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On the Definiteness of Univariate and Bivariate Two-Side Matrix Polynomials over Unitary Complex Numbers

Graziano Chesi

Abstract—Establishing whether a two-side matrix polynomial is definite over unitary complex numbers is an important problem in modeling, analysis and design of control systems. This paper addresses this problem for univariate and bivariate two-side matrix polynomials, i.e., matrix functions that are polynomial in the variables and their inverse. Sufficient and necessary conditions are proposed in terms of linear matrix inequality (LMI) feasibility tests by exploiting trigonometric transformations and the theory of positive polynomials. Some numerical examples illustrate the proposed conditions.

I. INTRODUCTION

Frequency-domain methods are playing a key role in studying linear control systems since many years. These methods exploit the frequency response of the system, which is the evaluation of the transfer function onto the imaginary axis (i.e., over continuous-time frequencies) or onto the complex unit disc (i.e., over discrete-time frequencies). See for instance [10], [13] and references therein for details.

A problem arising in frequency-domain methods consists of establishing whether a two-side matrix polynomial is definite or semidefinite over unitary complex numbers. A two-side matrix polynomial is a matrix function that is polynomial in the variables and their inverse. Indeed, this problem can be met in system modeling, for instance when looking for approximations of a given transfer function, and can be met in system analysis and design, for instance when investigating or imposing bounds of the frequency response of the system. A possible way of addressing this problem is through the Kalman-Yakubovich-Popov lemma, see for instance [1]. However, this solution has some disadvantages, in particular nonconservatism is achieved only in the case of polynomials in one scalar variable. Other existing methods include [5], [9], which propose linear matrix inequality (LMI) conditions based on the representation of positive trigonometric polynomials as sums of squares of trigonometric polynomials.

This paper addresses the problem of establishing whether a two-side matrix polynomial is definite over unitary complex numbers. Specifically, two-side matrix polynomials are considered with either real or complex coefficients, in both cases of one scalar variable (univariate two-side matrix polynomial) and two scalar variables (bivariate two-side matrix polynomial). Without loss of generality, the paper focuses on the problem of establishing positive definiteness of the two-side matrix polynomial. By exploiting trigonometric transformations and the theory of positive polynomials, it is shown that sufficient and necessary conditions for this problem can be obtained in terms of LMI feasibility tests. In particular, the provided conditions exploit matrix polynomials that are sums of squares of matrix polynomials (SOS). Some numerical examples illustrate the proposed conditions. In particular, one of these examples show how one can consider the problem of approximating a given transfer function in two variables with one of lower degree to a desired accuracy.

The paper is organized as follows. Section II introduces the problem formulation and some preliminaries. Section III describes the proposed conditions. Section IV presents some illustrative examples. Lastly, Section V concludes the paper with some final remarks.

II. PRELIMINARIES

The notation is as follows:
- \( \mathbb{N}, \mathbb{R}, \mathbb{C} \): natural, real, and complex number sets;
- \( j \): imaginary unit, i.e., \( j^2 = -1 \);
- \( \mathbb{I} \): identity matrix (of size specified by the context);
- \( \mathbb{R}(A), \mathbb{I}(A) \): real and imaginary parts of \( A \), i.e., \( A = \mathbb{R}(A) + j\mathbb{I}(A) \);
- \( \bar{A} \): conjugate of \( A \), i.e., \( \bar{A} = \mathbb{R}(A) - j\mathbb{I}(A) \);
- \( A^T \): transpose of \( A \), i.e., \( (A^T)_{ij} = A_{ji} \);
- \( A^H \): conjugate transpose of \( A \), i.e., \( (A^H)_{ij} = \bar{A}_{ji} \);
- \( \text{Hermitian matrix } A \): a square matrix satisfying \( A^H = A \);
- \( \star \): corresponding block in Hermitian matrices;
- \( A > 0 \), \( A \geq 0 \): Hermitian positive definite and Hermitian positive semidefinite matrix \( A \);
- \( \lambda_{\text{min}}(A) \): minimum eigenvalue of \( A \);
- \( |a| \): magnitude of \( a \).

