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Decentralized Control of Compartmental Networks With $H_\infty$ Tracking Performance

Ping Li, Member, IEEE, and James Lam, Fellow, IEEE

Abstract—This paper studies the decentralized $H_\infty$ control problem for a class of compartmental networks, where the controller gain matrix is required to be diagonal and elementwise restricted. To solve the problem, we first propose a novel characterization under which the closed-loop system is asymptotically stable with a prescribed $H_\infty$ performance. Then, a necessary and sufficient condition for the existence of a required controller is obtained, and an iterative linear matrix inequality approach is developed to solve the design condition. Moreover, a $D-K$ iteration method is presented to optimize the initial value such that the solvability of the original problem can be further enhanced. Finally, the obtained results are illustrated through an application to a simple air traffic flow network model.

Index Terms—Compartmental networks, decentralized structure, $H_\infty$ control, iterative algorithm, positive systems.

I. INTRODUCTION

C ompartmental networks consist of a finite number of homogeneous and well-mixed subsystems (or compartments), which exchange materials with conservation laws between compartments and the environment. They have been widely adopted to model practical processes, such as chemical reaction, pharmacokinetics, and ecology [1], [2]. Recently, an interesting application of compartmental networks to congestion control problem in communication networks has been proposed in [3]. In addition to the conservation property, a key physical characterization of such systems is that the transfers between compartments are intrinsically nonnegative. Thus, differential equations for modeling the dynamics of such systems are subject to certain structural constraints, which make their solutions positive orthant invariant. Therefore, compartmental systems are a class of positive systems and possess some unique features that general dynamic systems do not have [4], [5].

The dynamics of compartmental networks have been extensively studied in the past few decades [6], [7], and a good tutorial paper can be found in [8]. In contrast with the abundance in the dynamic analysis, the synthesis of compartmental networks has received relatively less attention, mainly because these dynamic systems are defined on cones rather than linear spaces.

In this case, many well-established results for general linear systems may not be applicable to positive systems [9]–[12]. For example, reachability property is always preserved for general linear systems under similarity transformations, whereas it may not hold for compartmental linear systems due to the structural constraints on system matrices. Therefore, an increasing number of researchers have been devoted to the synthesis problem of such kind of systems. To name a few, positive observer design problem for compartmental networks has been studied by means of structural decomposition in [4]. The optimal output feedback controller design for set-point regulation of positive linear systems and compartmental systems has been considered in [13]. Recently, the $H_\infty$ filtering problem for compartmental systems with positivity preserved has been proposed in [14].

On the other hand, decentralized control strategy has become more and more favorable for large-scale systems due to either the lack of centralized information or the lack of centralized computing capability. Decentralized control for interconnected systems with time delays and uncertainties is studied in [15]–[17], and robust decentralized controller design with performance specification is explored in [18]. In addition, distributed collaborative control for industrial automation with wireless sensor and actuator networks has been considered in [18]. It should be pointed out that, with the appearance of large-scale systems, decentralized control has become increasingly important in real applications for economic and, possibly, reliability reasons, and it has been well known that the decentralized constraint added to the controller generally renders the control design problem to be nonconvex or even intractable [20], [21]. Therefore, considerable efforts have been made to identifying specific classes of tractable problems, which can be easily solvable via convex optimization techniques; see [22] and references therein. Moreover, compartmental networks have been utilized to model and control air traffic flow networks in the traffic management community [23], [24], and a major feature of the current management system is the insufficient sharing of information between decision makers, each having somewhat disparate information to control traffic flows in the corresponding region [25]. Thus, the management policy to be designed should only rely on local information, which can be characterized as a decentralized control problem in air traffic flow networks [24], [26], [27]. It should be noted that, while great efforts have been devoted to studying decentralized control for general dynamic systems, surprisingly, the decentralized design problem for compartmental systems has received very little attention mainly due to difficulties in preserving the system’s structural constraints, which motivates the present study.
In this paper, we study the decentralized control problem for compartmental networks with $H_\infty$ tracking performance and view the control strategy as a recirculation rate to adjust the flow speed in each compartment. The objective is to design a decentralized controller such that the outflow of the closed-loop compartmental system can track a given flow output in an $H_\infty$ manner. In this sense, the controller gain matrix is required to be diagonal and elementwise restricted, and conventional approaches for decentralized controller design may not be applicable anymore, mainly because of the additional constraints embedded to the gain matrix. To tackle this, we develop a novel linear matrix inequality (LMI) characterization under which the closed-loop system is asymptotically stable with a prescribed $H_\infty$ performance level. Such a characterization successfully introduces a parameter matrix with fully flexible structure, which can thus be designated to be diagonal and employed to construct a required gain matrix, whereas no conservatism will be introduced accordingly. Moreover, an iterative LMI approach is proposed to solve the design condition, and a D–K iteration method is addressed to optimize the initial value such that the solvability of the original problem can be further improved.

