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Realizations of a Special Class of Admittances With Strictly Lower Complexity Than Canonical Forms

Michael Z. Q. Chen, Member, IEEE, Kai Wang, Zhan Shu, Member, IEEE, and Chanying Li

Abstract—This paper investigates the simplified realization problem of a special class of positive-real admittances similar to biquadratic functions but with an extra pole at the origin, which is widely used in the analysis of suspension systems. The results in this paper are motivated by passive mechanical control with the inerter. The concept of strictly lower complexity is first defined, whose indices in this paper are the total number of elements, the number of resistors (dampers), and the number of capacitors (inerters). We then derive a necessary and sufficient condition for this class of admittance to be realized by the networks that are of strictly lower complexity than the canonical realization by the Foster Preamble method. To solve this problem, it is shown that it suffices to consider the following: 1) networks with at most four elements, 2) irreducible five-element resistor-inductor (RL) networks, and 3) irreducible five-element resistor-inductor-capacitor (RLC) networks. Other cases are shown to be impossible. By finding their corresponding network configurations through a series of constraints and deriving the corresponding realizability conditions, the final condition can be obtained. Finally, the U-V plane and numerical examples are provided to illustrate the theoretical results.

Index Terms—Network synthesis, inerter, passivity, transformerless synthesis, canonical realization.

I. INTRODUCTION

SINCE the publication of Brune’s seminal paper [4], passive network synthesis has been an important branch of systems theory. From the 1930s to the 1970s, passive network synthesis experienced its “golden era” [1]–[3], [17], after which interest has declined despite developments in positive-real functions [14], [16], [26], [41], [44] and the stability of behavioral systems [28]. Although many beautiful results have been discovered, several problems including the minimal transformerless realization problem are far from being solved even for the biquadratic function. The most general transformerless synthesis procedure, the Bott-Duffin synthesis [3], appears to be highly nonminimal.

Recently, interest in passive network synthesis has been renewed due to a new two-terminal mechanical element named “inerter” [31], in which the relative acceleration of the two terminals is proportional to the force applied at them. The inerter has been applied to mechanical control systems based on the passive mechanism [8], such as vehicle suspension control [7], [19], [32], [37], the control of motorcycle steering instabilities [15], building control [38], and railway suspension control [20], [29]. Furthermore, the introduction of the inerter completes the analogy between the passive mechanical system components and the electrical ones (see Fig. 1). All the theories and methods in the passive electrical network synthesis can therefore be mapped into the mechanical systems (networks). However, for mechanical systems, the complexity of the configurations is much more essential than in electrical systems and the levers are not preferred. Therefore, a series of new results in passive network synthesis [5], [6], [9]–[11], [13], [18], [21]–[23], [40] have been developed in recent years. The realization problem of any positive-real admittance with one spring (inductor), one inerter (capacitor), and an arbitrary number of dampers (resistors) is studied in [9]. In [10], it derives a necessary and sufficient condition for any positive-real admittance to be realizable with one damper, one inerter, and an arbitrary number of springs, and provides a group of configurations that can cover the condition. In [11], positive-real criteria for a series of low-order real-rational functions are obtained. The realization problem of a special class of admittances with one damper, one inerter, and an arbitrary number of springs is studied in [13], and the realizability condition is in an explicit form in terms of the coefficients. It is proved in [40] that any positive-real biquadratic function is always realizable through a set of decomposition procedures without minimization that needs at most one surplus factor, and no more elements are used than the Bott-Duffin method for series-parallel networks. The works in [7], [37] well demonstrate that the results in network synthesis [9], [10], [21] are directly applicable to suspension designs. The need for a renewed attempt on passive network synthesis and its fundamental connection to new advances in systems theory have recently been highlighted by Kalman [24].

For suspension systems, the mechanical admittances of suspension struts [33] are always in a form similar to a biquadratic function but with an extra pole at \( s = 0 \). Smith [31] first discussed the realizability of this class of admittances. It can be verified that using the Foster Preamble method, it is generally realizable by either a network consisting of one capacitor (inerter),
two resistors (dampers), and two inductors (springs), or a network consisting of two resistors (dampers) and three inductors (springs), which is regarded as the canonical realization. However, the realization is not guaranteed to be minimal and the realization condition with strictly lower complexity has not yet been discussed. This study considers the class of realizations which are of strictly lower complexity than the canonical realization. In this setting, no transformer can be employed as large leverage ratios can cause practical difficulties.

