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</thead>
<tbody>
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Moving Horizon Estimation for Networked Systems With Quantized Measurements and Packet Dropouts

Andong Liu, Li Yu, Member, IEEE, Wen-An Zhang, and Michael Z. Q. Chen, Member, IEEE

Abstract—This paper is concerned with the moving horizon estimation (MHE) problem for linear discrete-time systems with limited communication, including quantized measurements and packet dropouts. The measured output is quantized by a logarithmic quantizer and the packet dropout phenomena is modeled by a binary switching random sequence. The main purpose of this paper is to design an estimator such that, for all possible quantized errors and packet dropouts, the state estimation error sequence is convergent. By choosing a stochastic cost function, the optimal estimator is obtained by solving a regularized least-squares problem with uncertain parameters. The proposed method can be used to deal with the estimation and prediction problems for systems with quantized errors and packet dropouts in a unified framework. The stability properties of the optimal estimator are also studied. The obtained stability condition implicitly establishes a relation between the upper bound of the estimation error and two parameters, namely, the quantization density and the packet dropout probability. Moreover, the maximum quantization density and the maximum packet dropout probability are given to ensure the convergence of the upper bound of the estimation error sequence. Finally, an illustrative example is given to demonstrate the effectiveness of the proposed method.

Index Terms—Moving horizon estimation (MHE), networked systems, packet dropout, quantization.

I. INTRODUCTION

THE PAST FEW decades have witnessed an ever increasing research interest in networked systems due to increasing applications of networks in engineering systems (see e.g., [1]–[3] and the references therein). However, perfect communication is not always possible in many engineering systems, especially in a networked environment, which may lead to undesirable packet dropout. The packet dropout is caused by a variety of reasons, for example, a failure in the measurement, network congestion, etc. In practice, the only available information of the system is obtained through a set of noisy measurements that seldom include all of the state variables. Thus, the unmeasured states need to be inferred from these measurements.

The estimation problem for networked systems with packet dropouts has been extensively considered by many researchers during the last few years [3]–[12]. A Markovian jump process [4]–[6] and a binary switching sequence [7]–[12] are commonly used to model the packet dropout phenomena. In [4], the packet dropout phenomenon was modeled as a Markovian jump linear system and the estimator guaranteeing expected estimation error covariance was designed based on jump Riccati equations. The stability issue related to the Kalman filtering for networked systems with bounded Markovian dropouts was investigated in [5], where a sufficient condition for peak covariance stability is obtained in terms of system dynamics and the probability transition matrix of the Markov chain. The $H_{\infty}$ filtering problem was studied in [6] for networked system with random packet dropout, and the general multiple-input-multiple-output (MIMO) filtering system was considered. Another approach is to model the packet dropout as a Bernoulli distributed white sequence taking values of 0 and 1. In [7], the optimal $H_{\infty}$ filtering was considered for a class of networked systems with multiple packet dropouts, where random dropout rates were transformed into stochastic parameters in the system’s representation. In [8], the robust information fusion Kalman filtering problem was considered for multi-sensor systems with random delays and missing measurements. In [9], the optimal linear estimators including filter, predictor and smoother were developed via a Riccati difference equation approach, and the sufficient condition for the convergence of the optimal linear estimator was given. In [10], the robust $H_{\infty}$ filtering problem was studied for stochastic uncertain systems with missing measurements, and the filtering error system was exponentially mean-square stable. The finite-horizon robust $H_{\infty}$ filtering problem was considered in [11] for discrete-time stochastic systems with packet dropouts, and a time-varying filter was obtained. Recently, the moving horizon state estimation problem was investigated in [12] for networked systems with packet dropouts, and a sufficient condition for the convergence of the estimation error sequence was presented.

