Development of classical boundary element analysis of fracture mechanics in gradient materials

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Abstract Over the last decade, the authors have extended the classical boundary element methods (BEM) for analysis of the fracture mechanics in functionally gradient materials. This paper introduces the dual boundary element method associated with the generalized Kelvin fundamental solutions of multilayered elastic solids (or Yue's solution). This dual BEM uses a pair of the displacement and traction boundary integral equations. The former is collocated exclusively on the uncracked boundary, and the latter is collocated only on one side of the crack surface. All the singular integrals in dual boundary integral equations have been solved by numerical and rigid-body motion methods. This paper then introduces two applications of the dual BEM to fracture mechanics. These research results include the stress intensity factor values of different cracks in the materials, some fracture mechanics properties of layered rocks in rock engineering.

Keywords boundary element method, generalized Kelvin solution, FGMs, fracture mechanics, singular integrals

1. Introduction

The functionally gradient materials (FGMs) are applicable to many engineering fields. In FGMs, the composite medium is processed in such a way that the material properties are continuous functions of the depth or thickness coordinate. The knowledge of fracture mechanics in FGMs is important in order to evaluate their integrity. Crack problems in FGMs have become one of the hottest topics of active investigation in fracture mechanics [1].

The boundary element method (BEM), also known as the boundary integral equation method, has firmly established in many engineering disciplines and is increasingly manifested to be an effective numerical approach. The attraction of BEM can be largely attributed to the reduction in the dimensionality of the problem and to the efficient modeling of the stress concentration. Thus, BEM can overcome the limitations associated with FEM or other numerical methods in accurately and efficiently analyzing the crack problems. Aliabadi [2] pointed out that fracture mechanics has been the most active specialized area of using BEM and probably the one mostly exploited by industry.

If the material properties of FGMs vary in a complicated form along a given direction, it would be difficult to obtain their fundamental solutions. This limits the application of the BEM to analyze fracture mechanics of FGMs. Yue [3] obtained the fundamental solutions for the generalized Kelvin problems of a multilayered elastic medium of infinite extent subjected to concentrated body force vectors, which is referred to as Yue's solution. The potential application of the solutions is to formulate the BEM suitable for the multilayered media and graded materials encountered in science and engineering.

Over the last decade, the authors have extended the classical boundary element methods (BEM) for analysis of the fracture mechanics in FGMs. The new BEM method incorporates Yue's solution into the classical BEM methods. The BEM developed by the authors can be classified into two types: multi-region BEM and single region BEM (i.e., dual BEM). In this paper, we introduce mainly the some developments of the Yue's solution based dual BEM and their applications in the fracture mechanics of the layered and graded materials.

2 A brief introduction of the multi-region BEM based on Yue's solution

Mathematical formulation and computational procedures of the multi-domain BEM have been published by Yue et al. [4,5]. The BEM discretizes a FGM layer as a system of n number of fully bonded dissimilar sub-layers. The Yue's solution is used as the fundamental solution to replace the classical Kelvin point load solution in conventional three-dimensional boundary element methods. As a result, any FGMs with arbitrary property gradient in depth can be examined using this BEM.

Since Yue's solution satisfies the continuous conditions at any interface, there is no need to consider any sub-layer interfaces as boundary surfaces or sub-domain interfaces in the numerical formulation of BEM. In other words, the crack problem can be straightforwardly carried out using the similar BEM procedure for the same crack in a homogeneous elastic solid of infinite extent. It is only need to generate the BEM meshes for the crack interfaces and their associated auxiliary surfaces. The auxiliary surfaces are needed since the conventional multi-region method is used in the BEM. In the computational formulation, the eight-node isoparametric elements are usually employed to discretize the boundary surfaces. The so-called traction-singular elements are used to model the singular fields around the crack tip.

Authors [5,6] used the multi-region BEM to analyze the stress intensity factors of a penny crack *parallel or perpendicular to* the interfacial layer of FGMs and the growth of the penny crack under remote inclined loads. Besides, authors [7,8] further analyzed the stress intensity factors of an elliptical crack *parallel or perpendicular to* the interfacial layer of FGMs and the growth of the penny crack under remote inclined loads.

