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Asynchronous Output-Feedback Stabilization of Discrete-Time Markovian Jump Linear Systems

Zhan Shu, Junlin Xiong, and James Lam

Abstract— Various constraints on signal processing and transmission in practice have posed a big issue to perfect synchronous switching control for Markovian jump linear systems (MJLSs), and thus designing a controller partially or totally independent of the plant switching becomes significant. In this paper, we propose an approach to synthesizing asynchronous switching control laws for discrete-time MJLSs. By utilizing a separation technique, a necessary and sufficient condition for asynchronous static output-feedback stabilizability is established in terms of a set of matrix inequality with a special structure for computation. Then, an iterative algorithm is employed to solve the condition. Appropriate optimization on initial values may improve the solvability. Numerical examples are provided to illustrate the effectiveness of the proposed approach.

I. INTRODUCTION

The past decades have witnessed the tremendous advances in the theory of Markovian jump linear systems and its widespread applications in power systems, manufacturing processes, fault detection, etc. A great number of results on MJLSs have been obtained. Stability issues have been treated thoroughly in [1], [2]. The early study of linear quadratic control and its recent advances are available in [3], [4]. The problems of $H_2$ and/or $H_{\infty}$ control have been discussed in [5], [6], and the results on the filtering problem can be found in [7], [8], [9], [10], [11]. Stability and stabilization of Markovian jump systems with stochastic noises have been investigated thoroughly in [12], [13]. As for the applications of MJLSs in robot manipulations, networked control, multiagent control, and power systems, we refer readers to [14], [15], [16], [17] and references therein.

Most existing controller/filter design approaches for MJLSs are based on the assumptions that the mode information is fully accessible, and the switching of controller/filter is synchronous with that of plant. In many practical situations, however, these assumptions may not be true, and this motivates the recent study on controller/filter synthesis with constrained mode information. In [18], a controller with delayed mode information is proposed for networked control. Mode-independent filter design has been discussed in [19] and [20].

In this paper, we consider the problem of designing a static output-feedback controller whose switching is asynchronous with that of plant. Both feedback gains and transition probability matrices are needed to be determined. By employing a separation technique, a necessary and sufficient condition for asynchronous output-feedback stabilizability is established in terms of matrix inequalities, which has a special structure for linearized computation. An iterative algorithm is then proposed to solve the condition. Several approaches are proposed to generate desired initial values for iterative computation. Two numerical examples are employed to illustrate the effectiveness of the proposed approach.

Notation: For real symmetric matrices $X, Y \in \mathbb{R}^{n \times n}$, the notation $X > Y$ means that the matrix $X - Y$ is positive definite. For a matrix $A \in \mathbb{R}^{n \times n}$, $\text{Sym}(A) = A + A^T$ and $\rho(A)$ represents the spectral radius of $A$. The symbol $\otimes$ denotes the Kronecker product. $\mathbb{E}\{\cdot\}$ stands for the mathematical expectation with some underlying probability measure $\text{Pr}(\cdot)$. Associated with a discrete-time Markov chain taking values in a finite set $\mathbb{S}$ with transition rate matrix $\Pi = [\pi_{ij}]$, $i, j \in \mathbb{S}$,

$$E_{i}(P) \triangleq \sum_{j \in \mathbb{S}} \pi_{ij} P_{j},$$

for a set of matrices $P_{j}, j \in \mathbb{S}$. The asterisk * is used to denote a matrix which will not be used in the development, and # is used to denote a matrix which can be inferred by symmetry. Matrices, if their dimensions are not explicitly stated, are assumed to have compatible dimensions for algebraic operations.

II. PRELIMINARIES AND PROBLEM FORMULATION

Consider the following class of DMJLSs:

$$\begin{cases} x(t+1) &= A_{r(t)} x(t) + B_{r(t)} u(t), \\ y(t) &= C_{r(t)} x(t), \end{cases}$$

(1)

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, and $y(t) \in \mathbb{R}^n$ are the system state, the control input and the measured output, respectively, and $A_{r(t)}, B_{r(t)}, C_{r(t)}$ are the system matrices of the stochastic jumping process $\{r(t), t \geq 0\}$; the parameter $r(t)$ represents a discrete-time, discrete-state Markov chain taking values in a finite set $\mathbb{S} = \{1,2,\ldots,n_r\}$ with one-step transition probability matrix $\Lambda_r = [p_{ij}]$, where $p_{ij} \geq 0$, and for any $i \in \mathbb{S}$, $\sum_{j=1}^{n_r} p_{ij} = 1$.