In a network environment, signals are usually quantized before being communicated, and then the quantization errors occur frequently due to rounding or truncation. Therefore, it is necessary to conduct analysis on the quantizers and understand how much effect the quantization makes on the estimation performance. In [13], both static and dynamic quantization schemes were studied for linear discrete-time dynamic systems by using the method of sector bound uncertainties, and the tradeoff between performance degradation due to quantization and quantization density or number of quantization levels was given. The quantized $H_{\infty}$ filtering problems for quantized measurements were considered in [14], where the quantizer...
is dynamic and composed of a dynamic scaling and a static quantizer. In [15], a quantized fault detection filter was designed and the quantizer was considered as a static logarithmic type. Summarizing the above discussion, it can be seen that the packet dropouts and quantization errors are two important factors for networked systems. Then these facts should be taken into account in order to achieve the required performance. In [16], the problem of robust \( H_{\infty} \) estimation was studied for uncertain systems with signal transmission delay, measurement quantization and packet dropout, and a parameter-dependent filter was designed. In [17], a quantized \( H_{\infty} \) filter with the minimized static quantizer ranges was designed to guarantee that the error system is exponentially mean-square stable with random sensor packet dropouts. The quantized \( H_{\infty} \) filtering problem was investigated in [18] for Markovian jump LPV systems with packet dropouts. It should be noted that the works mentioned above (e.g., [4]–[11], [13]–[18]) have not been concerned with the bounds or constraints on states or noises. However, it is well known that these bounds allow us to add more information about the physical characteristics of the states, such as concentrations or molecular weights that must be positive [19], [20]. Therefore, the bounds serve as another important factor that should be taken into account when designing the estimator for networked systems with quantized measurements and packet dropouts.

Moving horizon estimation (MHE) is based on the idea of finding a state estimate by using a moving, limited information [21]–[23]. The choice of a moving horizon approach is justified by the possibility of explicitly considering bounds or constraints of the system in the synthesis of the estimator [24], [25]. The constrained MHE problem was studied in [25], [26] to deal with inequality constraints of states and disturbances. In [27]–[29], the bounded states and disturbances were considered for MHE, and a weighted penalty term related to the prediction of the state was presented. In [30]–[32], a new distributed estimation algorithm was investigated based on the concept of MHE for linear and nonlinear constrained systems. Because MHE avoids the computational burden of a full information estimator by considering only a window of data, stability issue on the performance of MHE arises [33]. Stability properties of MHE for constrained linear and nonlinear systems have been investigated in [26]–[29], [34]–[36]. In [27]–[29], the existence of bounded sequence on the estimation error was proved. Rao et al. [34] showed that if the full information estimator is stable, then the MHE is also stable.

Summarizing the above discussion, in this paper, we are motivated to study the moving horizon estimation problem for a class of networked systems with quantized measurement, packet dropout and bounded noise. The measured output is quantized by a logarithmic quantizer and the packet dropout and bounded noise. The measured output is quantized before they are transmitted to the estimator. The packet dropouts and external noises occur in the channel. The measurement output with quantization and packet dropout is given by

\[
\begin{align*}
    z_k &= Cx_k \\
    y_k &= \gamma_k q(z_k) + n_k
\end{align*}
\]

where \( z_k \in \mathbb{R}^n \) is the system state, \( w_k \in \mathbb{W} \subseteq \mathbb{R}^n \) is the bounded noise input, \( A \) is a constant matrix with appropriate dimensions.

The quantized estimator consists of three parts: a quantizer, a digital communication channel and an estimator, as shown in Fig. 1. The measured signals are quantized before they are transmitted to the estimator. The packet dropouts and external noises occur in the channel. The measurement output with quantization and packet dropout is given by

\[
\begin{align*}
    x_{k+1} &= Ax_k + w_k \\
    y_k &= \gamma_k q(z_k) + n_k
\end{align*}
\]

where \( x_k \in \mathbb{X} \subseteq \mathbb{R}^n \) is the system state, \( w_k \in \mathbb{W} \subseteq \mathbb{R}^n \) is the bounded external noise, \( C \) is a constant matrix with appropriate dimensions. \( \gamma_k \) is an independent and identically distributed Bernoulli white sequence taking values of 0 and 1 with \( P(\gamma_k = 1) = \bar{\gamma}, P(\gamma_k = 0) = \gamma \). The stochastic variable \( \gamma_k \) models the packet loss process in the network channel. If a packet is successfully delivered, then \( \gamma_k = 1 \); otherwise, \( \gamma_k = 0 \). \( \bar{\gamma} \) denotes the packet received probability. \( q(\bullet) \) is a quantizer and is assumed to be of the logarithmic type [37], [38]. The set of quantized levels is described by

\[
\begin{align*}
    u = \{ \pm u_n, \quad u_i = \rho^i u_0, \quad i = 0, 1, \pm 2, \ldots \} \cup \{ 0 \}, \\
    u_0 > 0
\end{align*}
\]

where the parameter \( \rho \) is the quantization density, and the logarithmic quantizer \( q(\bullet) \) is defined as

\[
q(v) = \begin{cases} 
    u_i, & \text{if } v_i < v \leq \bar{u}_i, \\
    0, & \text{if } v = 0, \\
    -q(\bar{v}), & \text{if } v < 0
\end{cases}
\]

where \( u_i = (1/1 + \delta)u_i, \bar{u}_i = (1/1 - \delta)u_i, \delta = (1 - \rho/1 + \rho) \).
By the results in [38], the quantization effects can be transformed into sector bounded uncertainties
\[ q(z_k) - z_k = \Delta_k z_k, \quad |\Delta_k| \leq \delta \]  
(5)

By using (1), (2) and (5), one obtains the following system model
\[ x_{k+1} = Ax_k + w_k \]
\[ y_k = f(\gamma_k, x_k, \Delta_k) + v_k \]  
(6)

where \( f(\gamma_k, x_k, \Delta_k) = \gamma_k(\Delta_k + 1)C x_k \).

