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LMI-Based Computation of the Instability Measure of Continuous-Time Linear Systems with a Scalar Parameter

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Abstract—Measuring the instability is a fundamental issue in control systems. This paper investigates the instability measure defined as the sum of the real parts of the unstable eigenvalues, which has important applications such as stabilization with information constraint. We consider continuous-time linear systems whose coefficients are linear functions of a scalar parameter constrained into an interval. The problem is to determine the largest instability measure for all admissible values of the parameter. Two sufficient and necessary conditions for establishing upper bounds on the sought instability measure are proposed in terms of linear matrix inequality (LMI) feasibility tests. The first condition exploits Lyapunov functions, while the second condition is based on the determinants of some specific matrices. Some numerical examples are used to compare the proposed conditions.

I. INTRODUCTION

A fundamental issue in control systems consists of measuring the instability. One of the key measures that have been proposed in the literature for measuring the instability is defined as the sum of the real parts of the unstable eigenvalues in the case of continuous-time linear systems, and as the product of the magnitudes of the unstable eigenvalues in the case of discrete-time linear systems, see for instance [11]. This measure has important applications in control systems such as stabilization with information constraint in the input channel, see for instance [1], [10]. The information constraint can be modeled in several ways including data-rate constraint, quantization, and signal-to-noise ratio. Solutions for stabilization with information constraint in the input channel can be obtained in terms of this instability measure, see for instance [7], [8].

Unfortunately, the mathematical model of a control system is very often affected by unknown parameters, for instance representing physical quantities that cannot be measured exactly or that are subject to changes. As a consequence, one has to consider a family of admissible models of control systems depending on the parameters. Clearly, the instability measure becomes a function of the parameters, and the target is to determine the largest instability measure for all admissible values of the parameters.

A typical way in the literature of modeling a control system affected by unknown parameters consists of introducing a polytopic model, i.e., a system whose coefficients are functions (typically linear) of a vector constrained into a convex bounded polytope, see for instance [6] and references therein. In such a framework, conditions for establishing upper bounds of the largest instability measure based on linear matrix inequalities (LMIs) have been proposed in [4] for the case of discrete-time linear systems, and in [5] for the case of continuous-time linear systems. However, it is unclear a priori whether these upper bounds are tight.

This paper considers continuous-time linear systems whose coefficients are linear functions of a scalar parameter constrained into an interval. The problem is to determine the largest instability measure for all admissible values of the parameter. Two sufficient and necessary conditions for establishing upper bounds on the sought instability measure are proposed in terms of LMI feasibility tests. The first condition exploits Lyapunov functions, while the second condition is based on the determinants of some specific matrices. Some numerical examples are presented to compare the proposed conditions.

The paper is organized as follows. Section II introduces the problem formulation and some preliminaries. Section III describes the proposed results. Section IV presents some illustrative examples. Lastly, Section V concludes the paper with some final remarks.

II. PRELIMINARIES

In this section we introduce the problem formulation and some preliminaries about homogeneous matrix polynomials.

A. Problem Formulation

Notation:

- \( \mathbb{R} \): space of real numbers;
- \( \mathbb{C} \): space of complex numbers;
- \( 0_n \): \( n \times 1 \) null vector;
- \( I \): identity matrix (of size specified by the context);
- \( A' \): transpose of matrix \( A \);
- \( A > 0, A \geq 0 \): symmetric positive definite and symmetric positive semidefinite matrix \( A \);
• trace($A$): trace of matrix $A$;
• det($A$): determinant of matrix $A$;
• $j$: imaginary unit, i.e. $j = \sqrt{-1}$;
• $\mathbb{R}(a)$, $\Im(a)$: real and imaginary parts of $a \in \mathbb{C}$, i.e. $a = \Re(a) + j\Im(a)$;
• $a^2$, where $a = (a_1, \ldots, a_n)'$: $(a_1^2, \ldots, a_n^2)'$.