Definition 1: For $\lambda \geq 1$, the system in (1) is said to be $\lambda$-exponentially stable if, when $u(t) \equiv 0$, there exists a scalar $\varepsilon > 0$ such that, for any $x(0) = x_0$, $r(0) = r_0$,

$$\mathbb{E}\left\{\|x(t)\|^2 \mid x_0, r_0\right\} \leq \sigma (\lambda + \varepsilon)^{-t} \|x_0\|^2,$$
where \( \lambda + \varepsilon \) and \( \sigma \) are called decay rate and decay coefficient, respectively.

If a system is \( \lambda \)-exponentially stable, then the system has a decay rate larger than \( \lambda \). Similar concept has been used to treat stabilization and control of noise-driven stochastic systems in [21], [22]. The following lemma gives an LMI characterization for \( \lambda \)-exponential stability.

**Lemma 1:** The system in (1) is \( \lambda \)-exponentially stable if and only if there exist real matrices \( P_i > 0, i \in S \), such that

\[
\lambda A_i^T \sigma_i (P_i) A_i - P_i < 0.
\]

This lemma can be proved by following a similar line as used in [23], and thus omitted here. The asynchronous static output-feedback (ASOF) controller under consideration is of the form

\[
u(t) = K_{\theta(t)} y(t).
\]

Connecting controller (3) to system (1) yields the following closed-loop system:

\[
x(t + 1) = (A_{\theta(t)} + B_{\theta(t)} K_{\theta(t)} C_{\theta(t)}) x(t).
\]

In previous controller synthesis for MJLSs, it is often assumed that all the mode information is accessible and the switching of controller is completely synchronous with the plant, that is, \( r(t) \equiv s(t) \), whereas, in practice, these assumptions may not always be reasonable or feasible due to various constraints in mode detection and/or inevitable delays in signal processing and transmission. For these scenarios, constructing a control law with an asynchronous Markovian switching could be a possible solution, that is, \( s(t) \) being a Markov process independent of \( r(t) \) (see Fig. 1). To tell in details, assume that \( s(t) \) is a Markov chain taking values in \( S_s = \{1, 2, \ldots, n_s\} \) with one-step transition probability matrix \( \Lambda_s = [q_{ij}] \). Then, the task is to design both \( K_{\theta(t)} \) and \( \Lambda_s \) such that the closed-loop system in (4) is \( \lambda \)-exponentially stable. To this end, one may define \( \theta(k) \) as a joint Markov process \( (r(k),s(k)) \) taking values in an augmented mode space \( S_\theta = S_r \times S_s \) with one-step transition probability matrix \( \Lambda_\theta = [\pi_{ij}] \). Here, it is assumed that the \( k_s \)th mode in \( S_s \) and the \( k_r \)th mode in \( S_r \) form the \([k_r-1,n_r+k_s]\)th mode in \( S_\theta \). With this setting, it is easy to show that closed-loop system (4) is a new DMJLS with Markovian jumping parameter \( \theta(t) \). The following lemma gives an important relationship among \( \Lambda_r \), \( \Lambda_s \), and \( \Lambda_\theta \).

**Lemma 2:** For the joint Markov process \( \theta(t) \) aforementioned,

\[
\Lambda_\theta = \Lambda_r \otimes \Lambda_s.
\]

**Proof:** It follows from the independence of \( r(k) \) and \( s(k) \) that \( Pr(\theta(k+1) = (r_{k+1}, s_{k+1})| \theta(k) = (r_{k}, s_{k})) = Pr(r(k+1) = r_{k+1}, s(k+1) = s_{k+1}) = Pr(r(k+1) = r_{k+1} | r(k) = r_k) Pr(s(k+1) = s_{k+1} | s(k) = s_k) \), and thus one can obtain via some simple manipulations that (5) holds.