The following assumptions are needed in the derivation of the main results:

**Assumption 1:** The statistics of \( w_k \) and \( v_k \) are unknown, and the sets \( \mathcal{W} \) and \( \mathcal{V} \) are compact.

**Assumption 2:** The pair \((A, C)\) is completely observable in \( N \) steps.

**Assumption 3:** System (1) is quadratically stable, that is, there exists a positive matrix \( P \) such that
\[ A^T P A - P < 0. \]

Note that Assumption 3 ensures
\[ \text{tr}(PP^{-1}) = \text{tr}(P) \leq 2. \]

**Remark 1:** In most of the existing results, such as those in [4]–[11], [13]–[18], it is assumed that the statistics of the noises are known. However, it is difficult to know the exact statistics of the noises in practice. Moreover, there may exist constraints in the noises. Hence, we consider a unified framework that comprises quantized measurements, random packet dropouts and bounded unknown noises.

With the above notations and assumptions, we shall use the moving horizon strategy described in [28] to design an estimator for networked systems with packet dropouts. Specifically, at any stage \( k = N, N + 1, \ldots \), the objective is to find estimates of the state vectors \( x_{k-N}, \ldots, x_k \) based on the past measurement data and of a prediction state \( \hat{x}_k \) of the state \( x_k \). Let \( \hat{x}_{k-N}, \ldots, \hat{x}_k \) be the estimates of \( x_{k-N}, \ldots, x_k \), respectively. Due to packet dropouts, a natural criterion to derive the estimator is a stochastic least squares approach. Consider the following stochastic cost function as given
\[ J_k = E \left\{ \| \hat{x}_{k-N} - x_{k-N} \|^2_Q + \sum_{i=k-N}^k ||y_i - f(\gamma_i, \hat{x}_i, \Delta_i)||^2 \left\| \mathcal{M}_k \right\| \right\} \]  
(7)

where the first term, weighted by the matrix \( Q \), expresses our belief in the prediction \( \hat{x}_{k-N} \) as compared with the observation model. The matrix \( Q \) is assumed to be positive-definite and can be regarded as a design parameter. \( N \) is the moving horizon size. \( \mathcal{M}_k \) is the \( \sigma \)-algebra generated by \{ \( (\gamma_l, \hat{x}_l) \), \( k - N \leq l \leq k \) \}. It can be seen from (7) that, at stage \( k \), the cost \( J_k \) is a function of \( \hat{x}_{k-N}, \gamma_k, \Delta_k \), and \( \mathcal{M}_k \). As for the uncertainties in the system matrices, we shall follow a minmax approach. Then, at any stage \( k = N, N + 1, \ldots \), the following MHE optimization problem has to be solved.

**Problem 1:** Given a pair \( \{ \bar{x}_k, y_k^b \} \), a quantization density \( \rho \) and a packet received probability \( \bar{q}_i \), find the optimal estimate \( \hat{x}_{k-N} \) such that
\[ \hat{x}_{k-N}^* = \min_{\hat{x}_{k-N}} \max_{\Delta_k} J_k \left( \hat{x}_{k-N}, \gamma_k, \Delta_k \right) \]  
(8)

and the following constraints are satisfied
\[ \hat{x}_{i+1} = A \hat{x}_i, \quad i = k - N, \ldots, k - 1 \]  
(9)

Moreover, the optimal prediction is determined as
\[ \hat{x}_{k-N}^* = A \hat{x}_{k-N-1}^*, \quad k = N + 1, N + 2, \ldots \]  
(10)

**Remark 2:** The regularized robust least-squares MHE was first presented in [28] for discrete-time linear systems. However, signals are usually quantized before being communicated in network environments, and then the quantization errors occur frequently due to rounding or truncation. On the other hand, communication networks are usually unreliable, and may lead to packet dropout resulting from network traffic congestions and limited network bandwidth, which may significantly degrade the system performance. Then, the detrimental effects caused by quantization error and packet dropout should be taken into account in the estimator design for networked systems. Here, we use the aforementioned moving horizon strategy to estimate the states of networked systems with quantized measurements and packet dropouts. For the stage \( k \leq N \), the MHE optimization problem is determined by the batch state estimation formulation [34].