A. Problem Formulation

The first problem addressed in this paper considers univariate two-side matrix polynomials \( F : \mathbb{C} \rightarrow \mathbb{C}^{n \times n} \) defined as

\[
F(z) = \sum_{i=-d}^{d} F_i z^i
\]

where \( z \in \mathbb{C} \) is the complex variable, \( d \in \mathbb{N} \) defines the degree of \( F(z) \), and \( F_{-d}, \ldots, F_d \in \mathbb{C}^{n \times n} \) are the matrix coefficients of \( F(z) \). We want to investigate the definiteness of \( F(z) \) over unitary complex numbers, i.e., over the set

\[
D = \{e^{j\omega}, \ \omega \in \mathbb{R}\}.
\]

Since definiteness is defined for Hermitian matrices, without loss of generality we assume that \( F(z) \) is Hermitian over \( D \), i.e.,

\[
F(z) = F(z)^H \ \forall z \in D.
\]
This condition holds if and only if
\[ F_i = F_{-i}^H \quad \forall i = 0, \ldots, d. \]  
(4)

Let us also observe that, since \( F(z) \) is a two-side matrix polynomial, the problem of establishing whether \( F(z) \) is definite over \( \mathcal{D} \) is equivalent to that of establishing whether \( F(z) \) is either positive or negative definite over \( \mathcal{D} \). Hence, without loss of generality, we introduce the first problem addressed in this paper as follows.

**Problem 1.** The first problem consists of establishing whether \( F(z) \) is positive definite over \( \mathcal{D} \), i.e.,
\[ F(z) > 0 \quad \forall z \in \mathcal{D}. \]  
(5)

The second problem addressed in this paper considers bivariate two-side matrix polynomials \( F : \mathbb{C}^2 \to \mathbb{C}^{n \times n} \) defined as
\[ F(z_1, z_2) = \sum_{i=-d}^{d} \sum_{k=-d}^{d} F_{i,k} z_1^i z_2^k \quad (6) \]
where \( z_1, z_2 \in \mathbb{C} \) are the complex variables, \( d \in \mathbb{N} \) defines the degree of \( F(z_1, z_2) \), and \( F_{-d,-d}, \ldots, F_{d,d} \in \mathbb{C}^{n \times n} \) are the matrix coefficients of \( F(z_1, z_2) \). We want to investigate the definiteness of \( F(z_1, z_2) \) over \( \mathcal{D}^2 \). Similarly to (3), we assume without loss of generality that
\[ F(z_1, z_2) = F(z_1, z_2)^H \quad \forall z_1, z_2 \in \mathcal{D}. \]  
(7)
which holds if and only if
\[ F_{i,k} = F_{-i,-k}^H \quad \forall i, k = -d, \ldots, d. \]  
(8)

Let us also observe that, as in the case of univariate one-side matrix polynomials, the problem of establishing whether \( F(z_1, z_2) \) is definite over \( \mathcal{D}^2 \) is equivalent to that of establishing whether \( F(z_1, z_2) \) is either positive or negative definite over \( \mathcal{D}^2 \). Hence, without loss of generality, we introduce the second problem addressed in this paper as follows.

**Problem 2.** The second problem consists of establishing whether \( F(z_1, z_2) \) is positive definite over \( \mathcal{D}^2 \), i.e.,
\[ F(z_1, z_2) > 0 \quad \forall z_1, z_2 \in \mathcal{D}. \]  
(9)

**B. SOS Matrix Polynomials**

Here we provide some information about establishing whether a matrix polynomial is SOS via an LMI feasibility test.

Let \( P : \mathbb{R}^q \to \mathbb{R}^{n \times n} \) be a matrix polynomial of degree less than or equal to \( 2m \), with \( P(x) = P(x)^T \), \( x \in \mathbb{R}^q \). Then, \( P(x) \) is said to be SOS if there exist matrix polynomials \( P_i : \mathbb{R}^q \to \mathbb{R}^{n \times n}, i = 1, \ldots, k \) such that
\[ P(x) = \sum_{i=1}^{k} P_i(x)^T P_i(x). \]  
(10)