3. Yue's solution based dual boundary element method

3.1 General

As mentioned in Section 2, the multi-region BEM based on Yue's solution has been applied for the analysis of penny and elliptical cracks in a FGM system. It can be found that the proposed method has the following drawbacks: (1) The introduction of artificial boundaries is not unique and thus cannot be implemented into an automatic procedure. (2) The method generates a larger system of algebraic equations than strictly required. (3) If artificial boundaries are located in FGMs, the unknown quantities of linear equations increase greatly because of the variations of material properties. Dual BEM can overcome the above drawbacks. Dual boundary element formulation is based on a pair of boundary integral equations (BIE), namely, the displacement and traction BIEs. The method is a single-region based, thus it can model the solids with multiple interacting cracks or damage. Although dual BEMs have been widely applied, few of the crack problems in non-homogeneous media may be not involved. In the following, we will present a numerical implementation of the dual boundary integral equations based on Yue's solution [9].

3.2. Yue's solution based Dual BIEs

Fig. 1 shows a three-dimensional crack in a multilayered solid. By collocating the source point on the uncracked boundary, the conventional displacement BIE can be written as

$$c_{ij}(P_S)u_j(P_S)+\int_{S+\Gamma^++\Gamma^-} t_{ij}^Y(P_S,Q)u_j(Q)dS(Q)=\int_{S+\Gamma^++\Gamma^-} u_{ij}^Y(P_S,Q)t_j(Q)dS(Q), \qquad i,j=x,y,z \quad (1)$$

where P_s and Q are the source and field points, respectively; $t_{ij}^Y(P_S,Q)$ and $u_{ij}^Y(P_S,Q)$ are the tractions and displacements of Yue's solution, respectively; $t_j(Q)$ and $u_j(Q)$ are the tractions and displacements of the field point Q on the boundaries; S is the uncracked boundary of the cracked body; Γ^+ and Γ^- are two crack surfaces; $c_{ij}(P_S)$ is a coefficient dependent on the local boundary geometry at the source point P_s.

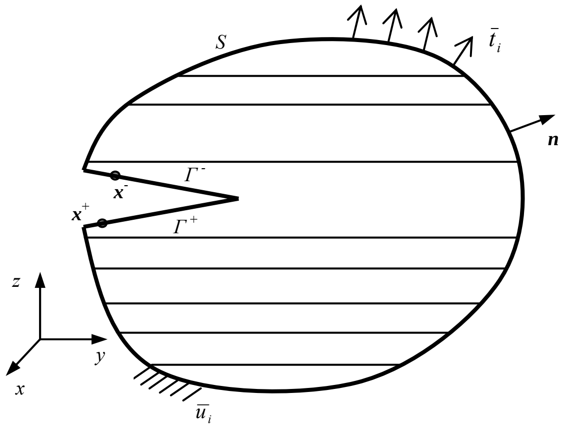


Fig. 1 A three-dimensional crack in a multilayered solid

Before loading, the points Q_{Γ^-} and Q_{Γ^+} on two crack surfaces are completely coincident and there are opposite outward normal directions on the two points. Thus, there exist the following relationships of kernel functions of the points on two crack surfaces

$$t_{ij}^Y(P_S,Q_{\Gamma^+})=-t_{ij}^Y(P_S,Q_{\Gamma^-}), \quad u_{ij}^Y(P_S,Q_{\Gamma^+})=u_{ij}^Y(P_S,Q_{\Gamma^-}) \quad (2)$$

Assume that there is a balanced relationship of tractions: $t_j(Q_{\Gamma^+})=-t_j(Q_{\Gamma^-})$. The relative crack opening displacement (COD) can be described as

$$\Delta u_j(\mathrm{Q}_{\Gamma^+})=u_j(\mathrm{Q}_{\Gamma^+})-u_j(\mathrm{Q}_{\Gamma^-}), j=x,y,z \quad (3)$$