The rest of this paper is structured as follows. Section II formulates the decentralized control problem for compartmental networks with $H_\infty$ tracking performance. In Section III, we develop novel results to design the decentralized $H_\infty$ controller. A four-compartment air traffic flow network is presented to show the efficiency of our results in Section IV. Conclusions are given in Section V.

Notation: Let $\mathbb{R}$ be the set of real numbers, $\mathbb{R}^n$ denotes the $n$-column real vectors, and $\mathbb{R}^{n \times m}$ is the set of all real matrices of dimension $n \times m$. For any real symmetric matrices $P$ and $Q$, the notation $P \succeq Q$ (respectively, $P \succ Q$) means that the matrix $P - Q$ is positive semidefinite (respectively, positive definite). For a matrix $A \in \mathbb{R}^{m \times n}$, $a_{ij}$ denotes the element located at the $i$th row and the $j$th column. $I$ represents the identity matrix with an appropriate dimension. The superscript $^T$ denotes matrix transpose. For a given matrix $B \in \mathbb{R}^{n \times m}$ ($n \leq m$) with rank$(B) = r$, $B^+ \in \mathbb{R}^{m \times (m - r)}$ denotes the right orthonormal complement of $B$ by $BB^+ = 0$ and $(B^+)^T B^+ = I$. In addition, $\text{Sym}(M) \triangleq M^T + M$ is defined for any matrix $M \in \mathbb{R}^{n \times n}$, $\cdot \cdot \cdot \cdot$ represents the Euclidean norm for vectors, and $\| \cdot \|$ represents the spectral norm for matrices. $\|G\|_{\infty}$ represents the $H_\infty$ norm of a transfer function matrix $G$. The symbol $\#$ is used to represent a matrix which can be inferred by symmetry. Matrices, if their dimensions are not explicitly stated, are assumed to have compatible dimensions for algebraic operations.

II. PROBLEM FORMULATION

A compartment is a conceptual storage tank containing an amount of some material. A compartmental network is a network consisting of a finite number of compartments, which is usually used to describe material exchanges between compartments and the environment. Consider a compartment network with $n$ compartments denoted as $1, \ldots, n$. For compartment $i$, its state $x_i(t)$ represents the amount of material at time $t$, which must be nonnegative all the time. Assume that the flow rate for compartment $i$ is linear. The traversal time is denoted as $\tau_i$; therefore, the natural flow rate can be defined as $x_i(t)/\tau_i$. Following the notation in Fig. 1 and using the principle of flow conservation, we denote $\alpha_{ji}, 0 \leq \alpha_{ji} \leq 1$, as the fraction of the inflow rate from compartment $j$ that flows into compartment $i$. Similarly, we let $0 \leq \beta_{ji} \leq 1$ as the fraction of the inflow rate from the $j$th environmental channel that flows into compartment $i$.

The control input for compartment $i$, denoted as $u_i(t)$, is viewed as a recirculation rate and can be usually implemented by changing the flow speed inside compartment $i$. That is, the control actions are modeled as taking part of the natural outflow of the region and recirculating it back into the same region. Therefore, the total outflow rate of compartment $i$ can be obtained as

$$f_i(t) = \frac{x_i(t)}{\tau_i} - u_i(t).$$  \hfill (1)

Indeed, the control input $u_i(t)$ is employed to vary the outflow rate of compartment $i$, which should further change the amount of material in the corresponding compartment. Thus, the dynamics of compartment $i$ can be described by

$$\dot{x}_i(t) = -\left[\frac{x_i(t)}{\tau_i} - u_i(t)\right] + \sum_{j=1}^{n} \alpha_{ji} \left[\frac{x_j(t)}{\tau_j} - u_j(t)\right] + \sum_{j=1}^{m} \beta_{ji} w_j(t) \quad (2)$$

for $i = 1, 2, \ldots, n$. The performance output is defined as $z = [z_1, \ldots, z_q]^T$ with $z_i = \sum_{j=1}^{n} \gamma_{ji} f_j$, which is a linear combination of the outflow rates. See [1] and [2] for more details about compartmental networks.

In this paper, we shall look for a general state-feedback controller for (2), which obeys the decentralized information structure constraint requiring that each compartment $i$ is controlled using only its own state information. That is, the controller should be designed as

$$u_i = k_i x_i, \quad \text{for } i = 1, \ldots, n.$$  \hfill (3)