Let us consider the uncertain system described by
\[ \dot{x}(t) = A(p)x(t) \] (1)
where $t \in \mathbb{R}$ is the time, $x(t) \in \mathbb{R}^n$ is the state, $p \in \mathbb{R}$ is an uncertain parameter constrained as
\[ p \in [p_-, p_+] \] (2)
and $A : \mathbb{R} \to \mathbb{R}^{n \times n}$ is an affine linear matrix function expressed as
\[ A(p) = A_0 + pA_1. \] (3)

Let us define the instability measure of a matrix $U \in \mathbb{R}^{n \times n}$ in the continuous-time case as
\[ m(U) = \sum_{i=1}^{n} \max \{0, \Re(\lambda_i(U))\} \] (4)
where $\lambda_i(U) \in \mathbb{C}$ is the $i$-th eigenvalue of $U$.

**Problem.** The problem that we consider in this paper consists of determining the largest instability measure of the system (1)–(2), i.e.,
\[ m^* = \sup_{p \in [p_-, p_+]} m(A(p)). \] (5)

### B. SOS Matrix Polynomials

Here we briefly review homogeneous matrix polynomials that can be written as sums of squares (SOS) of polynomials, see for instance [3] and references therein for details.

A homogeneous matrix polynomial $V : \mathbb{R}^r \to \mathbb{R}^{u \times u}$ is said to be SOS if there exist homogeneous matrix polynomials $V_1(s), \ldots, V_k(s), s \in \mathbb{R}^r$, such that
\[ V(s) = \sum_{i=1}^{k} V_i(s)'V_i(s). \] (6)

It turns out that one can establish whether a $V(s)$ is SOS via an LMI feasibility test.

Specifically, let $V(s)$ be symmetric, and let $2d$ be its degree for some nonnegative integer $d$. Then, $V(s)$ can be written as
\[ V(s) = (b(s) \otimes I)'(W + L(\alpha))(b(s) \otimes I) \] (7)
where $b \in \mathbb{R}^{\sigma(r,d)}$ is a vector containing all monomials of degree equal to $d$ in $s$, for instance according to
\[ b(s) = (s_1^d, s_2^{d-1}s_1, \ldots, s_d s_1^{d-1})', \] (8)
and
\[ \sigma(r,d) = \frac{(r + d - 1)!}{(r - 1)!d!}, \] (9)
\[ W \in \mathbb{R}^{u\sigma(r,d) \times u\sigma(r,d)} \] is a symmetric matrix satisfying
\[ V(s) = (b(s) \otimes I)'W(b(s) \otimes I) \] (10)
\[ L : \mathbb{R}^{\tau(r,2d,u)} \to \mathbb{R}^{u\sigma(r,d) \times u\sigma(r,d)} \] is a linear parametrization of the linear subspace
\[ \mathcal{L} = \{ \bar{L} = \bar{L}' : (b(s) \otimes I)'\bar{L}(b(s) \otimes I) = 0 \} \] (11)
where
\[ \tau(r,2d,u) = \frac{u}{2}(\sigma(r,d)(u\sigma(r,d) + 1) - (u + 1)\sigma(r,2d)), \] (12)
and $\alpha \in \mathbb{R}^{\tau(r,2d,u)}$ is a free vector.

The representation (7) is known as square matricial representation (SMR) of homogeneous matrix polynomials (in the scalar case, i.e., $u = 1$, this representation is also known as Gram matrix method). This representation is useful to establish whether $V(s)$ is SOS. Indeed, $V(s)$ is SOS if and only if there exists $\alpha$ satisfying the LMI
\[ W + L(\alpha) \succeq 0. \] (13)

### III. Instability Measure

Let us start by rewriting the system (1)–(2) into the canonical form
\[ \dot{x}(t) = B(s)x(t) \] (14)
where $B : \mathbb{R}^2 \to \mathbb{R}^{n \times n}$ is a linear matrix function and $s \in \mathbb{R}^2$ is an uncertain vector constrained as
\[ s \in \mathcal{S} \] (15)
where $\mathcal{S}$ is the simplex defined by
\[ \mathcal{S} = \{ s \in \mathbb{R}^2 : \gamma(s) = 1, s_1 \geq 0, s_2 \geq 0 \} \] (16)
with
\[ \gamma(s) = s_1 + s_2. \] (17)
Hence, it follows that the largest instability measure $m^*$ of the system (1)–(2) can be rewritten as
\[ m^* = \sup_{s \in \mathcal{S}} m(B(s)). \] (18)

Next, let us recall the following result, which provides an alternative formula for the determination of the instability measure in (4).