Using the above relationships, the two integrals in equation (1) can be written as

$$\int_{S+\Gamma^++\Gamma^-} t_{ij}^Y(P_S,Q)u_j(Q)dS(Q)=\int_S t_{ij}^Y(P_S,Q)u_j(Q)dS(Q)+\int_{\Gamma^+} t_{ij}^Y(P_S,Q)\Delta u_j(Q)dS(Q) \quad (4a)$$

$$\int_{S+\Gamma^++\Gamma^-} u_{ij}^Y(P_S,Q)t_j(Q)dS(Q)=\int_S u_{ij}^Y(P_S,Q)t_j(Q)dS(Q) \quad (4b)$$

Thus, equation (1) can be rewritten as

$$c_{ij}(P_S)u_j(P_S)+\int_S t_{ij}^Y(P_S,Q)u_j(Q)dS(Q)+\int_{\Gamma^+} t_{ij}^Y(P_S,Q)\Delta u_j(Q)dS(Q)=\int_S u_{ij}^Y(P_S,Q)t_j(Q)dS(Q) \quad (5)$$

The integral equation (5) is a general form of the displacement boundary integral equation based on Yue's solution.

By collocating the stress boundary integral equation on the source point P_{Γ^+} on the crack surface Γ^+ shown in Fig. 1, the conventional stress BIE can be written as

$$\frac{1}{2}\sigma_{ij}\left(P_{\Gamma^+}\right)+\frac{1}{2}\sigma_{ij}\left(P_{\Gamma^-}\right)+\int_{S+\Gamma^++\Gamma^-}T_{ijk}^{Y}\left(P_{\Gamma^+},Q\right)u_k(Q)dS(Q)=\int_{S+\Gamma^++\Gamma^-}T_{ijk}^{Y}\left(P_{\Gamma^+},Q\right)t_k(Q)dS(Q). \tag{6}$$

where T_{ijk}^{Y} and U_{ijk}^{Y} are the new kernel functions obtained by using the numerical difference of the derivatives of Yue's tractions and displacements.

Multiplying equation (6) by the outward unit normal $n_i\left(P_{\Gamma^+}\right)$ and noticing that $n_i\left(P_{\Gamma^+}\right)=-n_i\left(P_{\Gamma^-}\right)$, the traction boundary integral equation on the crack surface results in

$$\frac{1}{2}t_j\left(P_{\Gamma^+}\right)-\frac{1}{2}t_j\left(P_{\Gamma^-}\right)+n_i\left(P_{\Gamma^+}\right)\int_{S+\Gamma^++\Gamma^-}T_{ijk}^{Y}\left(P_{\Gamma^+},Q\right)u_k(Q)dS(Q)=n_i\left(P_{\Gamma^+}\right)\int_{S+\Gamma^++\Gamma^-}U_{ijk}^{Y}\left(P_{\Gamma^+},Q\right)t_k(Q)dS(Q) \tag{7}$$

The points Q_{Γ^-} and Q_{Γ^+} on two crack surfaces are completely coincident and there are opposite outward normal directions on the two points. Thus, there exist the following relationships of kernel functions of the points on two crack surfaces

$$T_{ijk}^{Y}\left(P_S,Q_{\Gamma^+}\right)=-T_{ijk}^{Y}\left(P_S,Q_{\Gamma^-}\right),\quad U_{ijk}^{Y}\left(P_S,Q_{\Gamma^+}\right)=U_{ijk}^{Y}\left(P_S,Q_{\Gamma^-}\right). \tag{8}$$

Applying expressions (8), the two integrals in equation (7) can be written as

$$\int_{S+\Gamma^++\Gamma^-}T_{ijk}^{Y}\left(P_{\Gamma^+},Q\right)u_k(Q)dS(Q)=\int_{S}T_{ijk}^{Y}\left(P_{\Gamma^+},Q\right)u_k(Q)dS(Q)+\int_{\Gamma^+}T_{ijk}^{Y}\left(P_{\Gamma^+},Q\right)\Delta u_k(Q)dS(Q) \tag{9a}$$

$$\int_{S+\Gamma^++\Gamma^-}U_{ijk}^{Y}\left(P_{\Gamma^+},Q\right)t_k(Q)dS(Q)=\int_{S}U_{ijk}^{Y}\left(P_{\Gamma^+},Q\right)t_k(Q)dS(Q) \tag{9b}$$