We first discuss the case where there exist zero coefficients for any given positive-real admittance, which can be realized using at most four elements with strictly lower complexity than the canonical realization. Then, a necessary and sufficient condition is established for this class of admittance with positive coefficients to be realized by a network which is of strictly lower complexity than the canonical realization. Networks with at most four elements and irreducible five-element networks are investigated, respectively. The possible network graphs are listed, and together with the constraints of the realization, the corresponding configurations can be obtained. In order to group the configurations, the concept of frequency-inverse dual (FID) is introduced. Through the derivation of their corresponding realizability conditions, the final realizability condition is obtained.

Section II reviews the research background, formulates the class of admittances to be investigated, and defines the concept of strictly lower complexity. In Section III, the realizations of this class of admittances with zero coefficients are discussed. Section IV derives a necessary and sufficient condition for this class of admittances to be realized by a network which is of strictly lower complexity than the canonical realization. In Section V, numerical examples are provided to illustrate the theoretical results. Finally, conclusions are drawn in Section VI. For brevity, the detailed proofs of some results in this paper are presented in the supplementary material [12].

II. BACKGROUND AND PROBLEM FORMULATION

The concept of passivity can be referred to [1], and the definition of positive-real function is given in [1], [2], [17]. The criteria of positive-real functions are provided in [1], [27], [42], [43]. In [1], we can see that for the real rational impedance or admittance function $Z(s)$ of a linear time-invariant two-terminal network, the network is passive if and only if $Z(s)$ is positive real. Since any positive-real admittance (impedance) can be realized with a finite number of inductors, capacitors, and resistors, any passive mechanism can be constructed with only springs, inductors, and dampers, which can be used in the passive control of various mechanical systems [8] for a relatively low cost with its advantages in spite of the development of the active control methods.

Consider a given admittance $Y(s)$ in the form of

$$Y(s) = \frac{a_0 s^2 + a_1 s + 1}{s(d_0 s^2 + d_1 s + 1)}, \quad (1)$$

where $a_0, a_1, d_0, d_1 \geq 0$, and $k > 0$. The positive-realness of (1) has been discussed in [11], [31] as follows.

**Lemma 1.** [11], [31] A real-rational function $Y(s)$ in the form of (1), where $a_0, a_1, d_0, d_1 > 0$, and $k > 0$, is positive-real if and only if $a_0 - a_1 \geq d_0 - d_1 \geq 0$, $a_1 - d_1 \geq 0$, and $a_0 - d_0 \geq 0$.

The resultant of $p(s) := a_1 s^2 + a_1 s + 1$ and $q(s) := d_0 s^2 + d_1 s + 1$ in $s$ is given as

$$a_0 a_1 1 0$$
$$a_0 a_1 1 0$$
$$d_0 d_1 1 0$$
$$0 d_0 d_1 1$$

$$= (a_0 - d_1)^2 - (a_0 d_1 - a_1 d_0)(a_1 - d_1)$$

$$\vdots : R_k. \quad (2)$$

It is well known that there exist common factors between $p(s)$ and $q(s)$ if and only if $R_k = 0$.

For any positive-real admittance $Y(s)$ in the form of (1) with $a_0, a_1, d_0, d_1 \geq 0$, and $k > 0$, [31] shows that it can be generally realized by a five-element network as shown in Fig. 2 by using the Foster Preamble. When $R_k > 0$, the network is shown in Fig. 2(a); when $R_k < 0$, the network is shown in Fig. 2(b). Since the canonical networks for $R_k = 0$ contain at most three elements by the Foster Preamble, which is fairly simple, they are not drawn. The configurations generally contain either two resistors (dampers), two inductors (springs), and one capacitor (inverter), or two resistors and three inductors. In this paper, the following definition is used to describe the complexity.

**Definition 1:** Given a set of indices to describe networks, if a network $N_1$ has at least one of the indices less than that of a network $N_2$ with its other indices no greater, then the network $N_1$ is of strictly lower complexity than the network $N_2$.

**Remark 1:** Since this class of admittances is mainly used in mechanical systems, the following three indices are mainly
concerned: (a) the total number of elements; (b) the number of dampers (resistors); (c) the number of inerters (capacitors). In this paper, the notations for electrical networks are used, but the results are also valid in the mechanical setting.