Before ending this section, let us recall the following lemma which will be used in next section.

**Lemma 1:** [39] Consider a regularized robust least-squares problem of the form
\[ \min_{z} \max_{\|s\| \leq \delta} \left\{ \|z - \hat{Q} + ||(B + \Delta B)z - (D + \Delta D)||^2 \right\} \]  
(11)

where \( \Delta B = H S E_{k}, \Delta D = H S E_{d} \), \( H, E_{k} \) and \( E_{d} \) are known matrices, \( S \) denotes an arbitrary contraction. Then problem (11) has a unique global minimum \( z^* \) given by
\[ z^* = (\hat{Q} + B^T \hat{R} B)^{-1} \left( B^T \hat{R} D + \lambda^o E_{k}^T E_{d} \right) \]  
(12)

where \( \hat{Q} = Q + \lambda^o E_{k}^T E_{k}, \hat{R} = R + RH(\lambda^o I - H^T R H)^+ H^T R \). The notation \( X^+ \) denotes the pseudo-inverse of \( X \), and the scalar parameter \( \lambda^o \) is determined as
\[ \lambda^o = \arg_{\lambda > \|H^T R H\|} \min_{\|s\| \leq \delta} \left\{ \|z(\lambda)||^2_Q + \lambda \| E_{k} z(\lambda) - E_{d} \|^2 + \| B z(\lambda) - D \|^2 \right\} \]  
(13)
where
\[ z(\lambda) = \left(\hat{Q}(\lambda) + B^T \hat{R}(\lambda) B \right)^{-1} \left( B^T \hat{R}(\lambda) D + \lambda^2 E_k^T E_d \right), \]
\[ \hat{Q}(\lambda) = Q + \lambda E_k^T E_b, \]
\[ \hat{R}(\lambda) = R + RH(\lambda I - H^T RH)^+ H^T R. \]

III. MOVING HORIZON ESTIMATOR

In this section, a method will be presented to solve the moving horizon state estimation problem and design the moving horizon estimator. The following notations will be useful in the remaining part of the paper.

Let \( G_k = \Gamma_k S_k + \Gamma_k, \) then, the observation vectors \( y_k \) can be rewritten as
\[ y_k = G_k F_k x_k + N_g + G_k H_k W_{k-1} + y_{k-1} \]  
and the cost (7) can be rewritten as follows
\[
J_k(x_k, \Gamma_k, S_k) = E\{ \left| y_k - G_k F_k x_k \right|^2 \} + \sum_{i=1}^{N} \left| \Delta_k \right| \left| S_k \right| \}
\]

Since \( |\Delta_k| \leq \delta, \) it can be seen that \( |S_k|^2 \leq \delta^2 \) and \( |S_k| \leq 1. \) On the other hand, \( |\Gamma_k|^2 \leq 1 \) since \( \gamma_k \) is a binary switching random variable taking values of 0 and 1. Then, given the quantization density \( \rho \) and packet received probability \( \gamma, \) the following inequality holds
\[ \left| \delta^{-1} \Gamma_k S_k \right| \leq 1 \]  

Denoting \( F_k = \delta F_k, S_k = \delta^{-1} \Gamma_k S_k, \) (15) can be rewritten as
\[
J_k(x_k, \Gamma_k, S_k) = E\{ \left| \delta^{-1} \Gamma_k S_k \right|^2 \}\]

Then, an alternative version of Problem 1 can be formulated in a compact way as follows.

Problem 2: Given a pair \( \{x_k, y_k\}, \) a quantization density \( \rho \) and a packet received probability \( \gamma, \) find the optimal estimate \( x_k \) such that
\[ x_k = \min_{x_k} \max_{S_k} J_k(x_k, \Gamma_k, S_k) \]

and the constraint (9) is satisfied. The optimal prediction \( \bar{x}_k \) is determined by (10).

Define
\[
\bar{x}_k = x_k - x_k - N \cdot B_k = \Gamma_k F_k N \;
\Delta B_k = \tilde{S}_k \tilde{F}_N, \quad D_k = y_{k-1} - \Gamma_k F_k x_k - N \;
\Delta D_k = \tilde{S}_k F_k \bar{x}_k - N \cdot E_k = \tilde{F}_N \;
E_d = - \tilde{F}_N \bar{x}_k - N \cdot H = I.
\]

Then, we are ready to state the following theorem.