For convenience, denote $K = \text{diag}(k_1, \ldots, k_n)$. In what follows, we shall formulate the constraint added to $k_i$. On the one hand, the controlled outflow rate $f_i(t)$ in (1) should be
nonnegative according to its physical meaning. Because \( x_i \geq 0 \)
represents the amount of material in compartment \( i \), we have
\[
  k_i \leq \frac{1}{\tau_i} \quad \text{for } i = 1, \ldots, n. \tag{4}
\]
In fact, (4) will preserve the positivity of system (2); see [28, (Theorem 2 on page 14)] for more details. On the other hand, there should be a minimal traversal time \( \tau_i \) for compartment \( i \), and thus, \( f_i(t) \) should be upper bounded by \( x_i(t)/\tau_i \), which further yields
\[
  \frac{1}{\tau_i} - \frac{1}{\tau_{\bar{i}}} \leq k_i \quad \text{for } i = 1, \ldots, n. \tag{5}
\]
Conditions (4) and (5) can be referred to as the positivity condition and capacity condition, respectively. Therefore, we have \( \bar{K} \leq K \leq \hat{K} \), where
\[
  \bar{K} = \text{diag} \left( \frac{1}{\tau_1}, \ldots, \frac{1}{\tau_n} \right) \tag{6}
\]
and
\[
  \hat{K} = \text{diag} \left( \frac{1}{\tau_{\bar{1}}}, \ldots, \frac{1}{\tau_{\bar{n}}} \right). \tag{7}
\]

To facilitate later developments, we combine the individual dynamics for each compartment in (2) and introduce the following general linear time-invariant system
\[
\begin{cases}
  \dot{x}(t) = A\bar{x}(t) + Bu(t) + B_w w(t) \\
  z(t) = C\bar{x}(t) + Du(t)
\end{cases} \tag{8}
\]
where the system matrices \( A, B, B_w, C, \) and \( D \) can be obtained directly from the interconnection of subsystems (2).

The control objective is to find a structural controller \( u(t) = \hat{K}\bar{x}(t) \) with \( \bar{K} \leq K \leq \hat{K} \) such that the closed-loop system in (8) is asymptotically stable and the output \( z \) behaves satisfactorily. More specifically, we assume that the reference output is generated by
\[
\begin{cases}
  \dot{\xi}(t) = F\xi(t) + Gw(t) \\
  z_{\text{ref}}(t) = H\xi(t)
\end{cases} \tag{9}
\]
where \( \xi(t) \in \mathbb{R}^r \) for any \( t > 0 \). Of course, the system parameters in (9) must be selected such that the reference output \( z_{\text{ref}}(t) \) is meaningful with respect to the physical system to be controlled.

Let \( \hat{x}(t) = [x^T(t), \xi^T(t)]^T \) and \( e(t) = z(t) - z_{\text{ref}}(t) \); then, from (8) and (9), we obtain the following augmented system
\[
\begin{cases}
  \dot{\hat{x}}(t) = \hat{A}\hat{x}(t) + \hat{B}w(t) \\
  e(t) = \hat{C}\hat{z}(t)
\end{cases} \tag{10}
\]
where
\[
\hat{A} = A + MK\bar{J} \quad \hat{B} = B \quad \hat{C} = C + \hat{N}K\bar{J}
\]
with
\[
\begin{align*}
  \hat{\bar{A}} &= \begin{bmatrix} A & 0 \\ 0 & F \end{bmatrix} \\
  \hat{\bar{B}} &= \begin{bmatrix} B_w \\ G \end{bmatrix} \\
  \hat{\bar{C}} &= \begin{bmatrix} C & -H \end{bmatrix} \\
  \hat{\bar{M}} &= \begin{bmatrix} B \\ 0 \end{bmatrix} \\
  \bar{J} &= \begin{bmatrix} I & 0 \end{bmatrix} \\
  \hat{\bar{N}} &= \bar{N}.
\end{align*}
\]

In this paper, we characterize the optimality criterion for controller satisfactory in terms of the \( H_\infty \) norm of system (10). More specifically, we want to find structural controllers such that \( \|T_{ew}\|_\infty < \delta \), where \( T_{ew} \) is the transfer function from \( w \) to \( e \) and \( \delta > 0 \) is a given \( H_\infty \) performance level.

Based on the foregoing discussions, we now formulate the decentralized \( H_\infty \) tracking problem for compartmental network (2) as follows.

Problem DHCCN (Decentralized \( H_\infty \) Control of Compartmental Networks): Given a reference output in (9), design a decentralized controller \( u(t) = \hat{K}\bar{x}(t) \) for system (8) such that the following requirements are fulfilled simultaneously.

1) \( K \) is diagonal with \( \bar{K} \leq K \leq \hat{K} \), where \( \bar{K} \) and \( \hat{K} \) are defined in (6) and (7), respectively.

2) The system in (10) is asymptotically stable and satisfies \( \|T_{ew}\|_\infty < \delta \), where \( \delta \) is a given \( H_\infty \) performance level.

Remark 1: In fact, the \( H_\infty \) control problem has been extensively studied for various kinds of systems in the literature; see, for instance, [29]–[35]. The major difference in this paper and those in the previous works is that the controller not only has to fulfill the \( H_\infty \) performance specification but also needs to be diagonal with elements restricted, which further implies that conventional approaches, such as similarity transformation and elimination technique, may not be applicable anymore.

The following result gives a fundamental characterization on the stability of (10) with \( H_\infty \) performance, which can be viewed as a special case of the well-known bound real lemma in [29].