It is obvious that for its canonical realization the three indices are five elements, two resistors, and one capacitor when \( R_k > 0 \); five elements, two resistors, and zero capacitor when \( R_k < 0 \). This paper focuses on investigating the condition for an admittance \( Y(s) \) in the form of (1) with \( a_0, a_1, d_0, d_1 \geq 0 \), \( k > 0 \), and \( R_k \neq 0 \) to be realized by a network of strictly lower complexity than the canonical network shown in Fig. 2. Since the complexity of realizations for \( R_k = 0 \) is fairly low and it is only a special case, we assume that \( R_k \neq 0 \).

III. CASES WITH ZERO COEFFICIENTS

We deal with the cases with zero coefficients in this section since they are relatively simple.

**Lemma 2:** An admittance \( Y(s) \) in the form (1) with \( a_0, a_1, d_0, d_1 \geq 0 \), and \( k > 0 \) is positive-real with at least one of \( a_0, a_1, d_0, \) and \( d_1 \) being zero if and only if at least one of the following two cases holds:

1. \( d_0 = 0 \) and \( a_1 - d_1 \geq 0 \);
2. \( a_1 = 0, d_1 = 0, \) and \( a_0 - d_0 \geq 0 \).

**Proof:** Refer to [12] for the detailed proof.

**Theorem 1:** Consider a positive-real admittance \( Y(s) \) in the form of (1) with \( a_0, a_1, d_0, d_1 \geq 0 \), \( k > 0 \), and \( R_k \neq 0 \). If at least one of the four coefficients is zero, then it can be realized with at most four elements. Furthermore, the realization is of strictly lower complexity than the canonical realization.

**Proof:** Refer to [12] for the detailed proof.

Hence, to further discuss the realizability condition, we assume that \( a_0, a_1, d_0, d_1, k > 0 \) in Section IV.

IV. MAIN RESULTS

Section IV-A presents several basic notions and lemmas, which will be used in the following discussion. Section IV-B discusses the networks with at most four elements. Section IV-C presents the discussion on resistor-inductor (RL) five-element networks. Section IV-D discusses resistor-inductor-capacitor (RLC) five-element networks. Section IV-E presents the final realizability condition by summarizing the above discussion.

A. Basic Notions and Lemmas

Consider any one-port network \( N \) consisting of at most three kinds of elements, which are resistors, inductors, and capacitors, with \( n_e \) elements and \( n_v \) voltage nodes. If each element corresponds to an edge of a graph and each voltage node corresponds to a vertex of a graph, then a graph of \( N \) is formulated with \( n_e \) edges and \( n_v \) vertices, which is called the network graph of \( N \). Denote the two terminals of \( N \) as \( a \) and \( a' \). \( \mathcal{P}(a, a') \) is the path [29] whose end vertices are \( a \) and \( a' \). \( \mathcal{C}(a, a') \) is the cut-set [29] which separates \( G_e \) into two connected subgraphs \( G_1 \) and \( G_2 \) containing \( a \) and \( a' \), respectively. The basic notions and conclusions of the graph theory are referred to [29].

**Definition 2:** For a given network \( N \), if \( N_1 \) is a network whose network graph is dual with \( N \), where its corresponding elements are of the same kind but have the inverse values, then \( N_1 \) is the frequency-inverse dual (FID) of \( N \).

**From the above definition, it can be verified that the admittances of these two networks satisfy** \( Y_1(s) = Y^{-1}(s^{-1}) \). Consider a network \( N \) whose admittance \( Y(s) \) is in the form of (1) with \( a_0, a_1, d_0, d_1, k > 0 \). Denote the class of this kind of admittances as \( T \). If \( N_1 \) is the frequency-inverse dual of \( N \), then the admittance of \( N_1 \) can be written as

\[
Y_1(s) = Y^{-1}(s^{-1}) = \frac{a_0}{a_0k} \cdot \frac{s^2/d_0 + d_1s/d_0 + 1}{s(s^2/a_0 + a_1s/d_0 + 1)} - k' a_0' s^2 + a_1' s + 1 \frac{s(d_0' s^2 + d_1' s + 1)}{
\]

which also belongs to \( T \). Therefore, the condition for \( Y_1(s) \) to be realized by \( N_1 \) can be obtained from that of \( Y(s) \) by \( N \) through the following transformation: \( a_0 \rightarrow 1/d_0', a_1 \rightarrow d_1'/d_0', d_0 \rightarrow 1/a_0', d_1 \rightarrow a_1'/a_0', \) and \( k \rightarrow a_0' k'/d_0' \).