**Remark 1:** In this paper, \( n_s \) is given and does not constitute a design parameter, while a better design is to treat \( n_s \) as a quantity to be synthesized. This, however, beyond the scope of the present study, and may consist of an interesting and significant problem for further investigation.

This section is ended by defining \( S_{\theta(k)} = \{l_1, l_2, \ldots, l_\nu\}, l_i \in S_\theta \) as the subset of \( S_\theta \) satisfying that \( \pi_{l_i k} \neq 0 \).

### III. MAIN RESULTS

#### A. Asynchronous Output-Feedback Stabilizability

**Theorem 1:** For the asynchronous control scheme mentioned previously, the following statements are equivalent:

1) System (4) is \( \lambda \)-exponentially stable.
2) There exist matrices \( P_{l_1 k}, P_{l_2 k}, Q_{l_1}, G_{l_1 k}, G_{l_2 k}, H_{l_1 k}, H_{l_2 k}, \) and \( S_j > 0 \) such that, for all possible combinations of \( i \in S_r, j \in S_s, k \in S_\theta, \) and \( l \in S_{\theta(k)} \),

\[
P_{l_1 k} > 0 \quad (6a)
\]

\[
\begin{bmatrix}
\Phi_{11k} & \# & \# \\
0 & X_p + Q_k X_{\pi kl} & 0
\end{bmatrix} < -4Q_k \quad (6b)
\]

where

\[
\Phi_{11k} = \text{Sym}(H_k A_k + S_k A_k) - \lambda^{-1} E^T P_k E,
\]

\[
\Phi_{21k} = G_k A_k - H_k^T,
\]

\[
\Phi_{22k} = - \text{Sym}(G_k) + \text{diag}(0, P_{2l_k}),
\]

\[
X_p = [P_{l_1}, P_{l_2}, \ldots, P_{l_\nu}]^T,
\]

\[
X_{\pi kl} = [\pi_{kl_1}, \pi_{kl_2}, \ldots, \pi_{kl_\nu}]^T
\]

\[
Q_k = \text{diag}(Q_{l_1}, Q_{l_2}, \ldots, Q_{l_\nu})
\]

and

\[
P_k = \begin{bmatrix} P_{l_1 k} & 0 \\ 0 & P_{2l_k} \end{bmatrix}, \quad A_k = \begin{bmatrix} A_i & B_i \\ K_C & -I \end{bmatrix},
\]

\[
G_k = \begin{bmatrix} G_{l_1 k} \\ G_{l_2 k} \\ 0 \end{bmatrix}, \quad H_k = \begin{bmatrix} H_{l_1 k} \\ 0 \\ 0 \end{bmatrix},
\]

\[
R_k = \begin{bmatrix} 0 & 0 \\ 0 & R_{l_k} \end{bmatrix}, \quad S_k = \begin{bmatrix} 0 & -C_i^T K_i^T S_j \\ 0 & S_j \end{bmatrix},
\]

\[
E = \begin{bmatrix} I \\ 0 \end{bmatrix}
\]
Proof: 2) ⇒ 1) Pre- and post-multiplying (6b) by \[ I 0 0 0 \]
and its transpose yield that
\[
\begin{bmatrix}
\Phi_{11k} \\
\Phi_{21k} \\
\Phi_{21k} \\
\end{bmatrix} + \text{diag} \left( \frac{1}{2} \text{Sym} \left( X_{nk}^T X_P \right), 0 \right)
\]
\[
= \begin{bmatrix}
\Phi_{11k} \\
\Phi_{21k} \\
\Phi_{21k} \\
\end{bmatrix} - \text{Sym} \left( G_k + \hat{P}_k \right)
\]
\[
< 0,
\]
(7)
where \( \hat{P}_k = \text{diag} \left( \delta_k \left( P_k \right), P_{2k} \right) \). Pre- and post-multiplying (7) by \[ I \ A_k \]
and its transpose further give that
\[
A_k^T \hat{P}_k A_k - \lambda^{-1} E^T P_k E + \text{Sym} \left( S_k A_k \right) < 0
\]
(8)
Noting that the left side of (8) can be factorized as
\[
A_k^T \hat{P}_k A_k - \lambda^{-1} E^T P_k E + \text{Sym} \left( S_k A_k \right) =
\begin{bmatrix}
I & -C_k K_k^T \\
0 & I
\end{bmatrix}
\begin{bmatrix}
A_{ck} & \# \\
C_k & \#
\end{bmatrix}
\times \begin{bmatrix}
B_k^T \delta_k \left( P_k \right) A_{ck} \\
0
\end{bmatrix}
\begin{bmatrix}
B_k^T \delta_k \left( P_k \right) B_k \\
- S_j
\end{bmatrix}
\times \begin{bmatrix}
I & 0 \\
-K_j C_i & I
\end{bmatrix}
\]
(9)
where \( A_{ck} = A_i + B_i K_j C_i \), one has that \( A_k^T \delta_k \left( P_k \right) A_{ck} - \lambda^{-1} P_{ck} < 0 \), which implies 1) by Lemma 1.

