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AMERICAN TYPE GEOMETRIC STEP OPTIONS

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Abstract. This paper studies American geometric step options. An European geometric step option is such a contract that the payoff will be deduced in a proportion way according to the occupation time of the underlying price outside a single (or double) barrier. This kind of options were introduced and studied by Linetsky [11] and Davydov *et al.* [2]. In this paper we consider the American geometric step options with perpetual expiration and finite expriation, that is, we assume that the policyholders can exercise the option at any time (or any time before the maturity date). In the perpetual case, using the Feynman-Kac formula, the joint Laplace transform of the first hitting time and occupation time is obtained. We also show that there exists an optimal level such that the time first hitting this level plays the role of the optimal exercise time. Then a closed pricing formula is derived. In the finite expiration case, we show that the price of the American geometric step option satisfies a variational inequalities. A modified explicit finite difference approach is developed to calculate the price of the option numerically.

1. **Introduction.** Barrier option is one of the most popular exotic options. Pricing barrier and other exotic options has been studied by many authors. For example, under a classic Black-Scholes model, pricing formulas of single barrier options were derived by Merton [13] for down-and-out calls and Rubinstein *et al.*[16] for all types of single barrier options. Kunitomo *et al.* [10], German *et al.* [4] and Schroder [17] considered double barrier options. They obtained pricing formulas in form of infinite series of normal functions, Laplace transform and trigonometric series respectively. Kou *et al.* [9], Cont *et al.* [1], Jeannin *et al.* [6] studied barrier options under various asset price models, such as spectrally negative Lévy process, double-exponential jump-diffusion process, Lévy process with hyper-exponential jumps and so on. For more references on barrier options see Rich [14], Douady [3], Hui *et al.* [5], Sidenius [19], Shreve [18] etc.

The reason of barrier options becoming attractive and popular in the market can be explained from the design of the product. If an investor believes that the asset price unlikely reaches a certain level, he/she would like to add a provision to the vanilla option so as to reduce the premium. The features of barrier options meet

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the demands of this kind of investors. The premiums saved may be considerable. However, standard barrier option have series disadvantages, especially whenever the asset price fluctuates near the barrier. Market manipulation between option buyers and sellers results in violent price fluctuations. The investor loses his/her entire premium once the asset price hits the barrier if the barrier option is a knock-out option.

Motivated by the risk behavior of barrier options, Linetsky [11] and Davydov *et al.* [2] introduced a reasonable improvement on barrier options. They suggested a finite step knock-out (knock-in) rate ρ. Take the up and out step contract as an example, if the asset price hits a prespecified barrier, the contract does not expire at once, but its value reduces gradually in an exponential rate if the underlying price remains above the barrier level. In other words, if the occupation time above the prespecified barrier prior to expiration time T is $\Gamma^+(T)$, then the payment at T is $e^{-\rho\Gamma(T)}(S_T-K)^+$ for the call (and $e^{-\rho\Gamma(T)}(K-S_T)^+$ for the put). They are named geometric step options, or exponential step options, or proportional barrier options in the literature Linetsky [11] and Davydov *et al.* [2]. In our paper, we follow the name: geometric step options. Similar to the geometric step options, some other occupation time options, such as simple step options, delayed step options also serve as alternatives of standard barrier options, see Davydov *et al.* [2].

The geometric step options which Linetsky [11] and Davydov *et al.* [2] considered are indeed of European types, because the options are exercised only at a prespecified time T. In the present paper, we consider American type geometric step options. In other words, the contracts allow the owners of the option to exercise at any time on or before T. We will consider the perpetual American step option firstly. It is well known, the price of a standard perpetual American put option is given by

$$v_* = \max_{\tau\in\mathcal{T}} \tilde{E}\left[e^{-r\tau}(K-S(\tau))\right],$$

where $\mathcal{T}$ is the set of all stopping times and $\tilde{E}$ is the expectation with respect to the risk-neutral probability. It has been proved that the optimal exercise time is when the underlying asset price falls below an optimal level L_*. The optimal exercise time is a first hitting time. If an investor believes that the underlying asset price will not reach a certain upper barrier B in a rather long period, then he/she will be interested in a step option, the price of an American geometric step put option at $t=0$ should be

$$v_*(x) = \max_{\tau\in\mathcal{T}} \tilde{E}\left[e^{-r\tau-\rho\Gamma_B^+(\tau)}(K-S(\tau))\right],$$

where $\Gamma_B^+(\tau)$ denotes the amount of time that the underlying price process stays above the level B during the time period $(0,\tau)$. The first part of this paper calculates the value of $v_*(x)$. To obtain the pricing formula, we first set a fixed exercise level $L<K$, and exercise the option at τ_L, in this case the option price is $(K-L)\tilde{E}\left[e^{-r\tau_L-\rho\Gamma_B^+(\tau_L)}\right]$. The joint Laplace transform of τ_L and $\Gamma_B^+(\tau_L)$ is derived by using Feynman-Kac formula, which is a key step in the pricing calculation. Then we show that there is an optimal barrier L_* 'far enough' from K such that exercising the put option at τ_{L_*} is the optimal policy. Moreover, we prove that τ_{L_*} is the optimal time among all possible exercise time. Hence,

$$v_*(x) = v_{L_*}(x) = \max_{L} \tilde{E}\left[e^{-r\tau_L-\rho\Gamma_B^+(\tau_L)}(K-S(\tau_L))\right].$$

In the finial section of this paper, we consider the American geometric step option with finite expiration time. The price function $v(t,x)$ is defined by

$$v(t,x) = \max_{\tau\in\mathcal{T}_{t,T}} E\left[e^{-r(\tau-t)-\rho(\Gamma_B^+(\tau)-\Gamma_B^+(t))}(K-S_\tau) \middle| S_t = x \right],$$

where $\mathcal{T}_{t,T}$ is the set of all stopping time after t and prior to T. Unlike the perpetual case, the closed form of the pricing formula is impossible. However, we can show that $v(t,x)$ satisfies the variational inequalities

$$\begin{cases} v(t,x) \geq (K-x)^+, \\ (r+\rho I_{(B,\infty)}(x))v(t,x) - v_t(t,x) - rxv_x(t,x) - \frac{1}{2}\sigma^2x^2v_{xx}(t,x) \geq 0, \end{cases}$$

and either inequality takes equal sign. Note that the expressions of the left hand side of the second inequality are different for $x > B$ and $x < B$. We introduce a modified explicit finite difference scheme to compute the option price $v(t,x)$.