Lemma 1: The system in (10) is asymptotically stable with \( \|T_{ew}\|_\infty < \delta \), if and only if there exists a matrix \( P > 0 \) such that
\[
\begin{bmatrix}
  P\hat{\bar{A}} + \hat{\bar{A}}^T P & P\hat{\bar{B}} & \hat{C}^T \\
  # & -\delta I & 0 \\
  # & # & -\delta I
\end{bmatrix} < 0. \tag{11}
\]

III. MAIN RESULTS

A. Novel Analysis Characterization and Design Condition

In this section, we first develop a novel condition in Theorem 1 under which system (10) will be asymptotically stable and satisfies the \( H_\infty \) performance \( \|T_{ew}\|_\infty < \delta \). Then, we further establish a necessary and sufficient condition for the existence of desired structural controllers in Theorem 2.

Theorem 1: Given the decentralized control gain \( K \), the system in (10) is asymptotically stable with \( H_\infty \) performance \( \|T_{ew}\|_\infty < \delta \), if and only if there exist matrices \( P > 0 \) and \( X > 0 \) such that
\[
\begin{bmatrix}
  P\hat{\bar{A}} + \hat{\bar{A}}^T P & P\hat{\bar{B}} & \hat{C}^T \\
  # & -\delta I & 0 \\
  # & # & -\delta I
\end{bmatrix} < 0 \tag{12}
\]
where
\[
\begin{align*}
  \bar{P} &= \begin{bmatrix} P & 0 \\ -\frac{1}{2}XK\bar{J} & \frac{1}{2}X \end{bmatrix} \\
  \bar{A} &= \begin{bmatrix} A & M \\ \bar{K}\bar{J} & -I \end{bmatrix} \\
  \bar{B} &= \begin{bmatrix} B \\ 0 \end{bmatrix} \\
  \bar{C} &= \begin{bmatrix} C & \bar{N} \end{bmatrix}.
\end{align*}
\]
Proof: (Sufficiency) Define a nonsingular transformation matrix as follows:

$$T = \begin{bmatrix} I & 0 & 0 & 0 \\ K & 0 & 0 & I \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \end{bmatrix}.$$  \hspace{1cm} (13)

Pre- and postmultiplying (12) by \(T^T\) and \(T\), we obtain

$$\Sigma^\Delta \triangleq T^T \Sigma T = \begin{bmatrix} P\hat{A} + \hat{A}^T P & \hat{P}\hat{B} & \hat{C}^T & \hat{P}M \\ \# & -\delta I & 0 & 0 \\ \# & \# & -\delta I & N \\ \# & \# & \# & -X \end{bmatrix} < 0.$$  \hspace{1cm} (14)

It follows from Lemma 1 and the third leading principal matrix of \(\Sigma\) that the system in (10) is asymptotically stable with \(H_\infty\) performance guaranteed.

(Necessity) If the system in (10) is asymptotically stable and satisfies the \(H_\infty\) performance, then it can be seen from Lemma 1 that there exists a matrix \(P > 0\) such that

$$\Pi = \begin{bmatrix} P\hat{A} + \hat{A}^T P & \hat{P}\hat{B} & \hat{C}^T \\ \# & -\delta I & 0 \\ \# & \# & -\delta I \end{bmatrix} < 0.$$  \hspace{1cm} (15)

Then, for any \(S > 0\), there exists a sufficiently large scalar \(\varrho\) such that

$$-\varrho S - \begin{bmatrix} \hat{P}\hat{M} & 0 \\ 0 & N \end{bmatrix} \Pi \begin{bmatrix} \hat{P}\hat{M} \\ 0 \end{bmatrix} < 0.$$  \hspace{1cm} (16)

By choosing \(X = \varrho S\) and applying Schur complement [36] to (15), we have

$$\Sigma = T^{-T} \Sigma T^{-1} < 0$$

which completes the whole proof. \hfill \Box

Remark 2: From (11) in Lemma 1, one can see that the controller gain matrix \(K\) is coupled with the Lyapunov matrix \(P\), and thus, additional constraints on \(P\) may be induced when \(K\) is parametrized, which will inevitably make the corresponding result conservative [24]. The advantage of condition (12) is twofold: First, we separate \(P\) from \(K\), which can be further parametrized by matrix \(X > 0\), and second, the existence of \(X\) is naturally satisfied and it can be designated to be with any structure, which will greatly facilitate the design of \(K\) subsequently.

We are now in a position to establish a necessary and sufficient condition for the existence of the desired structural controllers, which is stated in the following theorem.

Theorem 2: Problem DHCCN is solvable if and only if there exist matrices \(P > 0\) and \(U\) and diagonal matrices \(X > 0\) and \(L\) satisfying the conditions in (16) and (17), shown at the bottom of the page.

In this case, a desired decentralized controller can be chosen as

$$K = X^{-1} L.$$  \hspace{1cm} (18)

Proof: Expanding (12), we have

$$\begin{bmatrix} P\hat{A} + \hat{A}^T P & \hat{P}\hat{B} & \hat{C}^T \\ \# & -X & 0 \\ \# & \# & -\delta I \end{bmatrix} < 0.$$  \hspace{1cm} (19)

In what follows, we shall prove the sufficiency part and the necessity part, respectively.