**Theorem 1 (15):** Let $U \in \mathbb{R}^{n \times n}$. For any integer $k$, $1 \leq k \leq n$, define
\[ c_k = \frac{n!}{(n-k)!k!} \] (19)
and the linear matrix function $\Omega_k : \mathbb{R}^{n \times n} \to \mathbb{R}^{n \times n}$ satisfying
\[ \text{spec}(\Omega_k(U)) = \{ \lambda_i(U) + \ldots + \lambda_k(U) : 1 \leq i_j \leq n, i_j \neq i_l \forall j \neq l \}. \] (20)
Define also
\[ \psi_k(U) = \max_{\lambda \in \text{spec}(\Omega_k(U))} \Re(\lambda). \] (21)
Then,
\[ m(U) = \max_{k=1,\ldots,n} \max \{0, \psi_k(U)\}. \] (22)
Theorem 1 provides a certain equivalence of the instability measure \( m(U) \) with the spectrum of the matrices \( \Omega_k(U) \) obtained for \( k = 1, \ldots, n \). Specifically, \( \Omega_k(U) \) has the property that its spectrum is given by all the possible sums of \( k \) distinct eigenvalues of \( U \). These matrices can be directly built from \( U \) following the idea described in [2]. For instance,

\[
U = \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix}
\]

\[
\Omega_1(U) = U, \\
\Omega_2(U) = \text{trace}(U)
\]

and

\[
U = \begin{pmatrix} u_{11} & u_{12} & u_{13} \\ u_{21} & u_{22} & u_{23} \\ u_{31} & u_{32} & u_{33} \end{pmatrix}
\]

\[
\Omega_1(U) = U, \\
\Omega_2(U) = \begin{pmatrix} u_{11} + u_{22} & u_{23} & -u_{13} \\ u_{32} & u_{11} + u_{33} & u_{12} \\ -u_{31} & u_{21} & u_{22} + u_{33} \end{pmatrix}, \\
\Omega_3(U) = \text{trace}(U).
\]

Hereafter we propose two strategies for determining the largest instability measure \( m^* \) of the system (1)–(2) based on Theorem 1.

In order to introduce the first strategy, let us define for \( k = 1, \ldots, n \) the linear matrix functions

\[
C_k(s) = \Omega_k(B(s))
\]

and

\[
D_k(s) = C_k(s) - w\gamma(s)I
\]

where \( w \in \mathbb{R} \). For a symmetric homogeneous matrix polynomial \( F_k : \mathbb{R}^2 \rightarrow \mathbb{R}^{c_k \times c_k} \), let us define

\[
G_k(s) = \gamma(s)F_k(s) - F_k(s)D_k(s) - D_k(s)'F_k(s).
\]

The following result provides an LMI condition for establishing an upper bound of \( m^* \).

**Theorem 2:** Let \( w \in (0, \infty) \). Then,

\[
m^* < w
\]

if there exists a symmetric homogeneous matrix polynomial \( F_k : \mathbb{R}^2 \rightarrow \mathbb{R}^{c_k \times c_k} \) of degree \( d_k, k = 1, \ldots, n \), such that

\[
\forall k = 1, \ldots, n \left\{ \hat{F}_k(s^2) \text{ is SOS} \quad \hat{G}_k(s^2) \text{ is SOS} \right\}
\]

where

\[
\hat{F}_k(s) = F_k(s) - \gamma(s)^{d_k}I, \\
\hat{G}_k(s) = G_k(s) - \gamma(s)^{d_k+1}I.
\]

Moreover, if \( d_k \) is equal to

\[
e_k = \frac{1}{2}c_k(c_k + 1) - 1,
\]

the condition is not only sufficient but also necessary.