This paper is organized as follows. In Section 2, we describe the perpetual American geometric step options. In order to calculate the price of this kind options, we need the joint Laplace transform of the first hitting time and the occupation time of a standard Brownian motion, Section 3 is devoted to the calculation of this joint Laplace transform. In Section 4, assuming that a fixed level L below the strike price K is set and the option is exercised at the first hitting time τ_L, the price of the put option $v_L(x)$ is derived. Moreover, the optimal exercise barrier L_* which maximizes the option price is determined. In Section 5, by analyzing the price function $v_{L_*}(x)$, we show that the hitting time τ_{L_*} is the optimal exercise time among all stopping times, thus the price formula of the perpetual American step option is just $v_{L_*}(x)$. Section 6 is concerned with the finite expiration case. We show how to construct the explicit finite difference scheme and calculate the price numerically. The Appendix contains a proof of the result used in Section 4.

2. **American Geometric Step Options.** In this paper, we assume that the financial market satisfies the assumptions of the Black-Scholes model. Under the risk neutral probability, the underlying asset's price follows a geometric Brownian motion

$$dS_t = rS_tdt + \sigma S_tdB_t,$$

where r is the constant risk free interest rate, σ is the constant volatility, and B_t is a standard Brownian motion. Equivalently,

$$S_t = S_0\exp\{(r-\frac{\sigma^2}{2})t + \sigma B_t\}.$$

Consider an European up-and-out put option with strike price K and expiration time T. The contract stipulates that the option becomes worthless as soon as the asset price reaches a specified level, say B. So the payoff at expiration time T is

$$I\{\tau_B > T\}(K-S_T)^+,$$

where τ_B is the first time that the underlying asset price hits the barrier B.

In view of high risk near the barrier, the asset price may spike through the barrier in short term and then lead to a terrible loss to the option buyers, Linetsky [11] proposed step options, the pay off of a put step option with knock-up and out geometric rate ρ is

$$e^{-\rho\Gamma_B^+(T)}(K-S_T)^+,$$

where $\Gamma_B^+(T)$ is the occupation time of (B,∞) until time T, that is, the total time of the asset price stays above the barrier B. This means that the contract value will not become zero even the asset price up-crosses the barrier B, but reduce gradually in an exponential rate and be contingent on the occupation time until T. It is a reasonable modified version of the standard barrier option. Accordingly, replacing $\Gamma_B^+(T)$ by $\Gamma_B^-(T)$, the occupation time of $(-\infty, B)$, and put by call, we have a call step option with knock-down and out geometric rate.

The options described above are of European types since they specify a fixed expiration time and are exercised at the designed day. This paper considers American geometric step options. That is to say, option owners can exercise the option at any time. The payoff of at time τ should be

$$e^{-\rho\Gamma_B^+(\tau)}(K-S_\tau).$$

Since τ is an arbitrary time, the price of a perpetual geometric step put option is given by

$$v_*(x)=\max_{\tau\in\mathcal{T}}E\left[e^{-r\tau}e^{-\rho\Gamma_B^+(\tau)}(K-S_\tau)\right],$$

where x is the initial asset price, $\mathcal{T}$ is the set of all stopping times.

To calculate $v_*(x)$, the joint Laplace transform of the first hitting time τ_L and the occupation time $\Gamma_B^+(\tau_L)$ is needed, which is the main contribution of the next section.

3. **The Joint Laplace Transform of the First Hitting Time and Occupation Time.**

Assume that $\{B_t\}$ is a standard Brownian Motion. $b>0, l<0$ are some constants, denote the first hitting time T_l by

$$T_l=\inf\{t\ge 0: B_t=l\}.$$

Write $\Gamma_b^+(t)=\int_0^t I_{(b,\infty)}(B_s)ds$, which is the occupation time of (b,∞) until time t. Consider the joint distribution of T_l and $\Gamma_b^+(T_l)$, the occupation time of (b,∞) until T_l, we have the following Proposition.

Proposition 1. *For $\alpha,\beta,b>0$ and $l<0$, we have*

$$E^0\exp\{-\alpha T_l-\beta\Gamma_b^+(T_l)\}=\frac{(1-\sqrt{1+\frac{\beta}{\alpha}})e^{-\sqrt{2\alpha}b}+(1+\sqrt{1+\frac{\beta}{\alpha}})e^{\sqrt{2\alpha}b}}{(1-\sqrt{1+\frac{\beta}{\alpha}})e^{\sqrt{2\alpha}(l-b)}+(1+\sqrt{1+\frac{\beta}{\alpha}})e^{\sqrt{2\alpha}(b-l)}}.$$

Proof. For positive numbers α,β,γ, let

$$\begin{aligned} f(x) &= \beta I_{(b,\infty)}(x)+\alpha I_{(l,\infty)}(x)\\ \kappa(x) &= \beta I_{(b,\infty)}(x)+\gamma I_{(-\infty,l)}(x), \end{aligned}$$

Define

$$z(x)=E^x\int_0^\infty f(B_t)\exp\{-\alpha t-\int_0^t\kappa(B_s)ds\}dt,$$

where E^x is the expectation conditioning on $B_0=x$. Note that

$$z(0)=E^0\int_0^{T_l}f(B_t)\exp\{-\alpha t-\int_0^t\kappa(B_s)ds\}dt+E^0\int_{T_l}^\infty f(B_t)\exp\{-\alpha t-\int_0^t\kappa(B_s)ds\}dt.$$

For the second term, since $f(x)$ takes values on the set $\{0, \alpha, \alpha+\beta\}$, then

$$\left|E^0 \int_{T_l}^{\infty} f(B_t) \exp\{-\alpha t - \int_0^t \kappa(B_s)ds\}dt\right| \quad \leq \quad (\alpha+\beta)E^0 \int_{T_l}^{\infty} \exp\{-\alpha t - \int_0^t \kappa(B_s)ds\}dt.$$

Because $\int_0^t \kappa(B_s)ds = \beta\Gamma_b^+(t) + \gamma\Gamma_l^-(t)$ and $\Gamma_l^-(t) > 0$ a.s. on $\{T_l < t\}$ (See Karatzas *et al.* [7], Problem 2.7.19), letting $\gamma \to \infty$ yields

$$\lim_{\gamma\to\infty} E^0 \int_{T_l}^{\infty} \exp\{-\alpha t - \int_0^t \kappa(B_s)ds\}dt \to 0.$$