(Sufficiency) First, since \(X > 0\) and \(L\) are both diagonal, it follows from (18) that \(K\) is diagonal. Second, (17) and (18) indicate that \(K \leq K \leq K\) directly. Moreover, substitute \(L = XK\) into (16), and observe that, for any \(U\)

$$-J^T K X K^T \leq U^T X U - \text{Sym}(U^T L\bar{J})$$  \hspace{1cm} (20)

we have that (19) holds, which further indicates (12). Then, based on Theorem 1, we conclude that the system in (10) is asymptotically stable and satisfies the \(H_\infty\) performance. This completes the sufficiency proof.

(Necessity) If Problem DHCCN is solvable, then, for the given system in (10), it follows from Theorem 1 that there exist \(P > 0\) and diagonal \(X > 0\) such that (12) or, equivalently, (19) holds. By choosing \(U = K \bar{J}\) and letting \(L = XK\), one has that (16) holds. In addition, condition (17) can be obtained directly because \(K \leq K \leq K\) and \(L = XK\) with \(X > 0\) diagonal. This completes the whole proof. \hfill \Box

Remark 3: It should be noted that, although \(X\) is specified to be positive diagonal in Theorem 2, the design condition for the existence of a required controller is still necessary and sufficient. Moreover, due to the fact that the structure of \(X\) is rather flexible, it can be even designated to be a positive scalar matrix, whereas no conservatism will be introduced consequently. Therefore, if the controller gain matrix \(K\) is required to satisfy some constraints, such as block diagonal, triangular, positive, or symmetric, they can be readily dealt with in the same framework proposed in this paper.
B. Computational Approaches

Our aim in this section is to derive numerically tractable algorithms to synthesize the required controllers with the help of convex optimization techniques. Indeed, although the controller can be designed according to Theorem 2, it is still difficult to verify (16) due to the existence of cubic terms there. However, it can be noted that, when matrix\(U\) is fixed, (16) turns out to be linear with respect to the other variables. Therefore, a natural way is to fix \(U\) and solve (16) and (17) by existing LMI techniques [36]. For \(\Sigma(U)\) defined in (16), it follows from the proof in Theorem 2 that
\[
\Sigma(X^{-1}L\bar{J}) \leq \Sigma(U)
\]
holds for any fixed \(P, X,\) and \(L\). Therefore, it can be concluded that the scalar \(\mu\) satisfying
\[
\Sigma(U) \leq \mu I
\]
achieves its minimum when \(U = X^{-1}L\bar{J}\). Thus, the following iterative algorithm can be proposed to solve the decentralized \(H_{\infty}\) control problem.

Algorithm DHCCN:

1) Set \(j = 1\). For a given \(H_{\infty}\) performance level \(\delta > 0\), select the initial matrix \(U_1\) such that the following auxiliary system
\[
\begin{align*}
\dot{x}(t) &= \bar{A}x(t) + \bar{M}u(t) + \bar{B}w(t) \\
e(t) &= \bar{C}x(t) + \bar{N}u(t)
\end{align*}
\]
with
\[
u(t) = U_1\bar{x}(t)
\]
is asymptotically stable with \(\|\hat{T}_{ew}\|_{\infty} < \delta\), where \(\hat{T}_{ew}\) is the transfer function matrix from \(w\) to \(e\) in (21).

2) For fixed \(U_j\), solve the following convex optimization problem for the parameters in \(\mathbb{S} \triangleq \{P > 0, X > 0\text{ diagonal and } L\text{ diagonal}\}:
\[
\begin{align*}
\mu_j^* &= \min_{\mathbb{S}} \mu_j \\
\text{s.t. } &\Xi(U_j) < \mu_j I \\
&\text{(17) and } \mu_j \geq -\vartheta
\end{align*}
\]
where \(\vartheta \geq 0\) is an arbitrary scalar. Denote the corresponding values of \(X\) and \(L\) as \(X_j\) and \(L_j\), respectively.

3) If \(\mu_j^* \leq 0\), then a desired controller matrix \(K\) is obtained as \(K = X_j^{-1}L_j\). STOP.

Else if \(\left|\left(\mu_j^* - \mu_{j-1}^*\right)/\mu_j^*\right| < \varepsilon_1\), where \(\varepsilon_1\) is a prescribed tolerance, then go to Step 4).

Else update \(U_{j+1}\) as
\[
U_{j+1} = X_j^{-1}L_j\bar{J}
\]
set \(j = j + 1\), and go to Step 2).

4) A solution to Problem DHCCN may not exist. STOP.

Remark 4: Note that the constraint \(\mu_j \geq -\vartheta\) is only added to make \(\mu_j^*\) bounded from below by a negative scalar and will not affect the search of \(\mu_j^*\), since we are only interested in the case of \(\mu_j^* \leq 0\). Meanwhile, it can be seen that the sequence \(\mu_j^*\) is monotonically decreasing with respect to \(j\), i.e., \(\mu_{j+1}^* \leq \mu_j^*\). Therefore, the convergence of the iterative process is naturally guaranteed.