**Theorem 1:** Given a pair \( \{x_k, y_k\}, \) a quantization density \( \rho \) and a packet received probability \( \gamma, \) the solution to Problem 2 is given by
\[
\bar{x}_k = \left( Q + \tilde{R} \right)^{-1} \left( \tilde{Q} F_N^{T} \tilde{F}_N + \lambda \right) \tilde{x}_k \]

where \( \tilde{Q} = Q + \tilde{E}, \tilde{E} = \lambda^2 \delta^2 F_N^{T} F_N, \tilde{R} = \tilde{Q} F_N^{T} \tilde{R}_k F_N, \tilde{R} = I + (\lambda^2 I - I)^+ \), and the scalar parameter \( \lambda^2 \) is the unique solution to the following one-dimensional optimization problem
\[
\lambda^2 = \arg \min_{\lambda \geq 1} \left\{ \| z_k - \lambda \|_2 + \lambda\delta \| F_N \bar{x}_k - N \|_2 + M(\lambda) \right\}
\]

with
\[
M(\lambda) = \| z_k - \lambda \|_2 \| F_N \bar{x}_k - N \|_2 + \| y_k - N \|_2
\]

\[ x_k = x_k - \lambda \bar{x}_k \cdot \bar{x}_k \mid \bar{x}_k \| F_N \bar{x}_k - N \|_2 \]

\[ \bar{Q}(\lambda) = \left| \tilde{Q} F_N^{T} \tilde{F}_N \right| \bar{x}_k \]
\[ \bar{E}(\lambda) = \lambda^2 \delta^2 F_N^{T} F_N, \]
\[ \tilde{F}_N \bar{x}_k - N \cdot H = I \]

**Proof:** With the above notation, the cost (17) can be rewritten as follows
\[ J_k = \left\{ \| z_k \|_2 \right\}
\[ + \left\| \left( H_N + H \tilde{S}_k \tilde{F}_k \right) S_k - \left( D_k + H \tilde{S}_k E_d \right) \right\|^2 \left| \tilde{M}_k \right| \}
\]

By using Lemma 1, we can obtain the following unique global solution
\[ E\left\{ \left( \tilde{Q} + B_k^T \tilde{R}_k B_k \right) - \left( B_k^T \tilde{R}_k D_k + \lambda \tilde{E}_k^T E_d \right) \right\} = 0 \]

where \( \tilde{Q} = Q + \lambda \delta^2 F_k^T F_k \)

For \( \tilde{Q} \) in (22), we have
\[ E\{ \tilde{Q} \} = Q + \lambda \delta^2 F_k^T F_k \]
\[ = Q \]
Similarly, we have

\[
E \left\{ B_k^T \hat{R} B_k \right\} = E \left\{ F_N^T \Gamma_k^T \Gamma_k F_N \right\} = \gamma F_N^T \hat{R} F_N = \hat{R}
\]

(24)

\[
E \left\{ B_k^T \hat{R} D_k \right\} = E \left\{ F_N^T \Gamma_k^T \hat{R} \left( y_k^b - \Gamma_k F_N \bar{x}_k \right) \right\}
\]

\[
= \gamma F_N^T \hat{R} y_k^b - \gamma F_N^T \hat{R} F_N \bar{x}_k = \gamma F_N^T \hat{R} y_k^b - \hat{R} \bar{x}_k
\]

(25)

\[
E \left\{ \lambda^c \delta^T F_N^T \Gamma_k \right\} = - \lambda^c \delta^T \gamma F_N^T \bar{x}_k
\]

(26)

Then, by (23)–(26), one obtains the optimal solution \( \hat{z}_{k-N} \) as follows

\[
z_{k-N} = (\bar{Q} + \hat{R})^{-1} \left[ \gamma F_N^T \hat{R} y_k^b - (\hat{R} + \bar{E}) \bar{x}_k \right]
\]

(27)

Since the stochastic sequence \( \bar{x}_k \) exists in the matrices \( B_k \) and \( D_k \), the scalar parameter \( \lambda^o \) is determined as follows

\[
\lambda^o = \arg \min \{ 2 \left\| z_{k-N}(\lambda) \right\|_Q^2 + \left\| \Gamma_k z_{k-N}(\lambda) - E_d \right\|^2 + \left\| B_k z_{k-N}(\lambda) - D_k \right\|^2_2 \}
\]

(28)