This result gives that

$$\begin{aligned}
\lim_{\gamma\to\infty} z(0) &= E^0 \int_0^{T_l} f(B_t) \exp\{-\alpha t - \int_0^t \kappa(B_s)ds\}dt \\
&= E^0 \int_0^{T_l} (\beta I_{(b,\infty)}(B_t) + \alpha) \exp\{-\alpha t - \int_0^t \beta I_{(b,\infty)}(B_s)\}dt \\
&= E^0 \int_0^{T_l} \exp\{-\alpha t - \beta\Gamma_b^+(t)\}d(\alpha t + \beta\Gamma_b^+(t)) \\
&= 1 - E^0 \exp\{-\alpha T_l - \beta\Gamma_b^+(T_l)\}.
\end{aligned}$$

Hence

$$E^0 \exp\{-\alpha T_l - \beta\Gamma_b^+(T_l)\} = 1 - \lim_{\gamma\to\infty} z(0). \tag{1}$$

According to Feynman-Kac formula (See Karatzas *et al.* [7], Theorem 4.4.9), the function $z(\cdot)$ is piecewise C^2 function and satisfies the equation

$$\begin{cases}
(\alpha+\beta)z = \frac{1}{2}z'' + (\alpha+\beta), & x > b, \\
\alpha z = \frac{1}{2}z'' + \alpha, & l < x < b, \\
(\alpha+\gamma)z = \frac{1}{2}z'', & x < l.
\end{cases}$$

The unique solution is of the form

$$z(x) = \begin{cases}
A_1 e^{\sqrt{2(\alpha+\beta)}x} + A_2 e^{-\sqrt{2(\alpha+\beta)}x} + 1, & x > b, \\
B_1 e^{\sqrt{2\alpha}x} + B_2 e^{-\sqrt{2\alpha}x} + 1, & l < x < b, \\
C_1 e^{\sqrt{2(\alpha+\gamma)}x} + C_2 e^{-\sqrt{2(\alpha+\gamma)}x}, & x < l.
\end{cases}$$

Recall the definition of $z(\cdot)$, it should be a bounded function, so the form of $z(x)$ is simplified to

$$z(x) = \begin{cases}
A e^{-\sqrt{2(\alpha+\beta)}x} + 1, & x > b, \\
B_1 e^{\sqrt{2\alpha}x} + B_2 e^{-\sqrt{2\alpha}x} + 1, & l < x < b, \\
C e^{\sqrt{2(\alpha+\gamma)}x}, & x < l.
\end{cases}$$

The four constants can be obtained since z is continuously differentiable at points $x = l$ and $x = b$. Obviously, the constants depend on the parameter γ, then we denote them by $A(\gamma), B_1(\gamma), B_2(\gamma), C(\gamma)$. In particular,

$$\begin{aligned}
1 - \lim_{\gamma\to\infty} z(0) &= -\lim_{\gamma\to\infty}(B_1(\gamma) + B_2(\gamma)) \\
&= \frac{(1-\sqrt{1+\frac{\beta}{\alpha}})e^{-\sqrt{2\alpha}b} + (1+\sqrt{1+\frac{\beta}{\alpha}})e^{\sqrt{2\alpha}b}}{(1-\sqrt{1+\frac{\beta}{\alpha}})e^{\sqrt{2\alpha}(l-b)} + (1+\sqrt{1+\frac{\beta}{\alpha}})e^{\sqrt{2\alpha}(b-l)}}.
\end{aligned}$$

Due to the identity (1), we have

$$E^0 \exp\{-\alpha T_l - \beta\Gamma_b^+(T_l)\} = \frac{(1-\sqrt{1+\frac{\beta}{\alpha}})e^{-\sqrt{2\alpha}b} + (1+\sqrt{1+\frac{\beta}{\alpha}})e^{\sqrt{2\alpha}b}}{(1-\sqrt{1+\frac{\beta}{\alpha}})e^{\sqrt{2\alpha}(l-b)} + (1+\sqrt{1+\frac{\beta}{\alpha}})e^{\sqrt{2\alpha}(b-l)}}.$$

□

4. **Pricing American Geometric Step Options I: Perpetual Case.** In this section, we adopt a 'L level strategy' to exercise the put option: Firstly, we set a barrier $L < K$. If the initial asset price is at or below L, then exercise the option now. In this case the option value is $K - S_0$. If the initial asset price is above L, then exercise the put option at the first hitting time τ_L, which is is defined by $\tau_L = \inf\{t > 0 : S_t < L\}$. We use $v_L(x)$ to denote the option price, then $v_L(x) = E\left[e^{-r\tau}e^{-\rho\Gamma_B^+(\tau)}(K - S_\tau)\right]$. In what follows , the value of $v_L(x)$ will be considered. Secondly, we show that there is an optimal barrier L_* such that

$$v_{L_*}(x) = \max_L E\left[e^{-r\tau_L}e^{-\rho\Gamma_B^+(\tau_L)}(K - S_{\tau_L})\middle| S_0 = x\right].$$

Without loss of generality, we assume $S_0 \geq L$ in the following discussion, $B > 0$ is the upper barrier designed in the barrier contract. Let's introduce some notations

$$b = \frac{1}{\sigma}\log\frac{B}{S_0}, \quad l = \frac{1}{\sigma}\log\frac{L}{S_0}, \quad \nu = \frac{1}{\sigma}(r - \frac{\sigma^2}{2}),$$

then

$$v_L(x) = E\left[e^{-r\tau_L}e^{-\rho\Gamma_B^+(\tau_L)}(K - S_{\tau_L})\middle| S_0 = x\right] = (K-L)E\left[e^{-rT_l}e^{-\rho\Gamma_b^+(T_l)}\middle| B_0 = 0\right],$$

where T_l is the first hitting time of the Brownian motion with drift $\{B_t + \nu t\}$, $\Gamma_b^+(T_l)$ is the occupation time of (b, ∞) of $\{B_t + \nu t\}$ until T_l. According to Girsanov Theorem, $B_t + \nu t$ is a standard Brownian motion starting from 0 under the new probability

$$\tilde{P}(A) = \int_A Z(T)dP \quad \text{for all } A \in \mathcal{F}_T,$$

where $\mathcal{F}_T = \sigma\{B_t, 0 \leq t \leq T\}$ and $Z(t)$ is an exponential martingale defined by

$$Z(t) = e^{-\nu(B_t+\nu t)+\frac{1}{2}\nu^2 t}, \quad 0 \leq t \leq T.$$

Hence,

$$\begin{aligned} v_L(x) &= (K-L)\tilde{E}^0[Z^{-1}(T_l)e^{-rT_l}e^{-\rho\Gamma_b^+(T_l)}] \\ &= (K-L)e^{\nu l}\tilde{E}^0[e^{-(\frac{1}{2}\nu^2+r)T_l - \rho\Gamma_b^+(T_l)}], \end{aligned}$$