It follows from Step 1) that, if one cannot find such a matrix \(U_1\), then it can be concluded immediately that there does not exist a solution to Problem DHCCN. In fact, the initial matrix \(U_1\) can be viewed as a state-feedback \(H_{\infty}\) controller matrix and can be constructed by existing convex optimization approaches. We present the following result with proof omitted.

Theorem 3: System (21) with (22) is asymptotically stable with \(\|\hat{T}_{ew}\|_{\infty} < \delta\), if and only if there exist matrices \(Q > 0\) and \(V\) such that
\[
\begin{bmatrix}
\text{Sym}(\bar{A}Q + \bar{M}V) & \bar{B} & V^T\bar{N}^T + Q\bar{C}^T \\
\# & -\delta I & 0 \\
\# & \# & -\delta I
\end{bmatrix} < 0.
\]

Under this condition, an initial matrix \(U_1\) can be given by \(U_1 = VQ^{-1}\).

Theorem 3 presents a necessary condition to check the feasibility of Problem DHCCN and shows how the initial matrix \(U_1\) can be constructed. In addition, it is well known that the iterative process usually depends on the choice of \(U_1\), and thus, a natural question that one may raise is how to choose an “optimal” one such that the solvability of Algorithm DHCCN can be further improved. In what follows, we shall focus on the optimization of \(U_1\).

From inequality (20), we have
\[
-J^TK^TXK\bar{J} \leq U^TXU - \text{Sym}(U^TX\bar{K})
\]
\[
= -J^TK^TXK\bar{J} + (U - K\bar{J})^TX(U - K\bar{J}).
\]

Therefore, if Problem DHCCN is feasible, then (16) and (17) will be feasible as well, provided that matrix \((U - K\bar{J})^TX(U - K\bar{J})\) is small enough. Hence, a natural way to improve the solvability of the original problem is to reduce \(||(U_1 - K\bar{J})^TX(U_1 - K\bar{J})||\), which can be implemented by reducing \(||U_1 - K\bar{J}||\) as small as possible. The following theorem gives an explicit characterization of how to choose \(U_1\) properly.

Theorem 4: For a sufficiently small scalar \(\epsilon > 0\), the following statements are equivalent.

i) There exist \(K\) and \(U_1\) such that system (10) is asymptotically stable with \(\|\hat{T}_{ew}\|_{\infty}\).

ii) There exists \(U_1\) such that system (21) with (22) is asymptotically stable with \(\|\hat{T}_{ew}\|_{\infty} < \delta\) and \(\|U_1\bar{J}\| < \epsilon\).

Proof: (i) \(\Rightarrow\) (ii) First, define
\[
F = \begin{bmatrix}
\bar{A} & \bar{B} & 0 \\
0 & -\frac{\delta}{2}I & 0 \\
\bar{C} & 0 & -\frac{\delta}{2}I
\end{bmatrix}, \quad G = \begin{bmatrix}
\bar{M} \\
0 \\
\bar{N}
\end{bmatrix}, \quad H = \begin{bmatrix}
I^T \\
0 \\
0
\end{bmatrix}.
\]

\[\theta = \begin{bmatrix}
\bar{A} & \bar{B} & 0 \\
0 & -\frac{\delta}{2}I & 0 \\
\bar{C} & 0 & -\frac{\delta}{2}I
\end{bmatrix}, \quad G = \begin{bmatrix}
\bar{M} \\
0 \\
\bar{N}
\end{bmatrix}, \quad H = \begin{bmatrix}
I^T \\
0 \\
0
\end{bmatrix}.
\]

(24)
If system (10) is asymptotically stable with \( \|T_{cw}\|_\infty < \delta \), then it follows from Lemma 1 that there exists \( P > 0 \) such that
\[
\text{Sym} \left( (\mathcal{F} + \mathcal{G}K\mathcal{J}\mathcal{H})^T \mathcal{X} \right) < 0 \tag{25}
\]
where
\[
\mathcal{X} = \text{diag}(P \ I \ I). \tag{26}
\]

Therefore
\[
\text{Sym} \left( (\mathcal{F} + \mathcal{G}U_1\mathcal{H})^T \mathcal{X} \right) = \text{Sym} \left( (\mathcal{A} + \mathcal{G}K\mathcal{J}\mathcal{H})^T \mathcal{X} \right) + \text{Sym} \left( (\mathcal{G}(U_1 - K\mathcal{J})\mathcal{H})^T \mathcal{X} \right) < 0 \tag{27}
\]
provided that \( \|U_1 - K\mathcal{J}\| < \epsilon \) with \( \epsilon > 0 \) sufficiently small. This indicates that system (21) with (22) is asymptotically stable with \( \|T_{cw}\|_\infty < \delta \).