Lemmas 3–7 describe a series of properties on the realizations of (1) as well as an equivalence property (Lemma 6).

**Lemma 3:** If a positive-real admittance \( Y(s) \) in the form of (1) with \( a_0, a_1, d_0, d_1, k > 0 \) can be realized as a lossless network, then it can be written as \( Y(s) = k/s \).

**Proof:** Refer to [12] for the detailed proof.

**Lemma 4:** If any positive-real admittance \( Y(s) \) in the form of (1) with \( a_0, a_1, d_0, d_1, k > 0 \) can be realized by a network \( N \) with two terminals \( a \) and \( a' \), then there must be a path \( \mathcal{P}(a, a') \) and a cut-set \( \mathcal{C}(a, a') \) consisting of only inductors.

**Proof:** Refer to [12] for the detailed proof.

**Lemma 5:** A positive-real admittance \( Y(s) \) in the form of (1) with \( a_0, a_1, d_0, d_1, k > 0 \) is not realizable by either of the two kinds of networks shown in Fig. 3, whose network graphs are planar.

**Proof:** Refer to [12] for the detailed proof.

**Lemma 6:** [22], [25] The two networks shown in Fig. 4 must be equivalent with each other with \( a' = a(a + b)/b, b' = a + b, c' = c(a + b)/b^2 \), where \( Z_1(s) \) and \( Z_2(s) \) are any positive-real impedances and \( a, b, c \) are positive constants.

**Proof:** Refer to [12] for the detailed proof.

B. Realizations With at Most Four Elements

In this subsection, the realizability condition is derived for any positive-real admittance \( Y(s) \) in the form of (1) whose coefficients are all positive to be realized with at most four elements. Furthermore, it will be shown that the realizations can...
Fig. 4. Two equivalent networks discussed in Lemma 6.

Fig. 5. The networks corresponding to the case when $R_k = 0$.

Fig. 6. The four-element networks discussed in Lemma 9.

be of strictly lower complexity than that of the canonical network shown in Fig. 2.

Lemma 8: A positive-real admittance $Y(s)$ in the form of (1) with $a_0, a_1, d_0, d_1, k > 0$ can be realized with at most three elements whose values are positive and finite if and only if $R_k = 0$. Moreover, the networks shown in Fig. 5 can cover the condition.

Proof: Refer to [12] for the detailed proof.

Lemma 9: If a positive-real admittance $Y(s)$ in the form of (1) with $a_0, a_1, d_0, d_1, k > 0$ can be realized as shown in Fig. 6, then $R_k = 0$.

Proof: This lemma can be proved by the equivalence discussed in Lemma 6 and the condition in Lemma 8.

Theorem 2: A positive-real admittance $Y(s)$ in the form of (1) with $a_0, a_1, d_0, d_1, k > 0$ and $R_k \neq \emptyset$ can be realized with at most four elements whose values are positive and finite if and only if $Y(s)$ can be realized as the admittance of at least one of the networks shown in Fig. 7.

Proof: Refer to [12] for the detailed proof.

Theorem 3: An admittance $Y(s)$ in the form of (1) with $a_0, a_1, d_0, d_1, k > 0$ and $R_k \neq \emptyset$ can be realized by the network shown in Fig. 7(a) with all the elements positive and finite if and only if $a_0 - d_0 > 0$ and $a_1 - d_1 = 0$. Moreover, if the condition is satisfied, then the values of elements are expressed as $R_1 = a_1(a_0 - d_0)/(ka_0^2), L_1 = d_0/(ka_0), L_2 = (a_0 - d_0)/(ka_0^2)$, and $C_1 = ka_0^2/(a_0 - d_0)$.

Proof: Refer to [12] for the detailed proof.

Corollary 1: If a positive-real admittance $Y(s)$ in the form of (1) with $a_0, a_1, d_0, d_1, k > 0$ and $R_k \neq 0$ can be realized with at most four elements, then the configurations realized can be of strictly lower complexity than its canonical realization shown in Fig. 2.