where $\tilde{E}^0$ is corresponding expectation under $\tilde{P}$. Under $\tilde{P}$, T_l and Γ_b^+ become the first hitting time and occupation time of a standard Brownian motion starting from 0. If $S_0 < B$, by Proposition 1 we have

$$\begin{aligned} v_L(x) &= (K-L)e^{\nu l}\frac{(1-\sqrt{1+\frac{2\rho}{\nu^2+2r}})e^{-\sqrt{\nu^2+2r}b} + (1+\sqrt{1+\frac{2\rho}{\nu^2+2r}})e^{\sqrt{\nu^2+2r}b}}{(1-\sqrt{1+\frac{2\rho}{\nu^2+2r}})e^{\sqrt{\nu^2+2r}(l-b)} + (1+\sqrt{1+\frac{2\rho}{\nu^2+2r}})e^{\sqrt{\nu^2+2r}(b-l)}} \\ &= \frac{(K-L)(\frac{L}{x})^{\frac{r}{\sigma^2}-\frac{1}{2}}\left[c_1(\frac{B}{x})^{-\frac{r}{\sigma^2}-\frac{1}{2}} + c_2(\frac{B}{x})^{\frac{r}{\sigma^2}+\frac{1}{2}}\right]}{c_1(\frac{L}{B})^{\frac{r}{\sigma^2}+\frac{1}{2}} + c_2(\frac{B}{L})^{\frac{r}{\sigma^2}+\frac{1}{2}}}, \end{aligned}$$

where $c_1 = 1 - \sqrt{1 + \frac{2\rho}{\nu^2+2r}}$, $c_2 = 1 + \sqrt{1 + \frac{2\rho}{\nu^2+2r}}$. For simplicity, let

$$g(L) = \frac{(K-L)L^{\frac{r}{\sigma^2}-\frac{1}{2}}}{c_1(\frac{L}{B})^{\frac{r}{\sigma^2}+\frac{1}{2}} + c_2(\frac{B}{L})^{\frac{r}{\sigma^2}+\frac{1}{2}}},$$

then

$$v_L(x) = g(L)\left(c_1 B^{-\frac{r}{\sigma^2}-\frac{1}{2}}x + c_2 B^{\frac{r}{\sigma^2}+\frac{1}{2}}x^{-\frac{2r}{\sigma^2}}\right), \quad x < B. \tag{2}$$

If $S_0 \geq B$, by using Markov property and $E^0[e^{-\alpha T_a}] = e^{-a\sqrt{2\alpha}}$, $a > 0$ (see (2.8.6) of Karatzas *et al.* [7]), we have

$$\begin{aligned} v_L(x) &= (K-L)E\left[e^{-r\tau_L}e^{-\rho\Gamma_B^+(\tau_L)}\right] \\ &= (K-L)e^{\nu l}\tilde{E}e^{-(\frac{1}{2}\nu^2+r+\rho)T_b}\tilde{E}[e^{-(\frac{1}{2}\nu^2+r)T_{l-b}-\rho\Gamma_0^+(T_{l-b})}] \\ &= \frac{2(K-L)(\frac{L}{x})^{\frac{r}{\sigma^2}-\frac{1}{2}}(\frac{B}{x})^{\frac{\sqrt{(r+\frac{\sigma^2}{2})^2+2\rho\sigma^2}}{\sigma^2}}}{c_1(\frac{L}{B})^{\frac{r}{\sigma^2}+\frac{1}{2}} + c_2(\frac{B}{L})^{\frac{r}{\sigma^2}+\frac{1}{2}}} \\ &= 2g(L)B^{\frac{\sqrt{(r+\frac{\sigma^2}{2})^2+2\rho\sigma^2}}{\sigma^2}}x^{-\frac{r}{\sigma^2}+\frac{1}{2}-\frac{\sqrt{(r+\frac{\sigma^2}{2})^2+2\rho\sigma^2}}{\sigma^2}}, \quad x \geq B. \end{aligned} \tag{3}$$

In summary, we state the results of (2) and (3) as follows.

Proposition 2. *Suppose that $\{S_t\}_{t\geq 0}$ is the underlying asset price starting from $S_0 = x$. $v_L(x)$ is the value of the perpetual American geometric step put with exercise level L and strike price K, $L \leq \min\{K, x\}$, then*

$$v_L(x) = \begin{cases} g(L)\left(c_1 B^{-\frac{r}{\sigma^2}-\frac{1}{2}}x + c_2 B^{\frac{r}{\sigma^2}+\frac{1}{2}}x^{-\frac{2r}{\sigma^2}}\right), & x < B; \\ 2g(L)B^{\frac{\sqrt{(r+\frac{\sigma^2}{2})^2+2\rho\sigma^2}}{\sigma^2}}x^{-\frac{r}{\sigma^2}+\frac{1}{2}-\frac{\sqrt{(r+\frac{\sigma^2}{2})^2+2\rho\sigma^2}}{\sigma^2}}, & x \geq B, \end{cases}$$

where $c_1 = 1-\sqrt{1 + \frac{2\rho}{\nu^2+2r}}$, $c_2 = 1+\sqrt{1 + \frac{2\rho}{\nu^2+2r}}$ and $g(L) = \frac{(K-L)L^{\frac{r}{\sigma^2}-\frac{1}{2}}}{c_1(\frac{L}{B})^{\frac{r}{\sigma^2}+\frac{1}{2}}+c_2(\frac{B}{L})^{\frac{r}{\sigma^2}+\frac{1}{2}}}$.

In what follows, we aim to find an argument $L_* \in (0, \infty)$, which maximizes $v_L(x)$. We call L_* the optimal exercise level. It is obvious that any level above $\min\{K, x\}$ is impossible to be the optimal exercise level because of the definition of $v_L(x)$. So we restrict L to $(0, \min\{K, x\}]$.