Then, (28) can be rewritten as follows

\[
\lambda^o = \arg \min \{ 2 \left\| z_{k-N}(\lambda) \right\|_Q^2 + \left\| \Gamma_k z_{k-N}(\lambda) - E_d \right\|^2 + \left\| B_k z_{k-N}(\lambda) - D_k \right\|^2_2 \}
\]

(29)

By using \( z_{k-N} = \hat{z}_{k-N} - \bar{x}_{k-N} \) and \( z_{k-N}(\lambda) = \hat{z}_{k-N}(\lambda) - \bar{x}_{k-N} \), the unique solution \( \hat{z}_{k-N} \) is derived in (19) with the scalar parameter \( \lambda^o \) in (20). Thus, the proof is completed.

Remark 3: The optimal estimator is obtained by using Lemma 1. It can be seen from the optimal estimator in (19) that the estimate \( \hat{z}_{k-N} \) depends on the quantization density and packet dropout probability. Moreover, it is shown that the quantization errors and packet dropouts degrade the estimation performance.

As for the minimization in (20), if one excludes the boundary point \( \lambda = 1 \), as done in [39], one can explicitly solve the pseudo-inverse operation in the definition of \( \hat{R} \) as follows

\[
\hat{R} = \frac{\lambda^o}{\lambda^o - 1} I
\]

(31)

and hence the results of (19) can be rewritten as

\[
\hat{x}_{k-N} = \left( Q + \lambda^o \delta^2 F_N^T F_N + \frac{\bar{\lambda}^o}{\lambda^o - 1} F_N^T F_N \right)^{-1} \left( Q \bar{x}_{k-N} + \frac{\bar{\lambda}^o}{\lambda^o - 1} F_N^T y_k^b \right)
\]

(32)

It can be seen from (19) and (32) that the proposed estimator is nonlinear and time-varying because of the dependence on the scalar parameter \( \lambda^o \), which has to be determined on-line by means of a constrained line search. If this is infeasible, one can obtain a reasonable approximation of the optimal solution by assigning to the scalar parameter \( \lambda^o \) a fixed value \( 1 + \mu \) by following some similar lines as in [28], [39]. The scalar parameter \( \mu \) can be appropriately tuned off-line by means of numerical simulations. This leads to an approximate solution to Problem 2 given by

\[
\hat{x}_{k-N} = \left( Q + (1 + \mu) \delta^2 F_N^T F_N + \frac{1 + \mu}{\mu} \bar{\lambda}^o F_N^T F_N \right)^{-1} \left( Q \bar{x}_{k-N} + \frac{1 + \mu}{\mu} F_N^T y_k^b \right)
\]

(33)

In next section, we shall present stability results for the optimal estimator (32).

IV. STABILITY PROPERTIES

The stability properties for the estimation error of the proposed estimator are presented in this section. For the sake of clarity of exposition, we shall use the following definitions

\[
a = \| A \|, \quad q = \sigma(Q), \quad \bar{q} = \bar{\sigma}(Q),
\]

\[
h = \sigma(F_N^T F_N), \quad h_1 = 1 - \sigma(F_N^T F_N),
\]

\[
h_2 = \| F_N^T H_N \|, \quad h_3 = \| F_N^T \|, \quad w_m = \max |w_k|, \quad v_m = \max \| y_k \|
\]

\[
f^* = (\delta + \sqrt{\bar{q}})^2 h, \quad c_1 = \frac{2 + \sqrt{1 + \bar{q}}}{h} h_1,
\]

\[
c_2 = \frac{h_2 w_m \sqrt{(1 + \bar{q})h + h_3 v_m \sqrt{h + 1}}}{h},
\]

\[
c_3 = \frac{\bar{q}}{q} + f^* w_m, \quad \beta_{\infty} = \frac{w_m}{(1 - a_f^*) \sqrt{\sigma(P)}} c_1 + c_2 + c_3.
\]

Then, we have the following theorem.

