.

(p, n_1, n_2)	$\rho = 0.0$			$\rho = 0.4$			$\rho = 0.8$		
	$\sigma^2 = 0.5$	1.0	2.0	0.5	1.0	2.0	0.5	1.0	2.0
$(40, 80, 80)$	0.0834	0.0821	0.0814	0.2977	0.2948	0.2930	0.6779	0.6743	0.6716
$(80, 160, 160)$	0.0699	0.0770	0.0743	0.7138	0.7149	0.7245	1.0000	1.0000	1.0000
$(160, 320, 320)$	0.0648	0.0676	0.0638	1.0000	1.0000	1.00000	1.0000	1.0000	1.0000
$(320, 640, 640)$	0.0571	0.0586	0.0590	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
$(640, 1280, 1280)$	0.0562	0.0523	0.0543	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

Table 4: Sizes and powers of the proposed LZ test based on 20,000 independent replications with $y_1 = 10$ and $y_2 = 0.5$. Test sizes are calculated under the null hypothesis H_0: $\mathbf{\Sigma}_1 = \sigma^2\mathbf{\Sigma}_2$ with $\rho = 0$; the powers are estimated under the alternative hypothesis H_1: $\mathbf{\Sigma}_1 \neq \sigma^2\mathbf{\Sigma}_2$ with $\rho > 0$, where $\mathbf{\Sigma}_1 = \sigma^2\mathbf{I}_p$ and $\mathbf{\Sigma}_2 = (\rho^{|i-j|})_{p\times p}$.

Multivariate Gaussian data

(p, n_1, n_2)	$\rho = 0.0$			$\rho = 0.4$			$\rho = 0.8$		
	$\sigma^2 = 0.5$	1.0	2.0	0.5	1.0	2.0	0.5	1.0	2.0
$(50, 5, 100)$	0.0469	0.0461	0.0469	0.0743	0.0727	0.0775	0.1212	0.1221	0.1166
$(100, 10, 200)$	0.0591	0.0546	0.0549	0.1378	0.1429	0.1372	0.2665	0.2675	0.2677
$(200, 20, 400)$	0.0575	0.0582	0.0620	0.2752	0.2766	0.2740	0.6231	0.6197	0.6262
$(500, 50, 1000)$	0.0536	0.0531	0.0532	0.7774	0.7724	0.7861	0.9987	0.9982	0.9986
$(800, 80, 1600)$	0.0514	0.0515	0.0526	0.9786	0.9823	0.9847	1.0000	1.0000	1.0000

Multivariate gamma data

(p, n_1, n_2)	$\rho = 0.0$			$\rho = 0.4$			$\rho = 0.8$		
	$\sigma^2 = 0.5$	01.0	2.0	0.5	1.0	2.0	0.5	1.0	2.0
$(50, 5, 100)$	0.0486	0.0488	0.0493	0.0789	0.0748	0.0757	0.1180	0.1194	0.1131
$(100, 10, 200)$	0.0628	0.0627	0.0610	0.1392	0.1406	0.1413	0.2579	0.2635	0.2626
$(200, 20, 400)$	0.0617	0.0607	0.0620	0.2741	0.2749	0.2709	0.6102	0.5980	0.6064
$(500, 50, 1000)$	0.0580	0.0544	0.0608	0.8376	0.8158	0.8344	1.0000	0.9995	0.9999
$(800, 80, 1600)$	0.0548	0.0527	0.0543	0.9980	0.9864	0.9934	1.0000	1.0000	1.0000

Mixture of multivariate Gaussian data

(p, n_1, n_2)	$\rho = 0.0$			$\rho = 0.4$			$\rho = 0.8$		
	$\sigma^2 = 0.5$	1.0	2.0	0.5	1.0	2.0	0.5	1.0	2.0
$(50, 5, 100)$	0.0608	0.0593	0.0574	0.0806	0.0833	0.0771	0.1103	0.1091	0.1106
$(100, 10, 200)$	0.0838	0.0799	0.0823	0.1501	0.1544	0.1453	0.2463	0.2446	0.2484
$(200, 20, 400)$	0.0756	0.0793	0.0783	0.2698	0.2750	0.2759	0.5716	0.5683	0.5682
$(500, 50, 1000)$	0.0614	0.0608	0.0614	0.8050	0.8274	0.8115	1.0000	0.9978	0.9984
$(800, 80, 1600)$	0.0590	0.0564	0.0576	0.9964	0.9821	0.9735	1.0000	1.0000	1.0000

Table 5: Sizes and powers of the proposed LZ test based on 20,000 independent replications using real Gaussian variables with $n_1 = 2p$ and $n_2 = p+5, p+10, \ldots$. Test sizes are calculated under the null hypothesis H_0: $\mathbf{\Sigma}_1 = \sigma^2\mathbf{\Sigma}_2$ with $\rho = 0$; the powers are estimated under the alternative hypothesis H_1: $\mathbf{\Sigma}_1 \neq \sigma^2\mathbf{\Sigma}_2$ with $\rho > 0$, where $\mathbf{\Sigma}_1 = \sigma^2\mathbf{I}_p$ and $\mathbf{\Sigma}_2 = (\rho^{|i-j|})_{p\times p}$.