Theorem 4.1. *There exists a unique $L_* \in (0, \min\{K, x\}]$ which is the optimal exercise level in the sense: if $x > L_*$, then $v_{L_*}(x) = \max_L v_L(x)$; otherwise, if $x \leq L_*$, $K - x = \max_L v_L(x)$ (in this case $x < K$). As a result, under the 'L level strategy' the value of the perpetual American geometric step put option is given by*

$$v_{L_*}(x) = \begin{cases} K - x, & x \leq L_*; \\ g(L_*)\left(c_1 B^{-\frac{r}{\sigma^2}-\frac{1}{2}}x + c_2 B^{\frac{r}{\sigma^2}+\frac{1}{2}}x^{-\frac{2r}{\sigma^2}}\right), & L_* < x < B; \\ 2g(L_*)B^{\frac{\sqrt{(r+\frac{\sigma^2}{2})^2+2\rho\sigma^2}}{\sigma^2}}x^{-\frac{r}{\sigma^2}+\frac{1}{2}-\frac{\sqrt{(r+\frac{\sigma^2}{2})^2+2\rho\sigma^2}}{\sigma^2}}, & x \geq B. \end{cases}$$

Proof. By Proposition 2, each multiplicator in the expression of $v_L(x)$ is positive. So $v_L(x)$ attains its maximum at a level $L_* \in (0, \min\{K, x\}]$ if and only if $g(L)$ attains its maximum at that point. To find L_*, it suffices to study the function $g(L)$. In Appendix, we show that there is a unique constant L_* such that $g(L)$ increases from 0 to a maximum $g(L_*)$ and decreases from $g(L_*)$ to 0 in $[0, K]$, see

Figure 1. Consequently, if the initial asset price $x > L_*$, the optimal exercise time is τ_{L_*} and the option price is $v_{L_*}(x)$; whereas, if $x \leq L_*$, $\{v_L(x), L \in (0, \min\{K, x\}]\}$ attains its maximum at the point x. In this case, the optimal exercise level is x and we should exercise at once, so the option price is $K - x$. $\square$

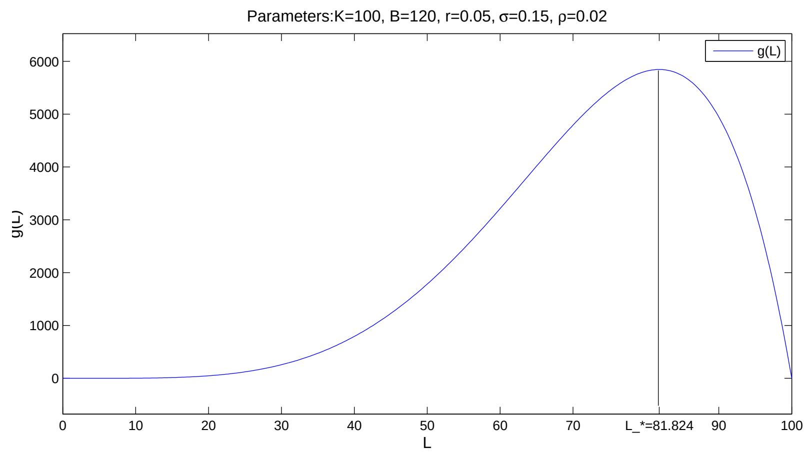

FIGURE 1. The figure of $g(L)$ between 0 and K

Remark 1. Theorem 4.1 states that under "L level strategy", if the initial asset price S_0 is lower than L_*, then it is best to exercise the option at once and receive the intrinsic value $K - S_0$; if the initial asset price is larger than L_*, waiting until the asset price hits the level L_*. Figure 2 shows the price $v_{L_*}(x)$ as a function of the initial asset price x.

Form the following Corollary, we also see that L_* could be uniquely determined.

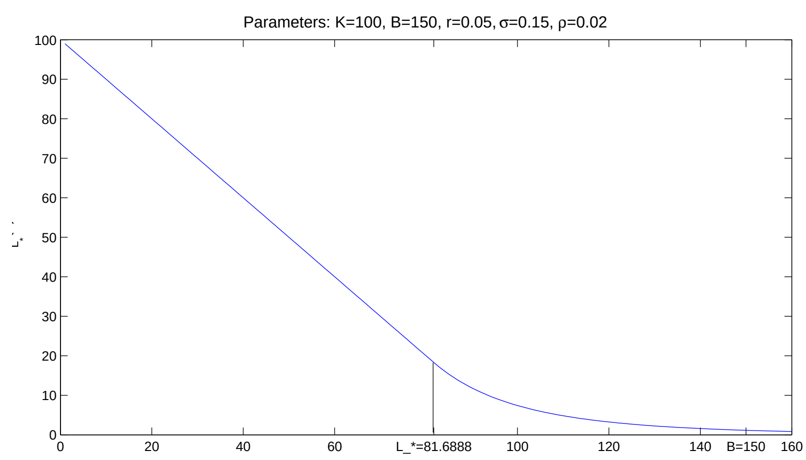

FIGURE 2. The price of the American geometric step put $v_{L_*}(x)$

Corollary 1. *L_* is the unique root of the following equation for $x \in [0, \frac{K}{B}]$.*

$$(1+\frac{2r}{\sigma^2})Bc_2x - \frac{2r}{\sigma^2}Kc_2 + Kc_1x^{\frac{2r}{\sigma^2}+1} = 0.$$

See Appendix for the proof.

5. **Hitting Time as the Optimal Exercise Time.** Using Itô formula, we shall show that τ_{L_*} is the optimal exercise time among all stopping times. However, application of Itô formula requires that $v_{L_*}(x)$ is a piecewise C^2 function. In other words, $v_{L_*}(x)$ is continuously differentiable and the second derivative is piecewise continuous. By Theorem 4.1, $v_{L_*}(x)$ is twice continuously differentiable except at the point L_* and B. In addition, the continuity at L_* and B of $v^{'}_{L_*}(x)$ can be verified by the following lemma. Thus $v_{L_*}(x)$ is a piecewise C^2 function.

Lemma 5.1. *When $x > L_*$, $v_{L_*}(x)$ satisfies the equation*

$$(-r-\rho I_{(B,\infty)}(x))v_{L_*}(x) + rxv^{'}_{L_*}(x) + \frac{\sigma^2}{2}x^2v^{''}_{L_*}(x) = 0.$$

Furthermore, the derivative of $v_{L_*}(x)$ is continuous at $x = L_*$ and $x = B$.

Proof. It is a simple calculation to verify that $v_{L_*}(x)$ satisfies the differential equation and $v^{'}_{L_*}(x)$ is continuous at B. So we omit the details here. Now we evaluate the right-hand derivative $v^{'}_{L_*}(L_*+)$ and prove that it agrees with the left-hand derivative $v^{'}_{L_*}(L_*-) = (K-x)^{'} = -1$. For convenience, $v_L(x)$ is also rewritten as $v(L,x)$. Note that $v(L,x)$ is the product of separate functions of of L and x, i.e. $v_L(x) = g(L)h(x)$. Since $g^{'}(L_*) = 0$, we have

$$\begin{aligned} v^{'}_{L_*}(L_*+) &= g(L_*)h^{'}(L_*) \\ &= g(L_*)h^{'}(L_*) + g^{'}(L_*)h(L_*) \\ &= \frac{d}{dL}v(L,L)|_{L=L_*} \\ &= \frac{d}{dL}(K-L)|_{L=L_*} \\ &= -1. \end{aligned}$$