Using expressions (8) and (9), equation (7) can be further written as

$$\begin{aligned}t_j\left(P_{\Gamma^+}\right)+n_i\left(P_{\Gamma^+}\right)\int_{S}T_{ijk}^{Y}\left(P_{\Gamma^+},Q\right)u_k(Q)dS(Q)+n_i\left(P_{\Gamma^+}\right)\int_{\Gamma^+}T_{ijk}^{Y}\left(P_{\Gamma^+},Q\right)\Delta u_k(Q)dS(Q)\\=n_i\left(P_{\Gamma^+}\right)\int_{S}U_{ijk}^{Y}\left(P_{\Gamma^+},Q\right)t_k(Q)dS(Q)\end{aligned} \tag{10}$$

The integral equation (10) is a general form of the traction boundary integral equation based on Yue's solution.

Equations (5) and (10) give explicit expressions of the dual boundary integral equations based on Yue's solution. These two integral equations do not contain the integrations on the interfaces of multilayered media because Yue's solution strictly satisfies the interface conditions. Collocating equation (5) on the uncracked boundary S and equation (10) on Γ^+ constitutes the dual boundary integral equations for crack problems. In these equations, the unknown quantities are the tractions and displacements on the uncracked boundary and the discontinuous displacements on the crack surfaces. When the dual BIEs are applied for the study of the crack problems in a multilayered medium of infinite extent, the displacement boundary integral equation is not required and then dual BEM degenerates into the discontinuous displacement method (DDM) [10,11].

3.3 Numerical implementation of the dual BIEs

In the context of the dual boundary integral equations, we apply four- to eight-node isoparametric elements to discretize the uncracked boundary and three types of nine-node quadralateral curved elements to discretize the crack surface. Once these quantities are determined, the displacements, tractions and CODs on boundary are known everywhere. Thus, we can now rewrite the dual boundary integral equations in a discretized form in terms of these parameters to be determined using the shape functions. According to the nature of the kernel and the relative position of the source point with respect to the element on which the integration is carried out, the integrals in the discretized dual BIEs are regular or non-regular. All these integrals have been calculated carefully [9].

Based on the numerical method, computer programs have been written in Fortran to calculate displacements, tractions and the CODs of a multilayered dissimilar elastic solid of finite or infinite extent containing cracks. The stress intensity factors (SIFs) can be calculated by using the CODs on the crack surface. We have examined the accuracy of the proposed DBEM in [9].

4 A square crack in the FGM interlayer

In Fig. 2, a square crack is located in the FGM interlayer bonded to the two homogeneous half-spaces and parallel to the interfaces between a homogeneous half-space and the FGM interlayer. The square crack has the side length 2*c*. The crack surfaces are subjected to uniform compressive stress *p*. Among three materials, materials 1 and 3 are homogeneous media and material 2 is a gradient medium. The elastic modulus of the materials is approximated by

$$E_2(z) = E_1 e^{\alpha z} \quad \text{and} \quad \alpha = 1/h \ln(E_3/E_1) \tag{11}$$

where *h* is the thickness of the interfacial layer, the constant α can be positive or negative, and E_1, E_2 and E_3 are the elastic moduli of three layers.

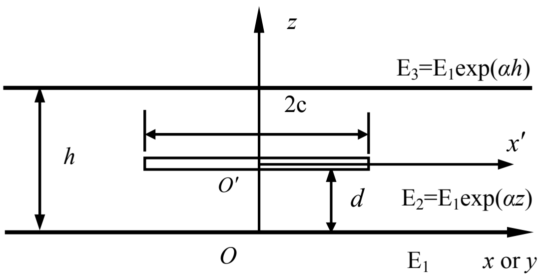


Fig. 2 A square crack in the FGM interlayer

Let $\alpha c = 2$, $h/c = 0.5$ and $\nu_1=\nu_2=\nu_3=0.3$. The five cases, $d/c = 0.05, 0.15, 0.25, 0.35, 0.45$, are analyzed. The crack surface is discretized into 100 nine-node elements. For the FGM described in expression (11), the FGM is closely approximated by *n* bonded layers of elastic homogeneous media. Each layer has the thickness equal to *h*/*n* and shear modulus equal to $E_2(z)$ at the top depth of the layer, i.e. for the *i*-th layer, $z=H_i$, where $H_i=ih/n$, $(i = 1, 2, \ldots, n)$. Two homogeneous materials bonded through the FGM are considered as semi-infinite domains for the layers H_0 and H_{n+1} respectively. For all the layers, the Poisson's ratios are the same and equal to 0.3. It can be observed that a close approximation of the elastic modulus variation can be obtained using a large number of *n* [5].