A necessary and sufficient condition for establishing whether \( P(x) \) is SOS can be obtained via an LMI feasibility test. Indeed, \( P(x) \) can be expressed as
\[ P(x) = (b(x) \otimes I)^T (Q + L(\alpha)) (b(x) \otimes I) \]  
(11)
where \( b(x) \in \mathbb{R}^{\sigma(q,m)} \) is a vector containing all monomials of degree less than or equal to \( m \) in \( x \), where
\[ \sigma(q,m) = \frac{(q + m)!}{q! m!}, \]  
(12)
\( Q \in \mathbb{R}^{\sigma(q,m) \times \sigma(q,m)} \), \( Q = Q^T \), is a matrix satisfying
\[ P(x) = (b(x) \otimes I)^T Q (b(x) \otimes I) \]  
(13)
\( L : \mathbb{R}^T(q,m,n) \in \mathbb{R}^{\sigma(q,m) \times \sigma(q,m)} \) is a linear parametrization of the linear subspace
\[ \mathcal{L} = \left\{ L = L^T : (b(x) \otimes I)^T L (b(x) \otimes I) = 0 \right\} \]  
(14)
and \( \alpha \in \mathbb{R}^T(q,m,n) \) is a free vector, where \( \tau(q,m,n) \) is the dimension of \( \mathcal{L} \) given by
\[ \tau(q,m,n) = \frac{1}{2} n (\sigma(q,m) \sigma(q,m) + 1) \]  
\[ - (n + 1) \sigma(q,2m). \]  
(15)
The representation (11) is known as square matrix representation (SMR) [3] and extends the Gram matrix method for (scalar) SOS polynomials to the matrix case. One has that \( P(x) \) is SOS if and only if there exists \( \alpha \) satisfying the LMI
\[ Q + L(\alpha) \succeq 0. \]  
(16)
Hence, establishing whether \( P(x) \) is SOS amounts to solving a convex optimization problem. See also [2], [6], [7], [12], [15] and references therein for details on SOS matrix polynomials, and [4], [8], [11] for the case of (scalar) SOS polynomials.

**III. PROPOSED RESULTS**

In this section we present the proposed conditions for establishing whether a two-side matrix polynomial is positive definite over unitary complex numbers. In particular, Section III-A addresses the case of univariate two-side matrix polynomials, while Section III-B addresses the case of bivariate two-side matrix polynomials.

**A. Univariate Two-Side Matrix Polynomials**

Let us start by recalling that
\[ e^{j\omega} = \frac{1 +jt}{1 -jt} \]  
(17)
where
\[ t = \tan \frac{\omega}{2} \]  
(18)
whenever
\[ \omega \neq \pi + 2k\pi, \ k = 0, \pm 1, \pm 2, \ldots \] (19)
Indeed, let us define \( \phi : \mathbb{R} \to \mathbb{C} \) as
\[ \phi(t) = \frac{1 + jt}{1 - jt} \] (20)
and the matrix function \( Q : \mathbb{R} \to \mathbb{C}^{n \times n} \) as
\[ Q(t) = F(\phi(t)). \] (21)
One has that
\[ F(e^{i\omega}) = Q(t) \] (22)
whenever \( t \) and \( \omega \) are related by (18) with \( \omega \in \Omega_0 \) where
\[ \Omega_0 = \{ \omega \in \mathbb{R} : (19) \text{ holds} \}. \] (23)
It turns out that \( Q(t) \) can be written as
\[ Q(t) = \frac{R(t)}{(1 + t^2)^d} \] (24)
where \( R : \mathbb{R} \to \mathbb{R}^{n \times n} \) is a matrix polynomial that satisfies
\[ R(t) = R(t)^H. \] (25)
Let us define the matrix polynomial \( U : \mathbb{R} \to \mathbb{R}^{2n \times 2n} \) as
\[ U(t) = \left( \begin{array}{cc} \Re(R(t)) & \Im(R(t)) \\ \Im(R(t)) & \Re(R(t)) \end{array} \right). \] (26)
The following result provides a sufficient and necessary condition for establishing whether (5) holds.