**Proof.** Let us suppose that there exists a symmetric homogeneous matrix polynomial \( F_k(s) \) such that (29) holds. From [6] it follows that

\[
\forall k = 1, \ldots, n \forall s \in \mathcal{S} \left\{ F_k(s) \geq I; G_k(s) \geq I \right\}
\]

This means that \( D_k(s) \) is Hurwitz for all \( s \in \mathcal{S} \) for all \( k = 1, \ldots, n \). Therefore,

\[
0 > \max_{\lambda \in \text{spec}(D_k(s))} \Re(\lambda) = \max_{\lambda \in \text{spec}(C_k(s))} \Re(\lambda) - w = \psi_k(B(s)) - w,
\]

and from Theorem 1 it follows that

\[
m^* = \sup_{s \in \mathcal{S}} m(B(s)) = \sup_{s \in \mathcal{S}} \max_{k=1,\ldots,n} \{0, \psi_k(B(s))\} < w
\]

since \( w > 0 \), i.e., (28) holds.

Next, let us suppose that (28) holds. Proceeding as in the previous part of the proof one has that

\[
\sup_{s \in \mathcal{S}} \max_{k=1,\ldots,n} \{0, \psi_k(B(s))\} < w.
\]

From Theorem 1 this implies that

\[
D_k(s) \text{ is Hurwitz } \forall s \in \mathcal{S} \forall k = 1, \ldots, n
\]

Also, \( D_k(s) \) is Hurwitz for all \( s \in \mathcal{S} \) if and only if the Lyapunov equation

\[
\gamma(s)F_k(s) - F_k(s)D_k(s) - D_k(s)'F_k(s) = I
\]

has a unique solution \( F_k(s) \) that satisfies

\[
F_k(s) \geq \varepsilon_1 I \forall s \in \mathcal{S} \forall k = 1, \ldots, n
\]

for some \( \varepsilon_1 > 0 \). This equation can be rewritten as

\[
\hat{A}_k(s)\hat{x}_k(s) = \hat{b}_k(s)
\]

where \( \hat{x}_k(s) \) is a vector with all the independent entries of \( F_k(s) \), whose number is \( c_k(c_k + 1)/2 \), and \( \hat{A}_k(s) \) and \( \hat{b}_k(s) \) are, respectively, a square matrix and a vector of suitable size. Since the solution for \( F_k(s) \) is unique, it follows that \( \hat{A}_k(s) \) is nonsingular for all \( s \in \mathcal{S} \), in particular one can choose \( \hat{A}_k(s) \) such that

\[
\varepsilon_2 \geq \det(\hat{A}_k(s)) \geq \varepsilon_3 \forall s \in \mathcal{S} \forall k = 1, \ldots, n
\]

for some \( \varepsilon_2, \varepsilon_3 > 0 \). Hence,

\[
\hat{x}_k(s) = \hat{A}_k(s)^{-1}\hat{b}_k(s)
\]

which implies that

\[
F_k(s) = \frac{\hat{F}_k(s)}{\det(\hat{A}_k(s))}
\]

where \( \hat{F}_k(s) \) is a symmetric homogeneous matrix polynomial of degree \( c_k \) satisfying

\[
\hat{F}_k(s) \geq \frac{\varepsilon_1}{\varepsilon_2} \forall s \in \mathcal{S} \forall k = 1, \ldots, n.
\]
Now, if we replace $F_k(s)$ as $$F_k(s) \rightarrow \frac{\beta \varepsilon_2}{\varepsilon_1} \tilde{F}_k(s)$$ in (27), where $\beta > 0$, it follows that $G_k(s)$ is replaced by

$$\tilde{G}_k(s) = \frac{\beta \varepsilon_2}{\varepsilon_1} \left( \gamma(s) \tilde{F}_k(s) - \tilde{F}_k(s) D_k(s) - D_k(s) \gamma(s) \tilde{F}_k(s) \right) = \frac{\beta \varepsilon_2}{\varepsilon_1} \det(\hat{A}_k(s)) I.$$