Figs. 3 and 4 illustrate the variations of the SIF values with the crack distance *d* to the FGM interlayer. In these figures, K_I and K_{II} are symmetrical to the x'-axis. It can be found that the crack distance *d* increasing, the K_I and K_{II} values increase. From $d = 0$ to $d = h$, the elastic modulus on the crack position becomes large and the constraint of the crack opening becomes strong so that the K_I values decrease. At the same time, the relative sliding of the crack surfaces along the x and y directions occurs so that the K_{II} values are not equal to zero and increase with the crack distance d increasing. Referring to the results of the case $\alpha c = 2$ and $h/c = 0.5$, the variations of the SIF values for the case $\alpha c = -2$ and $h/c = 0.5$ can be obtained easily from Figs. 3 and 4. For the case $\alpha c = -2$ and $h/c = 0.5$, the crack position d should be calculated from the plane $z/c = 0.5$, the K_I values are positive and the K_{II} values are negative, and the K_I values and the absolute values of K_{II} are the same as the case $\alpha c = 2$ and $h/c = 0.5$.

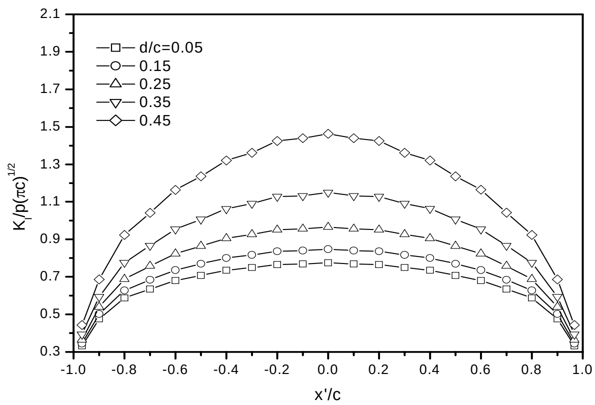


Fig.3 Variation of K_I with the crack distance

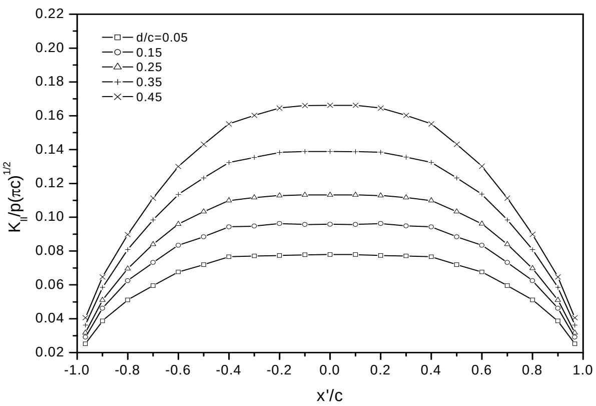


Fig. 4 Variation of K_{II} with the crack distance

5 A rectangular crack in layered rocks

As shown in Fig. 5, an elastic layer is bonded to two semi-infinite domains and contains a rectangular crack parallel to its surfaces. The two elastic half-spaces are assumed to have the same elastic properties. Two types of rock, i.e., fine-grained sandstone and mudstone, are selected. The elastic parameters of fine-grained sandstone E_1=56GPa and ν_1=0.3, and the ones of mudstone E_2=20GP and ν_2=0.25. Thus, there are four types of layered rocks shown in Table 1. The thickness of the mid-layer h=2m. The side lengths of the square crack are 2m and 4m and the crack surface is parallel to the interface of the layered rocks. The crack surfaces are smooth and are subjected to a linear load shown in Fig. 6. The crack surface is discretized into 100 nine-node elements.