**Theorem 1:** The univariate two-side matrix polynomial \( F(z) \) is positive definite over \( \mathcal{D} \) if and only if
\[ F(-1) > 0 \] (27)
and there exists a scalar \( \mu > 0 \) satisfying
\[ U(t) - \mu I_{2n} \text{ is SOS}. \] (28)

*Proof.* “\( \Rightarrow \)” Suppose that (27)–(28) hold. From (28) it follows that there exists a SOS matrix polynomial \( P(t) \) such that
\[ U(t) - \mu I_{2n} = P(t) \] (29)
which implies
\[ U(t) > 0 \ \forall t \in \mathbb{R} \] since \( \mu > 0 \). From the definitions of \( U(t) \) in (26) and \( Q(t) \) in (24), this implies that
\[ Q(t) > 0 \ \forall t \in \mathbb{R}. \]
Taking into account (22) and (18), one has that
\[ F(e^{i\omega}) > 0 \ \forall \omega \in \mathbb{R} \setminus \Omega_0 \]
Moreover, (27) implies that
\[ F(e^{i\omega}) > 0 \ \forall \omega \in \mathbb{R} \]
and, hence,
\[ F(e^{i\omega}) > 0 \ \forall \omega \in \mathbb{R} \]
i.e. \( F(z) > 0 \) for all \( z \in \mathcal{D} \).
“\( \Leftarrow \)” Suppose that \( F(z) \) is positive definite over \( \mathcal{D} \). This implies that (27) holds. Moreover, since
\[ \left\{ \phi(t), \ t = \tan \frac{\omega}{2}, \ \omega \in \Omega_0 \right\} = \mathbb{R}, \]
it follows from (22) that
\[ Q(t) > 0 \ \forall t \in \mathbb{R}. \]
This and (27) imply the existence of a scalar \( \mu > 0 \) satisfying
\[ Q(t) - \mu I_{2n} \geq 0 \ \forall t \in \mathbb{R} \]
from which we conclude that (28) holds since \( t \) is scalar variable, see [2] and references therein.

**Proof.**

Theorem 1 provides a sufficient and necessary condition for establishing whether \( F(z) \) is positive definite over \( \mathcal{D} \). This condition consists of two sub-conditions. The first sub-condition is the positive definiteness test in (27). The second sub-condition is a SOS test, in particular establishing the existence of a scalar \( \mu > 0 \) satisfying (28). This SOS test is equivalent to an LMI feasibility test as explained in Section II-B. The number of LMI scalar variables in this LMI test is
\[ \eta_1 = \tau(1, d, 2n) + 1 = nd(2d - 1) + 1 \]
where the first addend is the length of the vector \( \alpha \) in (11), and the second stands for the scalar \( \mu \).

**B. Bivariate Two-Side Matrix Polynomials**

Following the transformation introduced in the case of univariate two-side matrix polynomials, let us define
\[ \phi(t_i) = \frac{1 + j t_i}{1 - j t_i}, \ i = 1, 2 \]
where \( t_1, t_2 \in \mathbb{R} \), and let us introduce the matrix function \( Q : \mathbb{R}^2 \to \mathbb{C}^{n \times n} \) as
\[ Q(t_1, t_2) = F(\phi(t_1), \phi(t_2)). \]
One has that
\[ F(e^{i\omega_1}, e^{i\omega_2}) = Q(t_1, t_2) \]
whenever
\[ t_i = \tan \frac{\omega_i}{2}, \ i = 1, 2 \]
with \( \omega_1, \omega_2 \in \Omega_0 \). It turns out that \( Q(t_1, t_2) \) can be written as
\[ Q(t_1, t_2) = \frac{R(t_1, t_2)}{(1 + t_1^2)^d (1 + t_2^2)^d} \] (34)
where \( R : \mathbb{R}^2 \to \mathbb{R}^{n \times n} \) is a matrix polynomial that satisfies
\[ R(t_1, t_2) = R(t_1, t_2)^T. \] (35)
Let us define
\[ U(t_1, t_2) = \left( \begin{array}{cc} \Re(R(t_1, t_2)) & \Im(R(t_1, t_2)) \\ \Im(R(t_1, t_2)) & \Re(R(t_1, t_2)) \end{array} \right). \] (36)
Let us define
\[ F_1(z) = F(z, -1) \] (37)
and
\[ F_2(z) = F(-1, z). \] (38)

The following result provides a sufficient and necessary condition for establishing whether (9) holds.