This means that there exists a symmetric homogeneous matrix polynomial $\tilde{F}_k(s)$ of degree $c_k$ such that

$$\forall k = 1, \ldots, n \ \forall s \in S \left\{ \begin{array}{l} F_k(s) - \beta \gamma(s)^{c_k} I \geq 0 \\ G_k(s) - \beta \varepsilon_2 \varepsilon_3 \gamma(s)^{c_k+1} I \geq 0. \end{array} \right. $$

Let us choose $\beta$ as

$$\beta = \max \left\{ 1, \frac{\varepsilon_1}{\varepsilon_2 \varepsilon_3} \right\}.$$

It follows that

$$\forall k = 1, \ldots, n \ \forall s \in S \left\{ \begin{array}{l} F_k(s) - \gamma(s)^{c_k} I \geq 0 \\ G_k(s) - \gamma(s)^{c_k+1} I \geq 0. \end{array} \right. $$

or, equivalently from [6],

$$\forall k = 1, \ldots, n \ \forall s \in \mathbb{R}^2 \left\{ \begin{array}{l} F_k(s^2) - \gamma(s^2)^{c_k} I \geq 0 \\ G_k(s^2) - \gamma(s^2)^{c_k+1} I \geq 0. \end{array} \right. $$

Since a symmetric homogeneous matrix polynomial in two variables is positive semidefinite if and only if it is SOS (see [3] and references therein), it follows that (29) holds with $d_k = e_k$.

Theorem 2 provides a sufficient and necessary condition for establishing whether a scalar $\omega$ is an upper bound of $m^*$: This condition is an LMI feasibility test since the SOS conditions in (29) can be written as LMIs according to Section II-B. The condition provided by Theorem 2 is based on the use of the Lyapunov function defined by the symmetric homogeneous matrix polynomial $\tilde{F}_k(s)$. Let us observe that, while the necessity of this condition is achieved by choosing the degree $d_k$ of $\tilde{F}_k(s)$ equal to the quantity $e_k$, the sufficiency is guaranteed for any degree of $\tilde{F}_k(s)$.

Let us denote the best upper bound of $m^*$ provided by Theorem 2 (for $F_k(s)$ of degree $d_k$, $k = 1, \ldots, n$) as

$$\hat{m}^* = \inf_{w \in [0, \infty)} w \quad \text{s.t.} \quad \text{(29) holds for some } F_k(s) \text{ of degree } d_k. \quad (32)$$

Let us observe that $\hat{m}^*$ can be computed through a line search over the scalar $w$, in particular via a bisection algorithm in order to speed up the convergence. From Theorem 2 it follows that the upper bound $\hat{m}^*$ is tight at least when $d_k$ is equal to $e_k$.

The number of LMI scalar variables in (29) is given by the number of independent coefficients of $F_k(s)$, $k = 1, \ldots, n$, plus the length of the vectors $\alpha$ required to test whether $\tilde{F}_k(s^2)$ and $\tilde{G}_k(s^2)$ are SOS according to Section II-B, $k = 1, \ldots, n$. Hence, the number of LMI scalar variables in (29) is

$$\eta_1 = \sum_{k=1, \ldots, n} \frac{1}{2} \sigma(2, d_k) c_k (\sigma(2, d_k) c_k + 1) + \tau(2, 2d_k + 2, c_k).$$

(33)