1) ⇒ 2) According to Lemma 1, one has that there exist \( P_{1k} > 0 \), \( P_{2k} > 0 \), \( S_j > 0 \), \( L_j \), \( Q_{kl} \), \( Y_k > 0 \), \( M_k \), \( G_k \), \( H_k \), \( H_{2k} \), and \( \eta_{vw} \geq 0 \), \( \sum_{v=1} \eta_{vw} = 1 \), \( v, w \in \{ 1, \ldots, n_j \} \), such that the following equalities/inequalities hold for all possible combinations of \( i \in S_r \), \( j \in S_s \), \( k \in S_{gb} \), and \( l \in S_{bk} \):

\[
\begin{bmatrix}
\Omega_{11k} & \# & \# & \# \\
\Omega_{21k} & \Omega_{22k} & \# & \# \\
\Omega_{31k} & \Omega_{32k} & \# & \# \\
\Omega_{41k} & \Omega_{42k} & -Q_{2k} & \# \\
0 & 0 & \Omega_{53k} & 0 & -4Q_k
\end{bmatrix}
\geq
\begin{bmatrix}
Y_k \\
M_k^T L_j C_i \\
M_k^T L_j C_i \\
M_k^T L_j C_i \\
\vdots \\
M_k^T L_j C_i \\
\end{bmatrix}
\]
(11a)
\[
\begin{bmatrix}
\lambda^{-1} P_{1k} \\
\lambda^{-1} P_{2k} \\
\lambda^{-1} P_{3k} \\
\lambda^{-1} P_{4k} \\
- \lambda^{-1} P_{5k}
\end{bmatrix}
< 0
\]
(11b)
where \( \Lambda_{gb} \), \( \Lambda_{s} \), \( \Lambda_{v} \), and \( Q_k \) are defined as above, and
\[
\begin{align*}
\Omega_{11k} &= \text{Sym} \left( H_{1k} A_l \right) - Y_k + 2M_k^T S_j M_k - 2\text{Sym} \left( M_k^T L_j C_i \right), \\
\Omega_{21k} &= H_{2k} A_l + B_k^T H_{2k} + 2L_j C_i, \\
\Omega_{22k} &= \text{Sym} \left( B_k^T H_{2k} \right) - 2S_j, \\
\Omega_{31k} &= G_k A_l - H_{1k}^T, \\
\Omega_{32k} &= G_k B_l - H_{2k}, \\
\Omega_{53k} &= -\text{Sym} \left( G_k \right), \\
\Omega_{41k} &= G_{2k} A_l + L_j C_i, \\
\Omega_{42k} &= G_{2k} B_l - S_j, \\
\Omega_{53k} &= -2S_j + P_{2k}, \\
\Omega_{53k} &= \begin{bmatrix}
P_{l1} + \pi_{kl} Q_{kl} & P_{l2} + \pi_{kl} Q_{kl} & \cdots & P_{l1} + \pi_{kl} Q_{kl}
\end{bmatrix}^T.
\end{align*}
\]
Under the conditions, an ASOF control law can be obtained as \( K_j = S_j^{-1} L_j \) with corresponding transition matrix \( \Lambda_{v} = \left[ \eta_{vw} \right] \).