□

Proposition 3. *Let S_t be the price of the underlying asset and let τ_{L_*} be the first time of S_t reaching the level L_*. Then $\left\{e^{-rt-\rho\Gamma^+_B(t)}v_{L_*}(S_t)\right\}$ is a supermartingale and the stopped process $\left\{e^{-r(t\wedge\tau_{L_*})-\rho\Gamma^+_B(t\wedge\tau_{L_*})}v_{L_*}(S_{t\wedge\tau_{L_*}})\right\}$ is a martingale.*

Proof. Since $v_{L_*}(x)$ is continuously differentiable and has piecewise continuous second derivative, by Itô formula we have

$$\begin{aligned} & de^{-rt-\rho\Gamma^+_B(t)}v_{L_*}(S_t) \\ = \; & e^{-rt-\rho\Gamma^+_B(t)}\left[\left(-r-\rho I_{(B,\infty)}(S_t)\right)v_{L_*}(S_t) + rS_tv^{'}_{L_*}(S_t) + \frac{\sigma^2}{2}S_t^2v^{''}_{L_*}(S_t)\right]dt \\ & +\sigma S_te^{-rt-\rho\Gamma^+_B(t)}v^{'}_{L_*}(S_t)dB_t. \end{aligned}$$

By Lemma 5.1, for $x \geq L_*$,

$$(-r-\rho I_{(B,\infty)}(x))v_{L_*}(x) + rxv^{'}_{L_*}(x) + \frac{\sigma^2}{2}x^2v^{''}_{L_*}(x) = 0,$$

and for $x < L_*$, $v_{L_*}(x) = K - x$,

$$(-r - \rho I_{(B,\infty)}(x))v_{L_*}(x) + rxv_{L_*}^{'}(x) + \frac{\sigma^2}{2}x^2 v_{L_*}^{''}(x) = -rK < 0.$$

Hence, by taking expectation we have

$$E\left[e^{-rt-\rho\Gamma_B^+(t)}v_{L_*}(S_t)\right] \le v_{L_*}(S_0).$$

By the Markov property of $S(t)$, $\{e^{-rt-\rho\Gamma_B^+(t)}v_{L_*}(S_t)\}$ is a supermartingale.

If the initial price is above L_*, then $S(t) > L_*$ prior to time τ_L. From Itô formula, the drift term before dt is zero and hence the stopped process $\{e^{-r(t\wedge\tau_{L_*})-\rho\Gamma_B^+(t\wedge\tau_{L_*})}v_{L_*}(S_{t\wedge\tau_{L_*}})\}$ is a martingale. $\square$

Theorem 5.2. *Suppose that $\mathcal{T}$ is the set of all stopping times. We have*

$$v_{L_*}(x) = \max_{\tau\in\mathcal{T}} E\left[e^{-r\tau}e^{-\rho\Gamma_B^+(\tau)}(K - S_\tau)\right].$$

In another word, the hitting time τ_{L_*} is optimal among all exercise times. $v_{L_*}(x)$ is the price of the perpetual American geometric step put option.

Proof. By Proposition 3, $\left\{e^{-rt-\rho\Gamma_B^+(t)}v_{L_*}(S_t)\right\}$ is a supermartingale, by optional sampling Theorem for every stopping time $\tau \in \mathcal{T}$,

$$v_{L_*}(x) \ge E\left[e^{-r(t\wedge\tau)-\rho\Gamma_B^+(t\wedge\tau)}v_{L_*}(S_{(t\wedge\tau)})\right].$$

Letting $t \to \infty$ and using bounded convergence Theorem

$$v_{L_*}(x) \ge E\left[e^{-r\tau-\rho\Gamma_B^+(\tau)}v_{L_*}(S_\tau)\right] \ge E\left[e^{-r\tau-\rho\Gamma_B^+(\tau)}(K - S(\tau))\right], \tag{4}$$

where $v_{L_*}(x) \ge (K - x)^+$ is used. In fact, for any fixed x,

$$v_{L_*}(x) = \max_L v_L(x) \ge \begin{cases} v_x(x) = K - x, & x \le K, \\ v_K(x) = 0, & x > K. \end{cases}$$

Since the inequality (4) holds for arbitrary stopping time $\tau \in \mathcal{T}$,

$$v_{L_*}(x) \ge \max_{\tau\in\mathcal{T}} E\left[e^{-r\tau-\rho\Gamma_B^+(\tau)}(K - S(\tau))\right].$$

On the other hand, replacing τ by τ_{L_*} and using the fact that $\left\{e^{-r(t\wedge\tau_{L_*})-\rho\Gamma_B^+(t\wedge\tau_{L_*})}v_{L_*}(S_{t\wedge\tau_{L_*}})\right\}$ is a martingale, we have

$$v_{L_*}(x) = E\left[e^{-r\tau_{L_*}-\rho\Gamma_B^+(\tau_{L_*})}(K - S(\tau_{L_*}))\right].$$

Finally, it can be concluded that

$$v_{L_*}(x) = \max_{\tau\in\mathcal{T}} E\left[e^{-r\tau-\rho\Gamma_B^+(\tau)}(K - S(\tau))\right].$$

$\square$

6. **Pricing American Geometric Step Options II: Finite Expiration Case.** In this section, we consider the finite expiration case. The price of the American geometric step put with finite expiration is given by

$$v(t,x) = \max_{\tau \in \mathcal{T}_{t,T}} E\left[e^{-r(\tau-t)-\rho(\Gamma_B^+(\tau)-\Gamma_B^+(t))}(K-S_\tau) \middle| S_t = x \right], \tag{5}$$

where $\mathcal{T}_{t,T}$ is the set of all stopping time after t and prior to T. Recall a standard American put with strike K and expiration time T, an option which can be exercised at any time prior to time T. Similar to the perpetual American put, the optimal strategy is again a 'L level strategy'. The difference lies, the optimal level depends on the time to expiration, i.e. $L = L(T-t)$. It has been proved that the finite expiration American put price function $v(t,x)$ satisfies the linear complementarity condition

$$\begin{cases} v(t,x) \geq (K-x)^+ \\ rv(t,x) - v_t(t,x) - rxv_x(t,x) - \frac{1}{2}\sigma^2 x^2 v_{xx}(t,x) \geq 0 \end{cases}$$

for each $t \in [0,T)$ and $x \geq 0$. Either inequality above is an equity. Readers are referred to Chapter 8, Shreve [18] for details. The similar property holds by the American type geometric step options with finite expiration, we present the result without proof.