Proof: By Theorem 4, it is seen that $Y(s)$ must satisfy Case 2, that is, $R_k > 0$ and $(a_0 d_1 - a_1 d_0)(a_1 - d_1) = 0$. Since the networks shown in Fig. 7 can cover the condition of Case 2, which are all of lower complexity than that in Fig. 2(a), this corollary is proved.

C. Realizations of RL Five-Element Networks

In the previous subsection, the realization problem of the networks consisting of at most four elements, which are of strictly lower complexity than the canonical realization shown in Fig. 2, has been discussed. In order to solve the problem formulated in this paper, it suffices to focus on the irreducible five-element networks as the networks with more elements cannot satisfy the requirement.

By Lemma 3, it is obvious that admittance $Y(s)$ in the form of (1) with $a_0, a_1, d_0, d_1, k > 0$ and $R_k \neq 0$ cannot be realized by lossless networks. To discuss the realizations consisting of two kinds of elements, it suffices to investigate resistor-inductor
(spring-damper) networks as inductors must be contained according to Lemma 4. Considering the strictly lower complexity, the number of resistors cannot exceed two. Hence, only the networks containing one resistor and an arbitrary number of inductors or the ones containing two resistors and an arbitrary number of inductors are possible. It has been discussed in [31] that the admittance of the former kind of networks must satisfy \( R_k = 0 \). This means that it is eliminated. The next theorem gives the realizability condition for the latter case.

**Theorem 5:** A positive-real function \( Y(s) \) in the form of (1) with \( a_0, a_1, d_1, d_1, k > 0 \) and \( R_k \neq 0 \) can be realized as the admittance of a network \( N \) with two resistors and an arbitrary number of inductors if and only if \( R_k < 0 \). Moreover, if the condition is satisfied, then \( Y(s) \) is realizable by a network with two resistors and only three inductors as shown in Fig. 8 with values \( L_1 = 1/k, L_2 = B(B - D)/(k(A - B)(B - C)), L_3 = D(B - D)/(k(A - B)(C - D)), R_1 = (B - D)/(k(A - B)(B - C)), \) and \( R_2 = (B - D)/(k(A - D)(C - D)) \), where \( A, C = (a_1 \pm \sqrt{a_1^2 - 4a_0})/2 \) and \( B, D = (d_1 \pm \sqrt{d_1^2 - 4d_0})/2 \).

**Proof:** Refer to [12] for the detailed proof.

**Corollary 2:** If a positive-real admittance \( Y(s) \) in the form of (1), where \( a_0, a_1, d_0, d_1, k > 0 \) and \( R_k \neq 0 \) is realizable by resistor-inductor (RL) networks, then the RL realizations cannot be of strictly lower complexity than the canonical network shown in Fig. 2.

**Proof:** As discussed above, it is only possible for the RL network to contain one or two resistors, and the coefficients satisfy \( R_k = 0 \) if the number of resistors is one. Now, it remains to discuss the network consisting of two resistors and an arbitrary number of inductors. According to Theorem 5, it is known that \( R_k < 0 \), which does not satisfy the condition of Theorem 4. This means that the number of inductors must be at least three. Besides, the canonical realization must be the one in Fig. 2(b). Hence this corollary is proved.

### D. Realizations of RLC Five-Element Networks

For the irreducible five-element RLC networks that are of strictly lower complexity than the canonical network shown in Fig. 2, they must consist of one resistor, one capacitor, and three inductors. This subsection discusses the realizability problem of such networks.

**Lemma 10:** A positive-real function \( Y(s) \) in the form of (1) where \( a_0, a_1, d_0, d_1, k > 0 \) and \( R_k \neq 0 \) cannot be realized as the admittance of the series-parallel networks shown in Fig. 9 with the values of elements being positive and finite.