ρ	p	n_1	$n_2 = p+k$															
			$k=5$	10	15	20	25	30	35	40	45	50	55	60	65	70	75	80
	10	20	0.0018	0.0256	0.0362	0.0371	0.0449	0.0460	0.0493	0.0468	0.0512	0.0495	0.0491	0.0498	0.0520	0.0521	0.0528	0.0499
	25	50	0.0114	0.0417	0.0556	0.0602	0.0644	0.0664	0.0626	0.0656	0.0663	0.0649	0.0599	0.0629	0.0602	0.0598	0.0621	0.0589
0.0	50	100	0.0172	0.0533	0.0638	0.0729	0.0703	0.0722	0.0694	0.0696	0.0693	0.0689	0.0681	0.0664	0.0659	0.0664	0.0656	0.0933
	100	200	0.0184	0.0562	0.0684	0.0706	0.0728	0.0702	0.0759	0.0720	0.0741	0.0731	0.0722	0.0716	0.0706	0.0669	0.0691	0.0668
	10	20	0.0032	0.0425	0.0804	0.1099	0.1450	0.1727	0.2023	0.2322	0.2569	0.2771	0.2938	0.3130	0.3196	0.3453	0.3584	0.3664
	25	50	0.0131	0.0596	0.0875	0.1174	0.1595	0.2075	0.2590	0.3215	0.3739	0.4300	0.4865	0.5461	0.5940	0.6296	0.6741	0.7072
0.4	50	100	0.0175	0.0590	0.0856	0.1058	0.1384	0.1719	0.2178	0.2626	0.3251	0.3971	0.4716	0.5421	0.6087	0.6747	0.7386	0.7887
	100	200	0.0168	0.0600	0.0836	0.0954	0.1121	0.1354	0.1579	0.1887	0.2200	0.2691	0.3269	0.3956	0.4592	0.5274	0.6007	0.6762
	10	20	0.0035	0.0728	0.1690	0.2794	0.3906	0.4869	0.5744	0.6491	0.7037	0.7477	0.7873	0.8142	0.8340	0.8542	0.8755	0.8806
	25	50	0.0147	0.0769	0.1387	0.2372	0.3599	0.5161	0.6612	0.7842	0.8705	0.9258	0.9627	0.9817	0.9897	0.9948	0.9983	0.9989
0.8	50	100	0.0200	0.0703	0.1124	0.1684	0.2511	0.3705	0.5012	0.6488	0.7794	0.8829	0.9451	0.9775	0.9914	0.9968	0.9995	0.9997
	100	200	0.0187	0.0682	0.0933	0.1211	0.1634	0.2251	0.3049	0.4034	0.5193	0.6481	0.7702	0.8660	0.9365	0.9720	0.9901	0.9965

Table 6: Sizes and powers of the proposed LZ test based on 20,000 independent replications using Gamma variables data with $n_1 = 2p$ and $n_2 = p+5, p+10, \ldots$. Test sizes are calculated under the null hypothesis H_0: $\mathbf{\Sigma}_1 = \sigma^2\mathbf{\Sigma}_2$ with $\rho = 0$; the powers are estimated under the alternative hypothesis H_1: $\mathbf{\Sigma}_1 \neq \sigma^2\mathbf{\Sigma}_2$ with $\rho > 0$, where $\mathbf{\Sigma}_1 = \sigma^2\mathbf{I}_p$ and $\mathbf{\Sigma}_2 = (\rho^{|i-j|})_{p\times p}$.

ρ	p	n_1	$n_2 = p+k$															
			$k=5$	10	15	20	25	30	35	40	45	50	55	60	65	70	75	80
	10	20	0.0015	0.0244	0.0405	0.0465	0.0502	0.0532	0.0531	0.0601	0.0608	0.0582	0.0604	0.0634	0.0646	0.0626	0.0622	0.0602
	25	50	0.0109	0.0461	0.0569	0.0630	0.0683	0.0699	0.0670	0.0699	0.0721	0.0694	0.0696	0.0704	0.0687	0.0678	0.0701	0.0686
0.0	50	100	0.0164	0.0513	0.0653	0.0690	0.0738	0.0726	0.0698	0.0695	0.0698	0.0687	0.0737	0.0698	0.0663	0.0660	0.0713	0.0666
	100	200	0.0188	0.0572	0.0665	0.0757	0.0731	0.0743	0.0757	0.0764	0.0771	0.0747	0.0720	0.0688	0.0673	0.0677	0.0653	0.0687
	10	20	0.0025	0.0425	0.0789	0.1141	0.1397	0.1729	0.1927	0.2138	0.2322	0.2502	0.2672	0.2794	0.2974	0.3110	0.3181	0.3277
	25	50	0.0127	0.0585	0.0927	0.1253	0.1630	0.2049	0.2590	0.3185	0.3638	0.4188	0.4732	0.5215	0.5714	0.6088	0.6481	0.6882
0.4	50	100	0.0153	0.0606	0.0878	0.1114	0.1367	0.1712	0.2128	0.2714	0.3273	0.3903	0.4641	0.5397	0.6044	0.6706	0.7277	0.7715
	100	200	0.0185	0.0561	0.0789	0.0936	0.1116	0.1316	0.1598	0.1904	0.2300	0.2709	0.3302	0.3883	0.4659	0.5304	0.6004	0.6728
	10	20	0.0032	0.0731	0.1616	0.2642	0.3580	0.4386	0.5150	0.5801	0.6280	0.6763	0.7062	0.7379	0.7612	0.7866	0.8004	0.8158
	25	50	0.0117	0.0760	0.1445	0.2345	0.3575	0.5017	0.6323	0.7587	0.8466	0.9106	0.9503	0.9721	0.9845	0.9910	0.9952	0.9976
0.8	50	100	0.0179	0.0668	0.1128	0.1678	0.2489	0.3617	0.5019	0.6432	0.7670	0.8756	0.9420	0.9754	0.9896	0.9974	0.9992	0.9994
	100	200	0.0184	0.0628	0.0950	0.1225	0.1648	0.2198	0.3085	0.4030	0.5272	0.6542	0.7694	0.8649	0.9322	0.9725	0.9897	0.9967