In addition, we have
\[
\|U_1\mathcal{J}\| = \|(U_1 - K\mathcal{J})\mathcal{J}\| 
\leq \|U_1 - K\mathcal{J}\|\|\mathcal{J}\| < \epsilon.
\]

(ii) \( \Rightarrow \) (i) Recall \( \mathcal{J} = [J \ 0] \); we have \( \mathcal{J} = [0 \ 1]^T \). Choose \( K = U_1\mathcal{J}^T \); then
\[
(U_1 - K\mathcal{J})[\mathcal{J}^T \ \mathcal{J}] = [0 \ U_1\mathcal{J}].
\]

Since \( [\mathcal{J}^T \ \mathcal{J}] \) is an identity matrix, we obtain
\[
\|U_1\mathcal{J}\| = \|U_1 - K\mathcal{J}\| < \epsilon.
\]

Following a similar line in (i) \( \Rightarrow \) (ii), one can prove that system (10) is asymptotically stable with \( \|T_{cw}\|_\infty < \delta \). This completes the whole proof.

Theorem 4 indicates that \( U_1 \) should be selected according to Theorem 3, together with the requirement that \( \|U_1\mathcal{J}\| \) should be sufficiently small. In view of this, we propose the following \( D-K \) type optimization algorithm.

Algorithm \( D-K \) Optimization (DKO):
1) Set \( j = 1 \). Choose \( U_j \) according to Theorem 3.
2) For fixed \( U_j \), find \( P \) such that
\[
\text{Sym} \left( (\mathcal{F} + \mathcal{G}U_j\mathcal{H})^T \mathcal{X} \right) < 0 \tag{28}
\]
where \( \mathcal{F}, \mathcal{G}, \mathcal{H}, \) and \( \mathcal{X} \) are defined in (24) and (26).
3) For fixed \( P \) obtained in Step 2), solve
\[
\min_{\nu_j} \nu \quad \text{s.t.} \quad \begin{cases} 
\text{Sym} \left( (\mathcal{F} + \mathcal{G}U_j\mathcal{H})^T \mathcal{X} \right) < 0 \\
-\nu I \\
\# \\
-\nu I \\
\end{cases} < 0.
\]

Denote the minimum value \( \nu_j \) by \( \nu_j^* \) and the corresponding solution \( U \) by \( U_j^* \).
4) If \( \|x_j^* - \nu_{j-1}^*\| < \varepsilon_2 \), where \( \varepsilon_2 \) is a prescribed tolerance, then go to Step 5). If not, update \( U_{j+1} = U_j^* \) and \( j = j + 1 \), and go to Step 2).
5) A desired initial value \( U_{1\text{opt}} = U_j^* \) can be obtained. STOP.

Remark 5: It should be noted that the convergence of Algorithm DKO is naturally guaranteed, since \( \nu_j \) is monotonically decreasing with respect to \( j \), and bounded from below by zero. Moreover, since \( K \) is required to be diagonal with \( K \leq \tilde{K} \leq \bar{K} \) in this paper, it follows from the proof in (ii) \( \Rightarrow \) (i) that the constraint \( K \leq U_1\mathcal{J}^T \leq \bar{K} \) can also be incorporated in Step 3) when implementing Algorithm DKO.

IV. APPLICATION EXAMPLE

In this section, we provide an example to illustrate the effectiveness of the proposed decentralized \( H_\infty \) controller design method through an application to a simple air traffic flow network model.

The compartmental modeling approach is usually implemented by dividing the airspace into a number of interconnected compartments and then characterizing the flow dynamics in terms of the flow properties in these compartments. Fig. 2 shows a simple air traffic flow network, where there are four compartments \( A_1, A_2, A_3, \) and \( A_4 \) and two inlet ports \( I_1 \) and \( I_2 \). The arrow represents the flow direction in the network, for example, the flow from compartment \( A_1 \) can enter into \( A_2 \) and \( A_3 \), whereas it cannot enter into \( A_4 \). The objective is to regulate the arrivals at airport \( O_1 \) which constitute the system outflows.

Based on the discussion in Section II, the network dynamics can be represented by the following state-space model:

\[
\begin{align*}
\dot{x}(t) &= \begin{bmatrix}
-\frac{1}{\tau_1} & 0 & 0 & 0 \\
\frac{1}{\tau_1} & -\frac{1}{\tau_2} & 0 & 0 \\
0 & \frac{1}{\tau_2} & -\frac{1}{\tau_3} & 0 \\
0 & 0 & \frac{1}{\tau_3} & -\frac{1}{\tau_4}
\end{bmatrix} x(t) \\
& \quad + \begin{bmatrix}
1 \\
-\alpha_{12} \\
0 \\
0
\end{bmatrix} u(t) \\
\end{align*}
\]

\[
\begin{align*}
\dot{z}(t) &= \begin{bmatrix}
0 \\
0 \\
0 \\
\beta_1 & \beta_2
\end{bmatrix} x(t) + \begin{bmatrix}
0 & 0 & -1 & -1
\end{bmatrix} u(t)
\end{align*}
\]