Theorem 2: Given a pair \( \{ \bar{x}_{k-N}, y_k^b \} \), a quantization density \( \rho \) and a packet received probability \( \bar{p} \), if Assumptions 1–3 hold, then the norm of the estimation error \( \bar{x}_{k-N} = x_{k-N} - \hat{x}_{k-N} \) for the optimal estimator (32) is bounded by

\[
\| x_{k-N} \| \leq \varepsilon_{k-N}
\]

(34)

where the sequence \( \varepsilon_{k-N} \) is calculated recursively as

\[
\varepsilon_0 = \beta_0
\]

\[
\varepsilon_k = \alpha \varepsilon_{k-1} + \beta_k
\]
\[
\alpha = \frac{\bar{q}_n}{q + f^*}
\]
\[
\beta_0 = \alpha \| x_0 - \bar{x}_0 \| + c_1 \| x_0 \| + c_2
\]
\[
\beta_k = \frac{1}{\sqrt{\sigma(P)}} \left( \frac{a_p}{\| x_0 \| P} + \frac{1-a_p}{1-a_p} \right) \left( \frac{a_p}{\| x_0 \| P} + \frac{1-a_p}{1-a_p} \right) c_1 + c_2 + c_3
\]

Moreover, if matrix \( Q \) is selected such that
\[
\frac{\bar{q}_n}{q + f^*} < 1
\]
Then, the sequence \( \varepsilon_k \) converges exponentially to the steady value \( \varepsilon_\infty = \beta_\infty f/(1 - \alpha) \).

**Proof:** Define
\[
\Phi(\lambda^o) = \bar{Q}(\lambda^o) + \bar{R}(\lambda^o)
\]

Then, by the definition of \( \Phi(\lambda^o) \), the estimation error \( e_{k-N} \) is given by
\[
e_{k-N} - x_{k-N} = x_{k-N} - \bar{x}_{k-N}
\]
\[
\begin{align*}
&= \Phi^{-1}(\lambda^o) \left( Q \bar{x}_{k-N} + \frac{\bar{q}_n}{\lambda^o - 1} F_N^T y_{k-N} \right) \\
&= \Phi^{-1}(\lambda^o) \left( \Phi(\lambda^o) x_{k-N} - \bar{x}_{k-N} \right) \\
&\quad - \left( Q \bar{x}_{k-N} + \frac{\bar{q}_n}{\lambda^o - 1} F_N^T y_{k-N} \right)
\end{align*}
\]

Using (14), one obtains
\[
\begin{align*}
e_{k-N} &= \Phi^{-1}(\lambda^o) \left( \Phi(\lambda^o) x_{k-N} - Q \bar{x}_{k-N} - \frac{\bar{q}_n}{\lambda^o - 1} F_N^T y_{k-N} \right) \\
&= \Phi^{-1}(\lambda^o) \left( Q \bar{x}_{k-N} - \bar{x}_{k-N} \right) \\
&\quad + \left( \lambda^o \delta^2 F_N^T F_N + \frac{\bar{q}_n}{\lambda^o - 1} F_N^T F_N \right) x_{k-N} \\
&\quad - \frac{\bar{q}_n}{\lambda^o - 1} F_N^T y_{k-N}
\end{align*}
\]

Moreover, by applying the operator \( \| \cdot \| \) to the error dynamics and using triangular inequalities, one has
\[
\begin{align*}
\| e_{k-N} \| &\leq \left\| \Phi^{-1}(\lambda^o) Q A \right\| \| e_{k-N-1} \| + \| \Phi^{-1}(\lambda^o) Q w_{k-N-1} \| \\
&\quad + \left\| \lambda^o \delta^2 \Phi^{-1}(\lambda^o) F_N^T F_N \right\| \| x_{k-N} \| \\
&\quad + \left\| \frac{\bar{q}_n}{\lambda^o - 1} \Phi^{-1}(\lambda^o) F_N^T F_N \right\| \| x_{k-N} \| \\
&\quad + \left\| \frac{\bar{q}_n}{\lambda^o - 1} \Phi^{-1}(\lambda^o) F_N^T G_k F_N \right\| \| x_{k-N} \| \\
&\quad + \left\| \frac{\bar{q}_n}{\lambda^o - 1} \Phi^{-1}(\lambda^o) F_N^T G_k H_N w_{k-N} \right\| \\
&\quad + \left\| \frac{\bar{q}_n}{\lambda^o - 1} \Phi^{-1}(\lambda^o) F_N^T v_{k-N} \right\|
\end{align*}
\]
By (46)–(52), we obtain
\[
(53)
\]

\[
(54)
\]

\[
(55)
\]

Remark 4: Theorem 2 guarantees that the optimal estimator (32) is stable by a suitable choice of matrix \(Q\). A crucial issue for the design of the proposed estimator is the choice of the weighting matrix \(Q\) in the cost function (7). The parameter \(\rho\) can be chosen to tune the relative confidence in the prediction \(\bar{\sigma} \cdot k_{N}\).