Proof: (Sufficiency) By letting \( K_j = S_j^{-1} L_j \), one has that \( L_j = K_j S_j \). Substituting this into (11b), and noting (11a) and
\[
-C_k^T K_j^T S_j K_j C_i \leq M_k^T S_j M_k - \text{Sym} \left( M_k^T L_j C_i \right),
\]
one has that (6b) holds. The sufficiency follows immediately from Theorem 1.
(Necessity) Let \( Y_k = \lambda^{-1} P_k \) and \( M_k = K_j C_i \). Then, one can verify readily that \( -C_i^T K^T S_j K_j C_i = M_i^T S_j M_k - \text{Sym}(M_i^T L C_j) \). From this and Theorem 1, it follows that (11b) holds. This completes the proof.

**B. Computational Approach**

If one can seek a \( \lambda > 1 \) such that the conditions in Theorem 2 are satisfied, then an ASOF stabilizing controller can be constructed. In view of this, consider the following optimization problem:

**Problem 1:** Minimize \( \tau \triangleq \lambda^{-1} \) subject to the conditions in Theorem 2.

This is a nonlinear optimization problem, and is difficult to handle in general. However, if \( M_k \) and \( Q_{kl} \) are fixed, then the problem becomes, as verified by the following proposition, a standard generalized eigenvalue problem (GEVP), which can be treated readily by many efficient algorithms ([24]).

**Proposition 1:** If \( M_k \) and \( Q_{kl} \) are fixed, and \( \tau \) is sufficiently large, then there always exist \( P_{ik} > 0, P_{ik} > 0, S_j > 0, L_j, Q_{kl}, Y_k > 0, M_k \), and \( q_{vw} \geq 0 \), such that (11a)-(11b) hold.

**Proof:** Set \( Y_k = \lambda^{-1} P_{ik} \) and \( q_{vw} \geq 0 \) be any scalars satisfying \( \sum_{v=1}^{\nu} q_{vw} = 1 \). Then it suffices to prove that there exist variables such that (11b) holds. Define

\[
W_k \triangleq \frac{1}{4} \sum_{s=1}^{\nu} (P_{il} - \pi_{kl} Q_{ul})^T Q_{kl}^{-1} (P_{il} - \pi_{kl} Q_{ul}),
\]

\[
V_k \triangleq 2 \left( M_k - S_j^{-1} L_j C_i \right)^T S_j \left( M_k - S_j^{-1} L_j C_i \right),
\]

\[
\bar{A}_{ck} \triangleq A_i + B_i S_j^{-1} L_j C_i.
\]

Set \( H_{ik} = 0, H_{2k} = 0, G_{2k} = 0, G_{ik} = \delta_{ik} (P_i) + W_k, \) and \( P_{ik} = S_j \). From Schur complement equivalence, it follows that (11b) holds if and only if

\[
\begin{bmatrix}
-\tau P_{ik} - 2C_i^T L_j S_j^{-1} L_j C_i + V_k & \# & \# & \# \\
2L_j C_i & -2S_j & \# & \# \\
G_{ik} A_i & G_{ik} B_i - G_{ik} & \# & \# \\
L_j C_i & -S_j & 0 & -S_j
\end{bmatrix} < 0.
\]

Applying Schur complement equivalence again yields that the above inequality holds if and only if

\[
\begin{align*}
&\left[ -\tau P_{ik} - 2C_i^T L_j S_j^{-1} L_j C_i + V_k \right. \\
&\quad + \begin{bmatrix}
A_i & B_i \\
S_j^{-1} L_j C_i & -I
\end{bmatrix}^T \\
&\quad \times \begin{bmatrix}
G_{ik} & 0 \\
0 & S_j
\end{bmatrix} \\
&\quad \times \begin{bmatrix}
A_i & B_i \\
0 & S_j^{-1} L_j C_i & -I
\end{bmatrix} \\
&\quad \times \begin{bmatrix}
I & -C_i^T L_j S_j^{-1} \\
0 & I
\end{bmatrix} \\
&\quad \times \begin{bmatrix}
I & -C_i^T L_j S_j^{-1} \\
0 & I
\end{bmatrix} \\
&\quad < 0.
\end{align*}
\]

Choose \( S_j \) such that \( B_i^T G_{ik} B_i - S_j < 0 \). Since \( \tau \) is sufficiently large, it is obvious that the above inequality holds.