where \( x(t) = [x_1(t), \ldots, x_4(t)]^T \) denotes the number of aircraft in compartments \( A_1 \) to \( A_4 \) and \( u(t) = [u_1(t), \ldots, u_4(t)]^T \) is the control input added to regulate the traffic output \( z(t) \). Assume that the estimated parameters for (29) are

\[
\begin{align*}
\tau_1 &= \tau_2 = 0.2 & \tau_3 &= \tau_4 = 0.4 \\
\alpha_{12} &= 0.25 & \alpha_{23} &= 0.5 & \beta_1 &= 0.2 & \beta_2 &= 0.5
\end{align*}
\]
and the reference output is generated by
\[
\begin{cases}
\dot{x}(t) = -0.5x(t) + [1, 1]w(t) \\
\tau_{\text{ref}}(t) = 0.5x(t).
\end{cases}
\] (30)

In this example, we specify the $H_\infty$ performance level as $\delta = 0.4$, and the aim is to match the required arrival rate $\tau_{\text{ref}}(t)$ by resorting to Algorithm DKO with the constraint
\[
K
\]
solution when Algorithm DHCCN is implemented. However, which further yields the decentralized controller gain matrix as
\[
K = \text{diag}(5, 5, 2.5, 2.5),
\]
and thus, the final problem reduces to find a diagonal $K$ satisfying $K \leq \overline{K}$ such that system (29) is asymptotically stable and satisfies $\|T_{\text{ref}}\|_\infty < 0.4$ with $e(t) = z(t) - \tau_{\text{ref}}(t)$.

It can be easily verified that the $H_\infty$ performance level is 0.8 when there is no controller added to system (29). Also, it is worth pointing out that traditional $H_\infty$ controller design methods cannot guarantee us to obtain a desired controller, since they may violate the positivity constraint or capacity constraint in general. In what follows, we shall use the method proposed in this paper to design a required controller.

By resorting to Theorem 3, an initial matrix $U_1$ can be first obtained as
\[
U_1 = \begin{bmatrix}
4.6217 & -0.1208 & -0.4522 & 0.9453 & -0.1366 \\
0.2723 & 4.5842 & 3.2092 & -3.0690 & -0.2161 \\
0.5532 & -3.3629 & 3.3844 & 1.9084 & -0.2943 \\
-0.5006 & 3.3552 & -0.8915 & 0.5844 & -0.1918
\end{bmatrix}.
\]

In this case, it can be checked that there exists no feasible solution when Algorithm DHCCN is implemented. However, by resorting to Algorithm DKO with the constraint $K \leq U_1J^T \leq \overline{K}$ added, one can find an optimized $U_1$ as
\[
U_{1,\text{opt}} = \begin{bmatrix}
1.2205 & 0 & 0 & 0 & 0 \\
0 & -9.2562 & 0 & 0 & 0 \\
0 & 0 & 1.9567 & 0 & 0 \\
0 & 0 & 0 & 2.1395 & 0
\end{bmatrix}.
\]

By running Algorithm DHCCN again, a feasible solution can be obtained with
\[
X = \text{diag}(4.5863, 5.0920, 7.9523, 6.9406) \\
L = \text{diag}(5.5714, 4.1409, 15.4339, 14.6445)
\]
which further yields the decentralized controller gain matrix as
\[
K = X^{-1}L = \text{diag}(1.2148, 0.8132, 1.9408, 2.1100).
\]

The performance of the closed-loop flow is evaluated in a simulation, where the inflow rates from $I_1$ and $I_2$ are assumed to be sinusoidal and have a mean of 10 aircraft/time unit and an amplitude of three within the 20 time units as follows:
\[
w(t) = \begin{cases}
[10 + 3 \sin(t)], & 0 \leq t \leq 20 \\
0, & \text{otherwise}.
\end{cases}
\]

The initial conditions used in the simulation are
\[
x(0) = [2, 6, 1, 5]^T, \quad \xi(0) = 2.
\]

Fig. 3. Landing rate of the closed-loop system.

Fig. 3, which shows the landing rates of the closed-loop system for the desired, uncontrolled, and controlled cases, indicates that the controlled outflow rate is well regulated to match the desired one.

V. CONCLUSION

The decentralized $H_\infty$ control problem for a class of compartmental systems has been studied in this paper, where the controller gain matrix is required to be diagonal and element-wise restricted. A matrix-inequality-based characterization has been proposed to ensure the asymptotic stability of the controlled system with a prescribed $H_\infty$ performance level, and a necessary and sufficient condition for the existence of a desired controller has been established accordingly. Then, an iterative LMI algorithm has been developed to solve the design condition, and a $D-K$-type algorithm has been further given to improve the solvability of the original problem. Finally, a four-compartment air traffic network has been presented to illustrate the effectiveness of the obtained results. Future directions include the decentralized control for compartmental networks with time delays and/or missing measurements [37]–[39].

REFERENCES