**Proof:** Refer to [12] for the detailed proof.

**Proof:** Fig. 10 shows one half of all the possible network graphs, and other graphs are dual with them. Since the condition of Theorem 4 does not hold, the possible networks are irreducible, which means that there do not exist two elements of the same kind in series or in parallel. Therefore, the graphs in Fig. 10(a) and (e) are first eliminated. Network graphs in Fig. 10(b), (c), (f), (g), and (h) are eliminated as there must exist \( \mathcal{P}(a, a') \) consisting of only inductors by Lemma 4, which leads to at least two inductors in series for these graphs. Since \( \mathcal{C}(a, a') \) consisting of only inductors must also be required, it can be derived that Edge 1, Edge 2, and Edge 4 (Edge 5) must correspond to inductors for Fig. 10(l). Furthermore, Edge 3 and Edge 5 (Edge 4) must correspond to a capacitor and a resistor, otherwise it must be of the structure shown in Fig. 3(b). However, by the equivalence discussed in Lemma 6, the only possible network is still equivalent to the one shown in Fig. 3(b). Hence, the graph in Fig. 10(l) is eliminated. For Fig. 10(d), the only possible network is shown in Fig. 11, which can always be equivalent to a four-element network based on Lemma 6. For Fig. 10(i), Lemma 4 leads to Edge 1, Edge 2 (or Edge 3), and Edge 4 (or Edge 5) corresponding to inductors, and Edge 3 (or Edge 2) and Edge 5 (or Edge 4) must correspond to the resistor and capacitor. Therefore, the network must be of the structure shown in Fig. 3(a), which cannot realize \( Y(s) \) by Lemma 5. The only possible networks for Fig. 10(j) are shown in Fig. 9(a) and (c), and their frequency-inverse dual networks are shown in Fig. 9(b) and (d). It can be checked that all the possible networks whose graphs are shown in Fig. 10(k) are equivalent to one of the networks in Fig. 9 or four-element networks based on Lemma 6. It has been discussed in Lemma 10 that the networks in Fig. 9 cannot realize \( Y(s) \) in the form of (1) where \( a_0, a_1, d_0, d_1, k > 0 \) and \( R_k \neq 0 \). Therefore, there is no series-parallel network satisfying the requirement.

Finally, the bridge networks are considered, and the following equations are defined: \( W_1 := \alpha_1\alpha_2 - \alpha_3, W_2 := \alpha_2\beta_1 - \beta_3, \)
Fig. 10. One half of the network graphs for five-element one-port networks.

Fig. 11. The network whose network graph is shown in Fig. 10(d).

\[ W_3 := \alpha_1 \beta_1 - \beta_2, \quad \text{and} \quad W := 2\alpha_1 \alpha_2 \beta_1 + \beta_4 - \alpha_1 \beta_3 - \alpha_2 \beta_2 - \alpha_3 \beta_1. \]

**Lemma 12:** A positive-real function \( Y(s) \) can be realized as the admittance of the network shown in Fig. 12 with the values of the elements being positive and finite if and only if \( Y(s) \) can be expressed as

\[ Y(s) = \frac{\alpha_3 s^3 + \alpha_2 s^2 + \alpha_1 s + 1}{\beta_4 s^4 + \beta_3 s^3 + \beta_2 s^2 + \beta_1 s}, \]

where \( \alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3, \beta_4 > 0, W_1, W_2, W_3, W - 2\alpha_2 W_2 > 0, \) and the following two equations hold:

\[ W^2 - 4W_1 W_2 W_3 = 0, \]
\[ \beta_4 + \alpha_1 \beta_2 + \alpha_3 \beta_1 - \alpha_2 \beta_2 = 0. \]

Moreover, if the conditions hold, then the component values of the network can be expressed as follows:

\[ R_1 = \frac{\alpha_1 \alpha_2 - \alpha_3 \beta_2^2}{\alpha_1^2 (\alpha_2 \beta_1 - \beta_2)}, \]
\[ L_1 = \frac{\alpha_1 \alpha_2 \beta_1 - \alpha_3 \beta_3}{\alpha_1 (\alpha_2 \beta_1 - \beta_3)}, \]
\[ L_2 = \frac{\beta_3^2}{\alpha_1 (\alpha_2 \beta_1 - \beta_3)}. \]

**Proof:** Refer to [12] for the detailed proof.

**Lemma 13:** A positive-real function \( Y(s) \) in the form of (1) with \( a_0, a_1, d_0, d_1, k > 0 \) and \( R_k \neq 0 \) can be realized as the admittance of the network shown in Fig. 12 with the values of the elements being positive and finite if and only if

\[ (a_0 d_1 - a_2 d_0) (a_1 - d_1) - d_0^2 = 0. \]