Moreover, the following proposition lays a foundation for further optimization.

**Proposition 2:** Let \( \tau_\ast (M_k, Q_{kl}) \), \( L_j, S_j, P_{ik} \), and \( \pi_{kl} \) denote the optimal \( \tau \) and corresponding optimal decision variables to Problem 1 with \( M_k \) and \( Q_{kl} \) being fixed. Then

\[
\tau_\ast \left( S_j^{-1} L_j C_i, P_{il}, \pi_{kl} \right) \leq \tau_\ast (M_k, Q_{kl}).
\]

The proposition is an immediate result from the proofs of Theorems 1 and 2. Based on this, an iterative algorithm is constructed as follows to solve the conditions in Theorem 2.

**Algorithm 1:**

1) (Initialization) Set \( \kappa = 1 \). Choose initial \( M_k^{(k)} \) and \( Q_{kl}^{(k)} \) (details on this will be discussed later). Set \( \tau_\ast^{(k)} > 1 \) to be a large number.

2) (Iteration) For fixed \( M_k^{(k)} \) and \( Q_{kl}^{(k)} \), solve Problem 1. Denote \( \tau_\ast^{(k+1)}, P_{ik}^{(k)}, S_j^{(k)}, L_j^{(k)} \), and \( q_{vw}^{(k)} \) as the obtained optimal values of \( \tau, P_i, S_j, L_j, \) and \( q_{vw} \).

3) (Criterion) If \( \tau_\ast^{(k+1)} < 1 \), then an ASOF control law can be obtained as \( K_j = \left( S_j^{(k)} \right)^{-1} L_j^{(k)} C_i \) with corresponding transition matrix \( A_\ast = \left( q_{vw}^{(k)} \right)^{-1} \), and the closed-loop system is \( 1/\tau_\ast^{(k+1)} \)-exponentially stable. **STOP.**

Else if \( \tau_\ast^{(k)} - \tau_\ast^{(k+1)} < \epsilon \), where \( \epsilon \) is a prescribed tolerance, or \( \kappa > N \), where \( N \) is the prescribed maximal iteration number, then go to next step. Otherwise, update \( M_k^{(k)} \) and \( Q_{kl}^{(k)} \) as

\[
M_k^{(k)} = \left( S_j^{(k)} \right)^{-1} L_j^{(k)} C_i
\]

\[
Q_{kl}^{(k)} = \left( P_{ik}^{(k)} \right)^{-1} \pi_{kl}^{(k)}
\]

and set \( \kappa = \kappa + 1 \), then go to Step 2.

4) (Termination) There may not exist a solution. **STOP.** (or generate another set of initial conditions, and run the algorithm again)

**Remark 3:** During each iteration, it is possible that the obtained \( q_{vw}^{(k)} \) is almost zero, and thus the update in Step 3 will generate a quite “large” \( Q_{kl}^{(k)} \), which may cause some numerical problems. If this is the case, one may modify \( S_{6k} \) correspondingly, that is, removing the element \( l \) from \( S_{6k} \).

**Remark 4:** It follows from Proposition 2 that \( \tau_\ast^{(k)} \) is monotonic decreasing function, and therefore the convergence of the algorithm is guaranteed (the converged point may not be an optimal solution).

**Remark 5:** Since each iteration involves a GEVP problem and the iteration number is finite, the computational complexity of the algorithm is the same order as that of GEVP problem.

**C. Discussion on Initial Values**

Algorithm 1 can be viewed as a convex relaxation approach to Problem 1, and thus the initial values are critical to seeking an optimal solution. In this subsection, two approaches are proposed for generating appropriate initial values for Algorithm 1.

It follows from Proposition 2 that a desired initial \( M_k \) is nothing but a state-feedback stabilizing controller. According to Theorem 3 in [25], if there exist matrices \( P_k > 0, k \in \mathcal{S}_\theta \)
such that, \( \forall v, k \in S_\theta \) and all possible combinations of \( i \) and \( k \),
\[
(A_i + B_i M_k)^T P_i (A_i + B_i M_k) - P_k < 0 \tag{12}
\]
then for all possible transition probability matrices the closed-loop system \( x(t+1) = (A_1 + B_1 M_k)x(t) \) is \( 1 \)-exponentially stable. Therefore, a two-step approach can be constructed as follows.