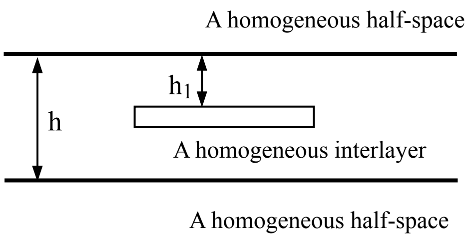


Fig.5 A rectangular crack in the interlayer of infinite domain

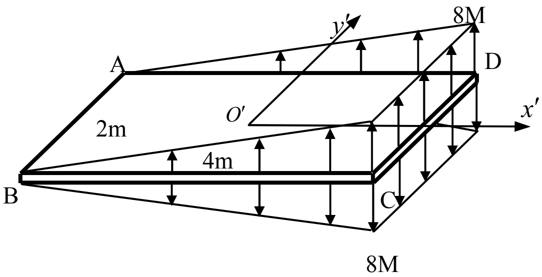


Fig. 6 A rectangular crack subject to linear load

In order to plot the values of the SIFs along crack front lines, a line coordinate L is used to measure the crack front lines from AB, BC to CD, shown in Fig. 6. The line coordinate L starts at the corner point A of the square crack (i.e. L=0). It increases along the line AB, BC to CD. Correspondingly, L increases from 0-2, from 2-6, and from 6-8, respectively.

Table 1 Cases of layered rocks containing a rectangular crack

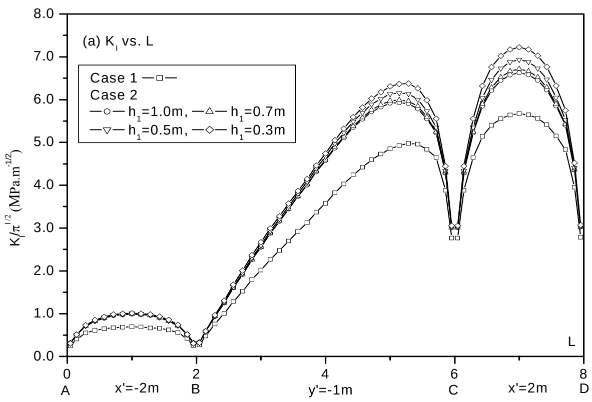
Case no.	Layered rocks of infinite extent
1	The homogeneous solid of infinite extent consists of fine-grained sandstone.
2	The sandwich solid consists of two semi-infinite extents of mudstone and the mid-layer of fine-grained sandstone.
3	The homogeneous solid of infinite extent consists of mudstone.
4	The sandwich solid consists of two semi-infinite extents of fine-grained sandstone and the mid-layer of mudstone.

(1) Case 1 and case 2

For case 1, there are no relative sliding displacements along the crack surfaces because of symmetry of geometry and loads. Due to the same reasons, there are no relative sliding displacements for case 2 (h_1=1m). For case 2, the COD values increase and the $K_{\rm I}$ values also increase in comparison with case 1. This is because the elastic module of mudstone in semi-infinite domain is less than the one of fine-grained sandstone.

For case 2 ($h_1 \neq$ 1m), the absolute values of displacements along the upper surface increase and the ones along the lower surface decrease. This leads to the appearance of the sliding discontinuous displacements along the crack surface. Due to the non-uniform load on the crack surfaces, there are different SIF values at the crack fronts $x' = \pm 2m$ whilst there are the same SIF values at the crack fronts $y' = \pm 1m$. In the following, the SIF values at the crack fronts $x' = \pm 2m$ and $y' = -1m$ are discussed.

Fig. 7 shows the variation of the SIF values at the crack fronts $x' = \pm 2m$ and $y' = -1m$. The $K_{\rm I}$ values are positive and the $K_{\rm II}$ values are negative. Obviously, these phenomena are related to the load on the crack surfaces and the relative position of the crack in this medium. Along the crack front $y' = -1m$, the larger the load, the larger the absolute values of $K_{\rm I}$ and $K_{\rm II}$. As the crack surface approaches to the interface, the absolute values of $K_{\rm I}$ and $K_{\rm II}$ increase.