**Theorem 2:** The bivariate two-side matrix polynomial \( F(z_1, z_2) \) is positive definite over \( \mathbb{D}^2 \) if and only if
\[ F_i(z) > 0 \quad \forall z \in \mathbb{D} \quad \forall i = 1, 2 \] (39)
and there exist a polynomial \( v(t_1, t_2) \) and a scalar \( \mu > 0 \) satisfying
\[ v(t_1, t_2) = v(t_1, t_2)U(t_1, t_2) - \mu I_{2n} \] \{ \text{are SOS} \} (40)

**Proof.** “⇒” Suppose that (39)–(40) hold. From the second constraint in (40) it follows that there exists a SOS matrix polynomial \( P(t_1, t_2) \) such that
\[ v(t_1, t_2)U(t_1, t_2) - \mu I_{2n} = P(t_1, t_2) \]
which implies
\[ U(t_1, t_2) > 0 \quad \forall t \in \mathbb{R}^2 \]
since \( v(t_1, t_2) \) is a SOS polynomial and \( \mu > 0 \). From the definitions of \( U(t_1, t_2) \) in (36) and \( Q(t_1, t_2) \) in (34), this implies that
\[ Q(t_1, t_2) > 0 \quad \forall t_1, t_2 \in \mathbb{R}. \]
Taking into account (32) and (33), one has that
\[ F(e^{j\omega_1}, e^{j\omega_2}) > 0 \quad \forall \omega_1, \omega_2 \in \Omega_0. \]
This and (39) imply that
\[ F(e^{j\omega_1}, e^{j\omega_2}) > 0 \quad \forall \omega_1, \omega_2 \in \mathbb{R} \]
i.e. \( F(z_1, z_2) \) is positive definite over \( \mathbb{D}^2 \).

“⇐” Suppose that \( F(z_1, z_2) \) is positive definite over \( \mathbb{D}^2 \). This implies that (39) holds. Moreover, since
\[ \{ \phi(t_i), \ t_i = \tan \frac{\omega_i}{2}, \ \omega_i \in \Omega_0 \} = \mathbb{R} \quad \forall i = 1, 2 \]
it follows from (22) that
\[ Q(t_1, t_2) > 0 \quad \forall t_1, t_2 \in \mathbb{R}. \]
This and (39) imply the existence of a scalar \( \bar{\mu} > 0 \) satisfying
\[ Q(t_1, t_2) - \bar{\mu} I_{2n} > 0 \quad \forall t_1, t_2 \in \mathbb{R}. \]
From [14] and the definition of positive definite matrix it follows that there exists a polynomial \( v(t_1, t_2), \) SOS and positive definite, such that
\[ v(t_1, t_2)Q(t_1, t_2) - \bar{\mu} I_{2n} \text{ is SOS}. \]
This implies that (40) holds for some \( \mu > 0. \)

Theorem 2 provides a sufficient and necessary condition for establishing whether \( F(z_1, z_2) \) is positive definite over \( \mathbb{D}^2 \). This condition consists of two sub-conditions. The first sub-condition is the pair of positive definiteness tests in (39). The second sub-condition is a SOS test, in particular it consists of establishing the existence of a nonzero polynomial \( v(t_1, t_2) \) and a scalar \( \mu > 0 \) satisfying (40). This SOS test is equivalent to an LMI feasibility test as explained in Section II-B. By denoting with 2m the degree of \( v(t_1, t_2) \), the number of LMI scalar variables in this LMI test is given by
\[ \eta_2 = \frac{1}{2} \sigma(2, m) \left( \sigma(2, m) + 1 \right) + \tau(2, d + m, 2n) + 1 \] (41)
where the first addend is the number of free coefficients in \( v(t_1, t_2) \), the second is the length of the vector \( \alpha \) in (11), and the third stands for the scalar \( \mu \). Let us observe that \( \eta_2 \) can be reduced by exploiting the structure of \( U(t_1, t_2) \), indeed as reported in Example 2 in the next section the actual number of LMI scalar variables is significantly smaller.

**IV. EXAMPLES**

This section provides some illustrative examples of the proposed conditions. The LMI feasibility tests are solved with the toolbox SeDuMi for Matlab [16].