**Algorithm 2:**
1) Solve the following LMIs:
\[
\begin{bmatrix}
-X_k & \# \\
A_i X_k + B_i N_k - X_k & -X_k
\end{bmatrix} < 0,
\tag{13}
\]
for any \( v, k \in S_\theta \) and possible combinations of \( i \) and \( k \).
Set \( N_k X_k^{-1} \) as the initial \( M_k \).
2) Choose an arbitrary stochastic matrix \( \Lambda_\theta \), and construct \( \Lambda_\theta \) as in Lemma 2. Set \( X_l^{-1} \pi_{ikl}^0, k \in S_\theta \) and \( l \in S_{\theta kl} \), as the initial \( Q_{kl} \).

The LMIs in (13) may not have a solution. If this is the case, the above algorithm can be modified as follows.

**Algorithm 3:**
1) Find a set of \( 0 < q_{ij} \leq 1 \) such that the inequalities
\[
\begin{bmatrix}
-X_k & \# \\
A_i X_k + B_i N_k - X_k
\end{bmatrix} < 0,
\tag{14}
\]
have a solution for \( k \in S_\theta \) and all possible combinations of \( i \) and \( k \) (\( \pi_{ikl} \) is constructed according Lemma 2. Set \( N_k X_k^{-1} \) as the initial \( M_k \).
2) For the obtained \( X_k, M_k \), and \( \pi_{ikl} \), solve the optimization problem: Minimize \( c \) subject to \( (A_i + B_i M_k)^T \pi_{ikl} X_k^{-1} (A_i + B_i M_k) - X_k^{-1} + \sum_{v \neq v'} \pi_{ivl} X_v^{-1} < cI \) to determine \( \pi_{ikl} \). Then, set \( X_l^{-1} \pi_{ikl}^0, k \in S_\theta \) and \( l \in S_{\theta kl} \), as the initial \( Q_{kl} \).

In fact, (14) is a necessary condition to the problem of ASOF stabilization, and seeking a desired \( \pi_{ikl} \) can be transformed to a GEVP, which is relatively easy to solve as mentioned previously.

The other approach to seeking initial values is based on the following proposition.

**Proposition 3:** If \( \| \Lambda_\theta \| \| A_k \| < \lambda^{-1} < 1 \) for \( k \in S_\theta \), where \( A_k = A_i + B_i K_i C_i \), then the closed-loop system in (4) is \( \lambda \)-exponentially stable.

**Proof:** By following a similar line as used in the proof of Theorem 1 in [23], one can show that the closed-loop system is \( \lambda \)-exponentially stable if and only if the spectral radius of \( \mathcal{M} \triangleq (\Lambda_\theta \otimes I) \text{diag} (A_{t_1} \otimes A_{t_2} \otimes A_{t_3} \otimes A_{t_4} \otimes A_{t_5}) \) is less than \( \lambda^{-1} \), where \( n_\theta = n_3, n_r \). Since
\[
\rho(\mathcal{M}) \leq \| \mathcal{M} \| \leq \lambda_{\max} \left( (A_\theta^T A_\theta) \otimes I \right) \times \max_k \left\{ \lambda_{\max} \left( (A_{ck}^T A_{ck}) \otimes (A_{ck}^T A_{ck}) \right) \right\}
\]
\[
= \sqrt{\lambda_{\max} (A_\theta^T A_\theta) \max_k \left\{ \lambda_{\max} (A_{ck}^T A_{ck}) \right\}}
\]
\[
= \| \Lambda_\theta \| \max_k \left\{ \| A_{ck} \|^2 \right\}
\]
the result follows immediately.

Therefore, the following computational procedures can be employed to generate initial values.