Moreover, the values of the elements can be expressed as

\[ R_1 = \frac{a_1 (T^2 + a_1 T + a_0)}{k (a_1 + T)^2 ((a_1 - d_1) T + (a_0 - d_0))}, \]
\[ L_1 = \frac{(a_1 - d_1) T^2 + (a_2 - a_3) d_1 - d_0) T + a_3 (a_2 - d_0)}{k (a_1 + T) ((a_1 - d_1) T + (a_0 - d_0))}, \]
\[ L_2 = \frac{a_0 T}{d_1 T + d_2}, \]
\[ L_3 = \frac{k ((a_1 - d_1) T + (a_0 - d_0))}{d_3 T + d_4}, \]
\[ C_1 = k ((a_1 - d_1) T + (a_0 - d_0)), \]

where

\[ T = \sqrt{\frac{a_0 d_1 - a_2 d_0}{a_1 - d_1}}. \]

**Proof:** Refer to [12] for the detailed proof.

**Lemma 14:** A positive-real function \( Y(s) \) in the form of (1) with \( a_0, a_1, d_0, d_1, k > 0 \) and \( R_k \neq 0 \) cannot be realized as the admittance of the networks whose elements are positive and finite as shown in Fig. 13.

**Proof:** Refer to [12] for the detailed proof.

**Theorem 6:** Consider any positive-real function \( Y(s) \) in the form of (1) where \( a_0, a_1, d_0, d_1, k > 0, R_k \neq 0, \) and the condition of Theorem 4 does not hold. Then it can be realized as the admittance of networks with one resistor, one capacitor, and
three inductors, and the values of them are positive and finite, if and only if it is the admittance of the network in Fig. 12, that is, the coefficients satisfy 
\[ (a_0d_1 - a_1d_0)(a_1 - d_1) = d_0^2. \]

**Proof:** Sufficiency. It obviously holds by Lemma 13.

Necessity. Since the condition of Theorem 4 does not hold, then it cannot be satisfied by series-parallel networks with one resistor, one capacitor, and three inductors, and the values of them are positive and finite by Lemma 11. Based on Lemma 4, the only possible bridge networks are shown in Figs. 12 and 13. Furthermore, by Lemma 13 and Lemma 14, only the network in Fig. 12 can realize \( Y(s) \) and the coefficients must satisfy 
\[ (a_0d_1 - a_1d_0)(a_1 - d_1) = d_0^2. \]

\[ \blacksquare \]

**E. The Final Condition**

**Theorem 7:** A positive-real function \( Y(s) \) in the form of (1), where \( a_0, a_1, d_0, d_1, k > 0 \) and \( R_k \neq 0 \), can be realized as the admittance of a network which is of strictly lower complexity than the canonical network shown in Fig. 2 if and only if it is the admittance of the network in Fig. 12, that is, the coefficients satisfy 
\[ (a_0d_1 - a_1d_0)(a_1 - d_1) = d_0^2. \]

**Proof:** Sufficiency. It can be shown by Theorem 4, Corollary 1, and Theorem 6.

Necessity. If \( Y(s) \) is realizable with at most four elements, then by Theorem 4 and Corollary 1, one has \( R_k > 0 \) and \( (a_0d_1 - a_1d_0)(a_1 - d_1) = 0 \). If \( Y(s) \) is realizable by the irreducible five-element networks consisting of only two kinds of elements, then it could only be RL networks. Then by Corollary 2, the networks cannot be of strictly lower complexity than the canonical networks shown in Fig. 2. For five-element RLC networks that are of strictly lower complexity than the canonical network, they could only contain one resistor, one capacitor, and three inductors. By Theorem 6, one has \( R_k > 0 \) and \( (a_0d_1 - a_1d_0)(a_1 - d_1) = d_0^2 \). Combine all the discussion above, the necessity part is proved.