**A. Example 1**

Let us start by considering the univariate two-side matrix polynomial
\[ F(z) = \left( \begin{array}{cc} (1 - 2j)z^{-1} + 6 + (1 + 2j)z & 4z^{-1} + 3j + 2z \\ 2z^{-1} - 3j + 4z & 5z^{-2} + 12 + 5z^2 \end{array} \right). \]

The problem is to establish whether \( F(z) \) is positive definite over \( \mathbb{D} \).

First of all, let us verify that (3) holds, i.e., \( F(z) \) is Hermitian over \( \mathbb{D} \). We have \( d = 2 \) and
\[
F_0 = F_0^H = \left( \begin{array}{cc} 6 & 3j \\ * & 12 \end{array} \right) \quad \text{and} \quad F_1 = F_1^H = \left( \begin{array}{cc} 1 - 2j & 4 \\ 2 & 0 \end{array} \right).
\]

This implies that (4) holds and, hence, (3) holds. In particular,
\[ \Re \left( F(e^{j\omega}) \right) = \left( \begin{array}{cc} 6 + 2\cos \omega & 6 \cos \omega - 2 \sin \omega \\ * & 12 + 10 \cos(2\omega) \end{array} \right) \]
and
\[ j \Im \left( F(e^{j\omega}) \right) = \left( \begin{array}{cc} 0 & j(3 + 2 \cos \omega - 2 \sin \omega) \\ * & 0 \end{array} \right). \]

Second, let us use Theorem 1 in order to establish whether \( F(z) \) is positive definite over \( \mathbb{D} \). We have that the sub-condition (27) holds since
\[ F(-1) = \left( \begin{array}{cc} 4 & -6 + 3j \\ * & 22 \end{array} \right) > 0. \]

However, it turns out that the second sub-condition does not hold since there does not exist any \( \mu > 0 \) satisfying (28). Indeed, the largest \( \mu \) for which \( U(t) - \mu I_{2n} \) is SOS is given by
\[ \mu = -1.720. \]
Therefore, from Theorem 1 we conclude that $F(z)$ is not positive definite over $D$. The number of LMI scalar variables in (28) is given by $\eta_1$ and is equal to 20.

Figure 1 attempts to verify graphically the obtained results, in particular showing the quantity

$$\delta(\omega) = \lambda_{\min}(F(e^{j\omega}))$$

on a grid in $[-\pi, \pi]$. As expected from the results just obtained with Theorem 1, $\delta(\omega) < 0$ for some points of the grid.

![Fig. 1. Example 1. Positivity index $\mu(\omega)$ over a grid in $[-\pi, \pi]$. As expected, $\delta(\omega) < 0$ for some points of the grid.](image)

**B. Example 2**

Let us consider the bivariate two-side matrix polynomial

$$F(z_1, z_2) = -z_2^{-3} + z_2^{-1} + 8 + z_1 - z_2^{-3} + j(-2z_2^{-1} + 2z_2 - z_1^{-1}z_2^{-1} + z_1z_2).$$

The problem is to establish whether $F(z_1, z_2)$ is positive definite (i.e., positive since $F(z_1, z_2)$ is a scalar in this case) over $D^2$.

First of all, let us observe that $F(z_1, z_2)$ satisfies (8) holds, and hence (7) holds, i.e. $F(z_1, z_2)$ is Hermitian over $D^2$.

Second, let us use Theorem 2 in order to establish whether $F(z_1, z_2)$ is positive definite over $D^2$. We have that the subcondition (39) holds, indeed

$$F_1(z) = -z^{-3} + 6 - z^{-3} + j(-2z^{-1} + 2z)$$

and

$$F_2(z) = z^{-1} + 10 + z$$

are positive definite over $D$. This is verified by using Theorem 1, indeed one has for $i = 1$

$$F_1(-1) = 8 > 0$$

(28) holds with $\mu = 3.916 > 0$

and for $i = 2$

$$F_2(-1) = 8 > 0$$

(28) holds with $\mu = 11.500 > 0$.

Moreover, (40) holds with

$$\mu = 7.148 > 0$$

by simply choosing the degree of the polynomial $v(t_1, t_2)$ equal to 0 (i.e., $v(t_1, t_2)$ is a constant). Therefore, from Theorem 2 we conclude that $F(z_1, z_2)$ is positive definite over $D^2$. The number of LMI scalar variables in (40) is 88 and turns out to be significantly smaller than the upper bound $\eta_2$ in this case equal to 317.