Theorem 6.1. *The price $v(t,x)$ defined by (5) has continuous first order partial derivatives and piecewise continuous second partial derivative with respect to x. It satisfies the linear complementary condition*

$$\begin{cases} v(t,x) \geq (K-x)^+ \\ (r+\rho I_{(B,\infty)}(x))v(t,x) - v_t(t,x) - rxv_x(t,x) - \frac{1}{2}\sigma^2 x^2 v_{xx}(t,x) \geq 0 \end{cases} \tag{6}$$

for each $t \in [0,T)$ and $x > 0$. Moreover, either of them is an equality.

Different from the standard American option, the second inequality in (6) is different for $x > B$ and $x < B$. In general, the closed form of the price function is not possible. In relating literatures, there are some of widely used methods, including finite difference scheme, Monte Carlo simulation and binomial tree, to numerically price American option, see Tsitsiklis *et al.* [20], Longstaff *et al.* [12] and Karatzas *et al.* [8]. The finite difference scheme is one of the popular approaches and it is easy to implement for American type options. Theorem 6.1 provides the inequalities to determine a unique price function $v(t,x)$. In what follows, we show how to compute $v(t,x)$ by the finite difference scheme.

Without loss of generality, we take the exercise price $K=1$ because it's not hard to show that $v_{K=K_0,B=B_0}(t,x) = K_0 v_{K=1,B=B_0/K_0}(t,\frac{x}{K_0})$. We first transform the original (t,x) to (s,y) by $s = T - \frac{2}{\sigma^2}t$, $x = e^y$, and let $v(t,x) = w(s,y)$. Then by Theorem 6.1 we have

$$\begin{cases} w_s(s,y) \geq w_{yy}(s,y) + (k-1)w_y(s,y) - kw(s,y) - \frac{2\rho}{\sigma^2} I_{(b,\infty)}(y)w(s,y), \\ w(s,y) \geq (1-e^y)^+, \end{cases}$$

either inequality must be an equality, where $k = \frac{2r}{\sigma^2}, b = \log\frac{B}{K}$.

Let $\alpha = -\frac{1}{2}(k-1), \beta = -\frac{1}{4}(k+1)^2$ and $w(s,y) = e^{\alpha y + \beta s}u(s,y)$, then $u(s,y)$ satisfies

$$\begin{cases} u_s(s,y) \geq u_{yy}(s,y) - \frac{2\rho}{\sigma^2} I_{(b,\infty)}(y)u(s,y) \\ u(s,y) \geq e^{-\alpha y - \beta s}(1-e^y)^+, \end{cases} \tag{7}$$

as before, either inequity takes equal sign. In fact, equal signs in the formulations above correspond to whether it is optimal to exercise the option ('=' holds in the second inequality) or not ('=' holds in the first inequality). Let's consider the initial and boundary conditions. No matter whenever the time is, as long as $y \to -\infty$, the corresponding price x lies below the optimal level $L(T-t)$. In this case it is optimal to exercise at once, so if y^- is small enough

$$u_s(s, y^-) \sim e^{-\alpha y^- - \beta s}(1 - e^{y^-})^+. \tag{8}$$

When $y \to \infty$, the time cost by the asset price from a high price to the level K is quite long (since the optimal exercise level is below K), a longtime discount makes the option value approaches to zero. Hence the second boundary condition is: for y^+ large enough,

$$u_s(s, y^+) \sim 0.$$

Finally, since $v(T, x) = (1-x)^+$, we have the initial condition

$$u(0, y) = e^{-\alpha y}(1 - e^y)^+. \tag{9}$$

Note that $u(s, y)$ has different expressions for $y > b$ and $y \le b$, in order to avoid technical complications, we adopt the 'explicit' finite difference method. For more details on the finite difference method for American option, we refer to Chapters 8 and 9, Wilmott et al. [21]. We divide the (s, y) plane into a regular mesh, approximate terms of the form $u_s - u_{yy}$ by finite differences with step sizes ds and dy. Truncating so that x lies between $-N_2 * dy$ and $N_1 * dy$, where N_1 and N_2 are suitably large numbers. Define $u(m, n) = u(m * ds, n * dy)$.

Define

$$\begin{aligned}
Eq1(m,n) &= \frac{u(m+1,n) - u(m,n)}{ds} - \frac{u(m,n+1) - 2u(m,n) + u(m,n-1)}{(dx)^2}, \\
Eq2(m,n) &= u(m+1,n) - e^{-\alpha n*dy - \beta m*ds}(1 - e^{n*dy})^+.
\end{aligned}$$

If, at time-step m, $u(m, n)$ for all values of n are known, we can explicitly calculate $u(m+1, n)$ for $n * dy < b$ by using the following difference equation

$$\begin{cases} Eq1(m,n) \ge 0 \\ Eq2(m,n) \ge 0 \\ Eq1(m,n) \times Eq2(m,n) = 0. \end{cases}$$

and the conditions (8) and (9). We rearrange this to obtain an explicit form

$$\begin{cases} Eq3(m,n) \ge 0 \\ Eq4(m,n) \ge 0 \\ Eq3(m,n) \times Eq4(m,n) = 0, \end{cases}$$

where $a = ds/(dy)^2$ and

$$\begin{aligned}
Eq3(m,n) &= u(m+1,n) - au(m,n+1) + (1-2a)u(m,n) + au(m,n-1), \\
Eq4(m,n) &= u(m+1,n) - e^{-\alpha n*dy - \beta m*ds}(1 - e^{n*dy})^+.
\end{aligned}$$

For the sake of stability, a is limited to $(0, \frac{1}{2})$. The constraint $Eq3(m,n) \times Eq4(m,n) = 0$ can be implemented as follows

$$u(m+1,n) = \max\left(au(m,n+1) + (1-2a)u(m,n) + au(m,n-1), e^{-\alpha n*dy - \beta m*ds}(1 - e^{n*dy})^+\right)$$

	The Initial Price of Underlying Asset x							
Time t	0.0043		87.4447		114.3580		144.6217	
	std	step	std	step	std	step	std	step
0	99.9707	99.9707	16.2020	16.1989	5.3419	5.3049	1.3232	1.2476
0.1000	99.9707	99.9707	15.8998	15.8978	4.9346	4.9067	1.1030	1.0452
0.2000	99.9707	99.9707	15.5770	15.5759	4.4996	4.4795	0.8824	0.8450
0.3000	99.9707	99.9707	15.2317	15.2312	4.0328	4.0193	0.6793	0.6504
0.4000	99.9707	99.9707	14.8614	14.8612	3.5295	3.5211	0.4845	0.4665
0.5000	99.9707	99.9707	14.4611	14.4610	2.9833	2.9788	0.3101	0.3002
0.6000	99.9707	99.9707	14.0293	14.0293	2.3873	2.3853	0.1655	0.1611
0.7000	99.9707	99.9707	13.5640	13.5640	1.7341	1.7334	0.0625	0.0612
0.8000	99.9707	99.9707	13.0749	13.0749	1.0232	1.0231	0.0103	0.0102
0.9000	99.9707	99.9707	12.6361	12.6361	0.3060	0.3060	0.0001	0.0001
1.0000	99.9957	99.9957	12.5553	12.5553	0	0	0	0

TABLE 1. The option parameters: the expiration time is 1, the strike time is 100, the barrier is 120, the risk free interest rate is 0.05, the volatility is 0.3, the knock out rate is 0.2. In the implementation of the implicit finite difference scheme, a is set to be 0.2, take 200 time steps, and 700 asset price steps (400 step below K, 300 step above K). 'std' represents the standard American put, 'step' represents the American type geometric step option.