Let us now describe the second strategy proposed in this paper for determining the largest instability measure $m^*$ of the system (1)--(2). For $k = 1, \ldots, n$ and $w \in [0, \infty)$ let us define the homogeneous polynomials $\tilde{f}_k, g_k : \mathbb{R}^2 \rightarrow \mathbb{R}$ as

$$\begin{cases} \tilde{f}_k(s) = \det(-D_k(s)) \\ g_k(s) = \det(H(D_k(s))) \end{cases}$$

where $H(D_k(s))$ is defined by

$$H(D_k(s)) = \left( \begin{array}{cccc} u_{c_k-1}(s) & u_{c_k-3}(s) & u_{c_k-5}(s) & \cdots \\ 1 & u_{c_k-2}(s) & u_{c_k-4}(s) & \cdots \\ 0 & u_{c_k-1}(s) & u_{c_k-3}(s) & \cdots \\ 0 & 0 & u_{c_k-2}(s) & \cdots \end{array} \right).$$

(35)

and $u_0(s), \ldots, u_{n-1}(s)$ are the coefficients of the characteristic polynomial of $D_k(s)$ according to

$$\det(\lambda I - D_k(s)) = \lambda^{c_k} + u_{c_k-1}(s) \lambda^{c_k-1} + \ldots + u_0(s).$$

(36)

where $\lambda \in \mathbb{C}$. The following result provides an alternative LMI condition for establishing an upper bound of $m^*$.

**Theorem 3:** Let $w \in [0, \infty)$. Then, (28) holds if and only if the following two sub-conditions hold:

- for any arbitrarily chosen $s_0$ in $S$, one has that
  $$m(B(s_0)) < w; \quad (37)$$
- there exists a scalar $\varepsilon \in \mathbb{R}$ such that
  $$\forall k = 1, \ldots, n \left\{ \begin{array}{l} \tilde{f}_k(s^2) \text{ is SOS} \\ \tilde{g}_k(s^2) \text{ is SOS} \end{array} \right. \quad (38)$$
  where
  $$\begin{cases} \tilde{f}_k(s) = f_k(s) - \varepsilon \gamma(s)^{c_k} \\ \tilde{g}_k(s) = g_k(s) - \varepsilon \gamma(s)^{c_k}(c_k-1)/2. \end{cases}$$

(39)

**Proof:** Let us suppose that (37)–(38) hold. From (38) and [6] it follows that

$$\forall k = 1, \ldots, n \ \forall s \in S \left\{ \begin{array}{l} f_k(s) \geq \varepsilon \\ g_k(s) \geq \varepsilon \end{array} \right. \quad (40) \quad \varepsilon > 0.$$

The condition $f_k(s) \geq \varepsilon$ implies that

$$\{0\} \not\in \text{spec}(D_k(s)),$$

while $g_k(s) \geq \varepsilon$ implies that

$$\{j\omega\} \not\in \text{spec}(D_k(s)) \quad \forall \omega \in \mathbb{R}_0$$

since [9]

$$g_k(s) = \prod_{i=i+1, \ldots, c_k} \lambda_i(D_k(s)) \lambda_l(D_k(s)).$$
Consequently,
\[ \{ j\omega \} \not\subset \operatorname{spec} (D_k(s)) \quad \forall \omega \in \mathbb{R} \quad \forall s \in \mathcal{S} \quad \forall k = 1, \ldots, n. \]
Moreover, (37) implies that
\[ \max_{\lambda \in \operatorname{spec}(D_k(s_0))} \Re(\lambda) < 0 \quad \forall k = 1, \ldots, n \]
since from Theorem 1 one has that
\[ w > m(B(s_0)) \geq \max_{k=1,\ldots,n} \psi_k(B(s_0)) = \max_{k=1,\ldots,n} \max_{\lambda \in \operatorname{spec}(C_k(s_0))} \Re(\lambda) = w + \max_{k=1,\ldots,n} \max_{\lambda \in \operatorname{spec}(D_k(s_0))} \Re(\lambda). \]
Hence, \( D_k(s) \) is Hurwitz for all \( s \in \mathcal{S} \) for all \( k = 1, \ldots, n \) due to the continuity of the eigenvalues of \( D_k(s) \) with respect to \( s \). Proceeding as in the first part of the proof of Theorem 2 we conclude that (28) holds.