**Algorithm 4:**
1) Solve the following optimization problem: Minimize \( c_1 + c_2 \) subject to
\[
\begin{bmatrix}
-c_1 I & \# \\
\Lambda_\theta & -I
\end{bmatrix} < 0, \quad \begin{bmatrix}
-c_1 I & \# \\
A_i + B_i M_k & -I
\end{bmatrix} < 0
\]
Set the obtained \( M_k \) as the initial value.
2) For the obtained \( M_k \) and \( \Lambda_\theta \), solve the problem: Minimize \( c_3 \) subject to \( (A_i + B_i M_k)^T \sigma (P) (A_i + B_i M_k) - P_k < c_3 I \). Set \( P_l \pi_{ikl}^{-1}, k \in S_\theta \) and \( l \in S_{\theta kl} \), as the initial \( Q_{kl} \).

**IV. NUMERICAL EXAMPLES**

**Example 1:** Consider a 4-mode DMJLS with the following system matrices
\[
A_i = \begin{bmatrix}
-1 & a_i \\
0.3 & 1.2
\end{bmatrix}, \quad B_i = \begin{bmatrix}
1 \\
b_i
\end{bmatrix}, \quad C_i = \begin{bmatrix}
c_i & -1
\end{bmatrix},
\]
where \( a_i, b_i, \) and \( c_i \) are given in Table I. The transition probability matrix is
\[
\Lambda_\theta = \begin{bmatrix}
0.2 & 0.3 & 0.1 & 0.4 \\
0.3 & 0.2 & 0.3 & 0.2 \\
0.3 & 0.3 & 0.3 & 0.1 \\
0.5 & 0.3 & 0.1 & 0.1
\end{bmatrix}.
\]

It is easy to verify that the open-loop system is unstable. For \( n_s = 3 \), apply Algorithm 2 to generate initial values. Then, using Algorithm 1 yields a \( t_+^{(3)} = 0.8933 \) after two iterations, and the corresponding control law is given as
\[
K_1 = 0.5712, \quad K_2 = 0.2861, \quad K_3 = 0.1751
\]
\[
\Lambda_s = \begin{bmatrix}
0.2085 & 0.6599 & 0.1317 \\
0.2710 & 0.5099 & 0.2281 \\
0.0883 & 0.1484 & 0.7633
\end{bmatrix}.
\]

It can be verified that spectral radius of \( \mathcal{M} \) is 0.7754 < \( t_+^{(3)} \), which means that the closed-loop system is indeed \( 1/t_+^{(3)} \)-exponentially stable.

**Example 2:** Consider a 2-mode DMJLS with the following system matrices:
\[
A_1 = \begin{bmatrix}
1.5 & 0 & 2 \\
1 & 0 & 0.35 \\
0 & 0.2 & -0.6
\end{bmatrix}, \quad B_1 = \begin{bmatrix}
1 \\
0.2 \\
0.3
\end{bmatrix},
\]
\[
A_2 = \begin{bmatrix}
1.8 & 0.5 & 2 \\
0.2 & 0.5 & 1 \\
0.1 & -0.3 & -0.1
\end{bmatrix}, \quad B_2 = \begin{bmatrix}
0.1 \\
-0.2 \\
0.8
\end{bmatrix},
\]
\[
C_1 = \begin{bmatrix}
1 & 0.5 & 1
\end{bmatrix}, \quad C_2 = \begin{bmatrix}
1 & 1 & -0.5
\end{bmatrix}.
\]
The transition probability matrix is \( \Lambda_r = \begin{bmatrix} 0.5 & 0.5 \\ 0.8 & 0.2 \end{bmatrix} \). It is expected to design a mode-independent controller for the system. To this end, set \( n_l = 1 \), and apply Algorithm 3 to generate the initial values. Then, using Algorithm 1 yields a \( t_s^{(3)} = 0.8769 \) after two iterations, and the corresponding control law is \( K = \begin{bmatrix} -0.2473 \\ -0.5699 \end{bmatrix} \).

V. CONCLUSIONS

The problem of asynchronous output-feedback stabilization for DMJLs has been studied. A necessary and sufficient condition for asynchronous output-feedback stabilizability has been established in terms of matrix inequalities with separated Lyapunov matrices from matrix variables to be designed. Based on this, an iterative algorithm has been proposed to solve the condition, and several optimization procedures have been provided to generate desirable initial values. Numerical examples have been used to illustrate the effectiveness of the proposed approach.

REFERENCES