Let's consider the case of $n * dy > b$. Suppose that values of $u(s_0, y)$ for all y are known, where s_0 is a fixed time. Define $\tilde{u}(s_0 + s, y) = u(s_0 + s, y)e^{\frac{2\rho}{\sigma^2}s}$. If $\tilde{u}_s(s, y) = \tilde{u}_{yy}(s, y)$, then it is easy to verify that

$$u_s(s_0 + s, y) = u_{yy}(s_0 + s, y) - \frac{2\rho}{\sigma^2}u(s_0 + s, y).$$

The dynamic consists with (7) if the first inequality takes equal sign. This gives us an idea to calculate the approximated value of $u(m, n)$, as outlined in follows. As usual, suppose all values of $u(m, n)$ are known at time-step m. Set $\tilde{u}(m, n) = u(m, n)$ for all n of $n * dy > b$, through

$$\tilde{u}(m + 1, n) = a\tilde{u}(m, n + 1) + (1 - 2a)\tilde{u}(m, n) + a\tilde{u}(m, n - 1),$$

we obtain the values of $\tilde{u}(m + 1, n)$. At the same time $u(m + 1, n)$ is an immediate result of the relation

$$u(m + 1, n) = \tilde{u}(m + 1, n)e^{-\frac{2\rho}{\sigma^2} * ds}.$$

Based on the iterations above, all values of $u(m, n)$ are numerically calculated. Finally, by some simple transformations, each (t, x) corresponding to $u(m, n)$ is easily obtained.

Table 1 provides some numerical results calculated using the finite difference scheme. We assume that the option has strike price $K = 100$ and expiration time $T = 1$, the ratio of the barrier and strike price is $B_0 = 1.2$, which implies the barrier is set at $B = 120$, the risk free interest rate is assumed to be $r = 0.05$ and the volatility of the underlying asset is $\sigma = 0.3$ per annum, knock out rate ρ is set to be 0.2. In this case, According to $a = dt/dy^2 = 0.2$, we partition the time axis into $M = 200$ steps, and take 700 y-steps, including 400 step below K and 300 step above K. So the mesh grids are 201×701. Table 1 compares the prices of the standard American put option and the American type geometric step put. Agree with our intuition, the latter price is relatively cheaper than the former due to the barrier provision. For the same initial time, the lower the initial asset price, the smaller the relative difference between two kinds of option prices. We also see that if the initial asset price is the same, the difference between two kinds of option tends to 0 as $t \to T$ if $x \leq K$, but not for $x > K$.

Appendix. Proof of existence and uniqueness of L_* In this Appendix, we show there is a unique constant $L_* \in (0, K)$ such that $g(L)$ strictly increases in $(0, L_*)$ and strictly decreases in (L_*, K), where $g(L)$ is defined by

$$g(L) = \frac{(K-L)L^{\frac{r}{\sigma^2}-\frac{1}{2}}}{c_1(\frac{L}{B})^{\frac{r}{\sigma^2}+\frac{1}{2}} + c_2(\frac{B}{L})^{\frac{r}{\sigma^2}+\frac{1}{2}}}.$$

Let $y = \frac{L}{B}$, then

$$\begin{aligned} g(L)B^{-\frac{r}{\sigma^2}+\frac{1}{2}} &= \frac{(K-B\frac{L}{B})(\frac{L}{B})^{\frac{r}{\sigma^2}-\frac{1}{2}}}{c_1(\frac{L}{B})^{\frac{r}{\sigma^2}+\frac{1}{2}} + c_2(\frac{L}{B})^{-\frac{r}{\sigma^2}-\frac{1}{2}}} \\ &= \frac{(K-By)y^{\frac{r}{\sigma^2}-\frac{1}{2}}}{c_1 y^{\frac{r}{\sigma^2}+\frac{1}{2}} + c_2 y^{-\frac{r}{\sigma^2}-\frac{1}{2}}} \\ &= \frac{K-By}{c_1 y + c_2 y^{-\frac{2r}{\sigma^2}}} \\ &= g_1(y). \end{aligned}$$

For our purpose, it only needs to show $g_1^{'}(y) > 0$ in $(0, y^*)$ and $g_1^{'}(y) < 0$ in $(y^*, \frac{K}{B})$ for some $y^* \in (0, \frac{K}{B})$. We calculate the derivative of $g_1(y)$,

$$g_1^{'}(y) = \frac{-y^{-\frac{2r}{\sigma^2}-1}\left((1+\frac{2r}{\sigma^2})Bc_2 y - \frac{2r}{\sigma^2}Kc_2 + Kc_1 y^{\frac{2r}{\sigma^2}+1}\right)}{(c_1 y + c_2 y^{-\frac{2r}{\sigma^2}})^2}.$$

Let

$$\begin{aligned} k_1(y) &= (1+\frac{2r}{\sigma^2})Bc_2 y - \frac{2r}{\sigma^2}Kc_2, \\ k_2(y) &= -Kc_1 y^{\frac{2r}{\sigma^2}+1}, \end{aligned}$$

it is easy to check $k_1(0) < k_2(0), \lim\limits_{y\to\infty} k_1(y) - k_2(y) = -\infty, k_1(\frac{K}{B}) - k_2(\frac{K}{B}) > 0$. Since $k_1(y)$ is a linear function and $k_2(y)$ is a convex function, $k_1(y) - k_2(y)$ has and only has two zeros. One lies in $(0, \frac{K}{B})$, which is denoted by y^*, and the other one lies in $(\frac{K}{B}, \infty)$. As a consequence, $g_1^{'}(y) > 0$ in $(0, y^*)$ and $g_1^{'}(y) < 0$ in $(y^*, \frac{K}{B})$. Figure 1 shows the behavior of $g(L)$.

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