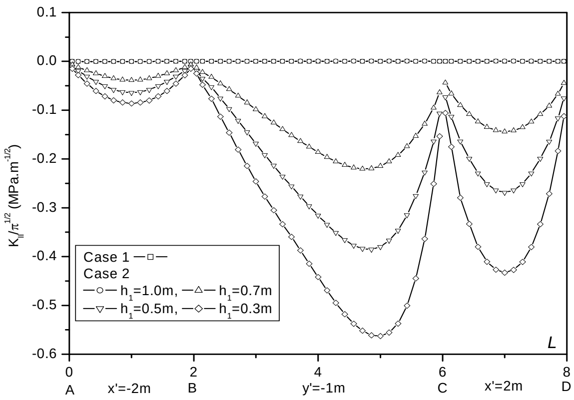


Fig. 7 SIF values of rectangular crack subjected to linear loads (Cases 1 and 2)

(2) Case 3 and case 4

Fig. 8 shows the variation of the SIF values at the crack fronts $x' = \pm 2m$ and $y' = -1m$. Herein, the K_{I} and K_{II} values are positive. Along the crack front $y' = -1m$, the K_{I} and K_{II} values become larger near the side $x' = 2m$. As the crack surface approaches to the interface, the absolute values of K_{I} and K_{II} decrease.

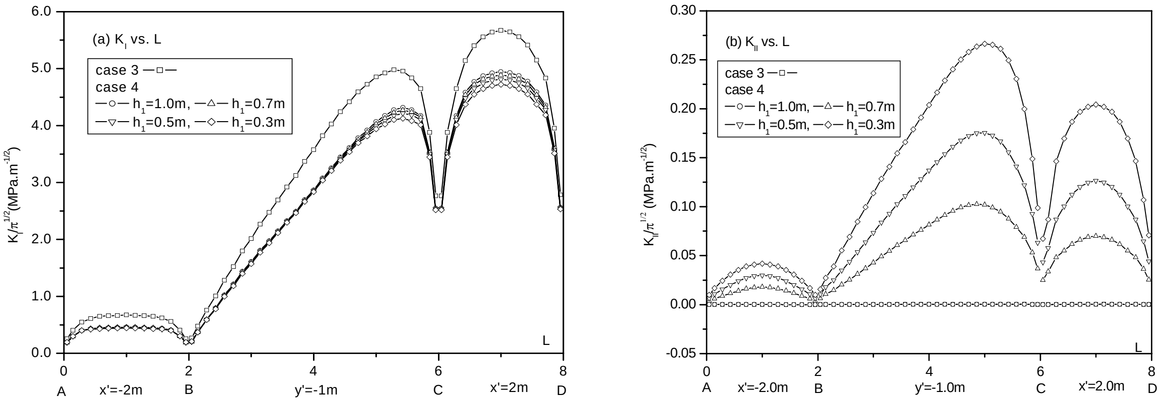


Fig. 8 SIF values of rectangular crack subjected to linear loads (Cases 3 and 4)

5 Concluding remarks

This paper introduces a novel dual boundary element method associated with Yue's solution. In analyzing the crack problems in FGMs and layered materials, the advantages of this approach are: (1) it is not necessary to introduce elements at the interface, (2) the method is applicable for multilayered solids with any layer number and (3) the high accuracy can be obtained for the crack in multilayered solids. Only the results of crack problems in infinite domains are presented in this paper and the numerical examples of crack problems in finite domains can be found in [9].

In 1995, Yue [12] developed the fundamental solution of a transversely isotropic bi-material. The authors also used this fundamental solution to develop the BEMs of the bi-material similar to the ones of Yue's solution. The proposed BEM includes the multi-domain BEM and the single domain BEM (i.e., dual BEM). The application of the BEMs to analyze a penny crack, an elliptical crack and a square crack in bi-materials has been presented [13-15].

In the book [16], the authors introduced the research results by using the fundamental solutions [3, 12] systematacially. The proposed BEMs and the results can be a powerful numerical tool, which can apply to various complex three-dimensional geometries of layered or gradient materials with cracks under mixed-mode loading.

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