"⇒" Suppose that (28) holds. This means that (37) holds for any arbitrarily chosen \( s_0 \) in \( \mathcal{S} \). From Theorem 1 we have that
\[ \psi_k(B(s)) < w \quad \forall k = 1, \ldots, n \quad \forall s \in \mathcal{S} \]
or, equivalently,
\[ \max_{\lambda \in \operatorname{spec}(C_k(s))} \Re(\lambda) < w \quad \forall k = 1, \ldots, n \quad \forall s \in \mathcal{S}. \]
Hence, \( D_k(s) \) is Hurwitz for all \( s \in \mathcal{S} \) for all \( k = 1, \ldots, n \). Proceeding as in the previous part of this proof, such a condition implies that there exists \( \varepsilon \in \mathbb{R} \) such that
\[ \forall k = 1, \ldots, n \quad \forall s \in \mathcal{S} \left\{ \begin{array}{l} f_k(s) \geq \varepsilon \\ g_k(s) \geq \varepsilon \\ \varepsilon > 0. \end{array} \right. \]
From [6] one has that
\[ \forall k = 1, \ldots, n \quad \forall s \in \mathcal{S} \left\{ \begin{array}{l} \tilde{f}_k(s) \geq 0 \\ \tilde{g}_k(s) \geq 0. \end{array} \right. \]
Since a homogeneous polynomial in two variables is positive semidefinite if and only if it is SOS (see [3] and references therein), it follows that (38) holds. \( \square \)

Theorem 3 provides an alternative sufficient and necessary condition for establishing whether a scalar \( w \) is an upper bound of \( m^* \). This condition is an LMI feasibility test since the SOS conditions in (38) can be written as LMIs according to Section II-B. The condition provided by Theorem 3 is based on the use of the determinants of some specific matrices.

By using Theorem 3, the largest instability measure \( m^* \) of the system (1)–(2) can be rewritten as
\[ m^* = \inf_{w \in [0, \infty)} w \quad \text{s.t. } (37)–(38) \text{ hold for an arbitrarily chosen } s_0 \text{ in } \mathcal{S} \quad \text{for some } \varepsilon \in \mathbb{R}. \quad (40) \]
Let us observe that (40) can be solved through a line search over the scalar \( w \), for instance via a bisection algorithm.

The number of LMI scalar variables in (38) is given by the length of the vectors \( \alpha \) required to test whether \( \tilde{f}_k(s^2) \) and \( \tilde{g}_k(s^2) \) are SOS according to Section II-B, \( k = 1, \ldots, n \), plus 1 for \( \varepsilon \). Hence, the number of LMI scalar variables in (38) is
\[ \eta_2 = 1 + \sum_{k=1,\ldots,n} \tau(2, c_k, 1) + \tau(2, c_k - 1, 1). \quad (41) \]

IV. EXAMPLES

In this section we present two illustrative examples of the proposed results. The computations have been done in Matlab by using the toolbox SeDuMi [12].

A. Example 1

Let us consider the system
\[
\begin{align*}
\dot{x}(t) &= A(p)x(t) \\
A(p) &= \begin{pmatrix} 1 & 4p \\ -3 & 1 - p \end{pmatrix} \\
p &\in [0, 2].
\end{align*}
\]
First of all, let us rewrite this system as into the canonical form (14)–(16). This can be done by defining
\[ p = 2s_2. \]
We obtain
\[
\begin{align*}
\dot{x}(t) &= B(s)x(t) \\
B(s) &= \begin{pmatrix} s_1 + s_2 & 8s_2 \\ -3s_1 - 3s_2 & s_1 - s_2 \end{pmatrix}.
\end{align*}
\]
Let us compute the best upper bound of \( m^* \) provided by Theorem 2, i.e., \( \hat{m}^* \) in (32). According to Theorem 2, \( \hat{m}^* \) is guaranteed to be tight by choosing
\[
\begin{align*}
\left\{ \begin{array}{l} d_1 = e_1 = 2 \\ d_2 = e_2 = 0, \end{array} \right.
\end{align*}
\]
For such a choice of \( d_k \), we find \( \hat{m}^* = m^* = 2.000 \). The number of LMI scalar variables in (29) is \( \eta_1 = 31 \).