In order to show the realizability conditions of \( Y(s) \), the following canonical form is introduced:

\[ Y_c(s) = \frac{M s^2 + 2 U \sqrt{M} s + 1}{s((1/M)s^2 + (2V/\sqrt{M})s + 1)}, \tag{18} \]

through \( Y_c(s) = \alpha Y(\beta s) \), where \( \alpha = 1/(k \sqrt{a_0d_0}) \) and \( \beta = 1/\sqrt{a_0d_0} \). It is seen that \( M = \sqrt{a_0d_0}, U = a_1/(2\sqrt{a_0}), \) and \( V = d_1/(2\sqrt{d_0}) \). By Lemma 1, it can be verified that \( Y_c(s) \) is positive-real if and only if \( M \geq 1, \sigma_c := VM - U \geq 0, \) and \( \sigma_c := VM - U \geq 0 \). Besides, it is verified that the resultant of the numerator and denominator of \( Y_c(s) \) is \( R_k = 4V^2 - 4U(M^2 + 1)V/\sqrt{M} + (M^2 - 1)^2 + 4M^2(2U)^2/2M^2 \).

Furthermore, it is given that \( Y_c(s) \) satisfies the condition of Theorem 7 if and only if \( \sigma_c = 0, \sigma_c = 0, \lambda_c = -4V^2 + 4U(M^2 + 1)V/\sqrt{M} - (M^2 - 1)^2/2M^2 = 0, \) and \( R_k > 0 \). The realizability condition of Theorem 7 is shown in Fig. 14. It can be seen that when \( M = 1.6 \) and \( U/V < 2 \), the condition is satisfied if and only if \( \sigma_c = 0, \sigma_c = 0, \) and \( \lambda_c = 0 \) (Fig. 14(a)); when \( M = 1.32 \) and \( U/V < 2, \) the condition is satisfied if and only if \( \sigma_c = 0 \) and \( \sigma_c = 0, \) since the curve \( \lambda_c = 0 \) is located outside the region \( R_k > 0 \) (Fig. 14(b)).

**V. Numerical Examples**

For the admittance \( Y(s) \) in the form of (1), if \( a_0 = 3, d_0 = 1, a_1 = 2, d_1 = 1, \) and \( k = 1 \), then it can be verified that \( Y(s) \) is positive-real and \( R_k = 3 > 0 \). By the Foster Preamble, \( Y(s) \) can be realized as the canonical network shown Fig. 2(a) with \( L_1 = 1H, L_2 = 1/2H, R_1 = 1/4\Omega, R_2 = 3/4\Omega, \) and \( C_1 = 8/3\Omega \). Furthermore, \( Y(s) \) satisfies the condition of Theorem 7 because of \( R_k > 0 \) and \( (a_0d_1 - a_1d_0)(a_1 - d_1) = d_0^2 \).

By Lemma 13, \( Y(s) \) can be realized as in Fig. 12 whose values are given as \( R_k = 49/81\Omega, L_1 = 7/9H, L_2 = 2/9H, L_3 = 14/3H, \) and \( C_1 = 3F \) by (12)–(17), which is of strictly lower complexity than its canonical network.

If \( a_0 = 3, d_0 = 2, a_1 = 7, d_1 = 6, \) and \( k = 1 \), then it can be verified that \( Y(s) \) is positive-real and \( R_k = -3 < 0 \), indicating that it does not satisfy the condition of Theorem 7 even though \( (a_0d_1 - a_1d_0)(a_1 - d_1) = d_0^2 \). By the Foster Preamble, \( Y(s) \) is realizable by the canonical network shown in Fig. 2(b) with \( L_1 = 1H, L_2 = 2H, L_3 = 16/3H, R_1 = 4\Omega, \) and \( R_2 = 4/3\Omega \). By Lemma 13, \( Y(s) \) can be realized as in Fig. 12 whose
values can be calculated as $R_1 = 49/81 \, \Omega$, $L_1 = 7/9 \, H$, $L_2 = 2/9 \, H$, $I_1 = 14/3 \, H$, and $C_1 = 3F$, which is not of lower complexity than the canonical network.

VI. CONCLUSION

This paper has studied the realization problem for a special class of admittances, which is widely used in suspension systems. By the method of Foster Preamble, this class of admittances is realizable by the configuration named the canonical realization. The concept of being of strictly lower complexity was defined, and a necessary and sufficient condition for this class of admittance to be realized by the network which is of strictly lower complexity than the canonical realization was given. By enumerating the possible network graphs and making use of the necessity conditions, the corresponding networks have been presented, which were classified by the concept of frequency-inverse dual. Deriving the realizability conditions of these corresponding networks, the final condition was summarized in a very concise form. Finally, the $U$-$V$ plane and numerical examples were given to illustrate the conclusions.

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