Figure 2 attempts to verify graphically the obtained results, in particular showing the quantity

$$\delta(\omega_1, \omega_2) = \lambda_{\min}(F(e^{j\omega_1}, e^{j\omega_2}))$$

on a grid in $[-\pi, \pi]^2$. As expected from the results just obtained with Theorem 2, $\delta(\omega_1, \omega_2) > 0$ for all points of the grid.

![Fig. 2. Example 2. Positivity index $\delta(\omega_1, \omega_2)$ over a grid in $[-\pi, \pi]^2$. As expected, $\delta(\omega_1, \omega_2) > 0$ for all points of the grid.](image)

**C. Example 3**

In this example we consider the problem of determining a polynomial $\hat{p}(z_1, z_2)$ of chosen degree that approximates a given polynomial $p(z_1, z_2)$ on $D$ to a desired accuracy $\gamma \in \mathbb{R}$, i.e.

$$|p(z_1, z_2) - \hat{p}(z_1, z_2)| < \gamma \quad \forall z_1, z_2 \in D.$$

The polynomial we want to approximate is given by

$$p(z_1, z_2) = 0.4 + 0.1z_2 + 0.3z_1^2 + 0.2z_1z_2 - 0.6z_2^2.$$

We want to find a polynomial $\hat{p}(z_1, z_2)$ of degree 1 (with real coefficients) that approximates $p(z_1, z_2)$ with $\gamma = 1$.

Let us observe that the approximation condition can be equivalently rewritten as

$$F(z_1, z_2) > 0 \quad \forall z_1, z_2 \in D$$

for some points of the grid.

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by defining the bivariate two-side matrix polynomial

\[ F(z_1, z_2) = \begin{pmatrix} \gamma \\ \frac{p(z_1^{-1}, z_2^{-2}) - \tilde{p}(z_1^{-1}, z_2^{-2})}{p(z_1, z_2) - \tilde{p}(z_1, z_2)} \end{pmatrix}. \]

Let us use Theorem 2. Since \( F(z_1, z_2) \) is linear in the unknown \( \tilde{p}(z_1, z_2) \), we have that the two sub-conditions in this theorem define a system of six LMIs. In particular, the sub-condition (39) introduces four LMIs through Theorem 1, while other two LMIs are introduced by the sub-condition (40). By simply choosing the degree of the polynomial \( v(t_1, t_2) \) equal to 0, we find that this system of LMIs is feasible with

\[ \mu = 0.136 \]

and

\[ \tilde{p}(z_1, z_2) = 0.386 - 0.050 z_1 - 0.100 z_2. \]

This is verified by Figure 3, which shows the approximation error

\[ \varepsilon(\omega_1, \omega_2) = |p(e^{j\omega_1}, e^{j\omega_2}) - \tilde{p}(e^{j\omega_1}, e^{j\omega_2})| \]

on a grid in \([-\pi, \pi]^2\).

![Figure 3: Example 3. Approximation error \( \varepsilon(\omega_1, \omega_2) \) on a grid in \([-\pi, \pi]^2\). As expected, \( \varepsilon(\omega_1, \omega_2) < 1 \) for all points of the grid.](image)

It is interesting to observe that the simple approximation obtained by taking the linear part of \( p(z_1, z_2) \) does not satisfy the considered criterion, i.e.

\[ |p(z_1, z_2) - \tilde{p}(z_1, z_2)| < \gamma \quad \forall z_1, z_2 \in D \]

with

\[ \tilde{p}(z_1, z_2) = 0.4 + 0.1 z_2. \]

Indeed,

\[ z_1 = z_2 = -1 \Rightarrow |p(z_1, z_2) - \tilde{p}(z_1, z_2)| = 1.1. \]

V. CONCLUSIONS

This paper has addressed the problem of establishing whether a two-side matrix polynomial in one or two scalar variables is definite over unitary complex numbers. In particular, sufficient and necessary LMI conditions have been proposed by exploiting trigonometric transformations and SOS matrix polynomials. Future work will investigate the relationships with existing LMI conditions such as [5], [9]. In particular, preliminary studies suggest that the proposed methodology could have some advantages in terms of computational burden.

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REFERENCES