It is worth remarking that, in this example, \( \hat{m}^* \) is not tight if one simply chooses \( d_k = 0 \) for all \( k = 1, 2 \). Indeed, the minimum values of \( d_k \) that allow one to obtain \( \hat{m}^* = m^* \) are
\[
\begin{align*}
\left\{ \begin{array}{l} d_1 = 1 \\ d_2 = 0, \end{array} \right.
\end{align*}
\]
For such a choice of \( d_k \), the number of LMI scalar variables in (29) is \( \eta_1 = 14 \).

Next, let us compute \( m^* \) by using Theorem 3. In particular, from (40), we find \( m^* = 2.000 \). The number of LMI scalar variables in (29) is \( \eta_2 = 1 \).
B. Example 2

Let us consider the system

\[
\begin{align*}
\dot{x}(t) &= A(p)x(t), \\
A(p) &= \begin{pmatrix}
1 - 2p & 1 & 0 \\
-2 & 5p - 2 & p \\
p - 1 & 0 & -1
\end{pmatrix}, \\
p &\in [-1, 1].
\end{align*}
\]

First of all, let us rewrite this system as into the canonical form (14)–(16). This can be done by defining

\[p = s_2 - s_1.\]

We obtain

\[
\begin{align*}
\dot{x}(t) &= B(s)x(t), \\
B(s) &= \begin{pmatrix}
3s_1 - s_2 & s_1 + s_2 & 0 \\
-2s_1 - 2s_2 & -7s_1 + 3s_2 & -s_1 + s_2 \\
-2s_1 & 0 & -s_1 - s_2
\end{pmatrix}.
\end{align*}
\]

Let us compute the best upper bound of \(m^*\) provided by Theorem 2, i.e., \(\hat{m}^*\) in (32). According to Theorem 2, \(\hat{m}^*\) is guaranteed to be tight by choosing

\[
\begin{align*}
d_1 &= e_1 = 5, \\
d_2 &= e_2 = 5, \\
d_3 &= e_3 = 0.
\end{align*}
\]

For such a choice of \(d_k\), we find \(\hat{m}^* = m^* = 2.850\). The number of LMI scalar variables in (29) is \(\eta_1 = 589\).

It is worth remarking that, in this example, \(\hat{m}^*\) is not tight if one simply chooses \(d_k = 0\) for all \(k = 1, 2, 3\). Indeed, the minimum values of \(d_k\) that allow one to obtain \(\hat{m}^* = m^*\) are

\[
\begin{align*}
d_1 &= 1, \\
d_2 &= 0, \\
d_3 &= 0.
\end{align*}
\]

For such a choice of \(d_k\), the number of LMI scalar variables in (29) is \(\eta_2 = 40\).

Next, let us compute \(m^*\) by using Theorem 3. In particular, from (40), we find \(m^* = 2.850\). The number of LMI scalar variables in (29) is \(\eta_2 = 11\).

V. Conclusion

This paper has investigated the instability measure defined as the sum of the real parts of the unstable eigenvalues for continuous-time linear systems whose coefficients are linear functions of a scalar parameter constrained into an interval. Two sufficient and necessary conditions for establishing upper bounds of the largest instability measure for all admissible values of the parameter have been proposed in terms of LMI feasibility tests. These conditions exploit Lyapunov functions and the determinants of some specific matrices.

Comparisons between the proposed conditions have been carried out in some numerical examples. Such comparisons have shown that the numerical complexity of the condition based on determinants can be significantly smaller than that of the condition based on Lyapunov functions. This is especially true when the degree of the Lyapunov functions is chosen in order to guarantee nonconservatism a priori.

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