Remark 5: It can be seen from Theorem 2 that the sequence of the estimation error is decay-rate-dependent, where the exponential decay rate \(\alpha\) depends on the quantization density and packet dropout status with given parameter \(Q\). Thus, condition (37) implicitly establishes a relation between quantization density \(\rho\), packet received probability \(\bar{\sigma}\) and the upper bound of \(\varepsilon_{k_{N}}\).

Remark 6: In Theorem 2, we have provided the stability properties of the optimal estimator (32). In a similar way, we can also obtain the stability properties of the approximate estimator (33) by replacing the scalar parameter \(\lambda^c\) with a fixed value 1 + \(\mu\).

**Corollary 1:** Given a pair \(\{\varepsilon_{k_{N}}, \bar{\sigma}_{k_{N}}\}\), a quantization density \(\rho\) and a packet received probability \(\bar{\sigma}\), if Assumptions 1–3 hold, then the norm of the estimation error \(\varepsilon_{k_{N}} = x_{k_{N}} - \hat{x}_{k_{N}}\) for the approximate estimator (33) is bounded by
\[
(56)
\]

where the sequence \(\varepsilon_{k_{N}}\) is calculated recursively as
\[
(57)
\]

Moreover, if matrix \(Q\) is selected such that
\[
(58)
\]

then, the sequence \(\varepsilon_{k_{N}}\) converges exponentially to the steady value \(\varepsilon_{\infty} = \bar{\sigma} \cdot \varepsilon_{\infty}/(1 - \alpha)\), where
\[
(59)
\]

In order to ensure the convergence of the upper bound of the state estimation error sequence, the following proposition is given to guarantee condition (37).

**Proposition 1:** Given a quantization density \(\rho\) and a packet received probability \(\bar{\sigma}\), condition (37) is satisfied for any value of \(a\) by choosing a suitable matrix \(Q\) as follows:

1. When \(a < 1\), it is always possible to choose \(Q\) such that the condition (37) is fulfilled.
2. When \(a > 1\), condition (37) holds if the region of the plane \(a = \bar{\sigma} \cdot \varepsilon_{\infty}/(1 - \alpha)\), the vertices of the triangle can be obtained by solving these two inequalities. It can be seen from Proposition 1 that the largest region of the plane \(a = \bar{\sigma} \cdot \varepsilon_{\infty}/(1 - \alpha)\), the vertices of the triangle can be obtained by solving these two inequalities. It can be seen from Proposition 1 that the largest region of the plane \(a = \bar{\sigma} \cdot \varepsilon_{\infty}/(1 - \alpha)\), the vertices of the triangle can be obtained by solving these two inequalities. It can be seen from Proposition 1 that the largest region of the plane \(a = \bar{\sigma} \cdot \varepsilon_{\infty}/(1 - \alpha)\), the vertices of the triangle can be obtained by solving these two inequalities.
In Theorem 2, we are concerned with the convergence of the estimation error sequence. Our attention is not focused directly on the estimation error sequence itself, but we first obtain an upper bound of the estimation error, and then analyze the convergence of the upper bound. Of course, the convergence of the upper bound guarantees the convergence of the estimation error sequence. It can be known from Theorem 2 that a larger quantization density leads to a larger $\alpha$ by given $Q$ and $\bar{\alpha}$, or even $\alpha > 1$ which infers that the upper bound of $||e_k||$ is divergent. Thus, it is necessary to determine the maximum quantization density $Q_{\text{max}}$ (or the minimum sector bound $\delta_{\text{min}}$) which ensures the convergence of the upper bound of the estimation error sequence by given $Q$ and $\bar{\alpha}$. On the other hand, it may be difficult to know the exact probability of packet dropout in some situations. Therefore, we focus on how the performance changes under different situations of packet dropout, and what is the maximum packet dropout probability (or the minimum packet received probability $\gamma_{\text{min}}$) that ensures the convergence of the estimation error sequence.

Proposition 2: Given a suitable matrix $Q$, the minimum sector bound $\delta_{\text{min}}$ and the minimum packet received probability $\gamma_{\text{min}}$ are given based on condition (37) as follows:

1) If the packet received probability $\gamma$ is known, and given a suitable matrix $Q$ with $\sqrt{\frac{(q_a - q)}{h} - \sqrt{\gamma}} > 0$, then the minimum sector bound is

$$\delta_{\text{min}} = \sqrt{\frac{(q_a - q)}{h} - \sqrt{\gamma}}$$

(59)

2) If the sector bound $\delta$ is known, and given a suitable matrix $Q$ with $\sqrt{\frac{(q_a - q)}{h} - \delta} > 0$, then the minimum packet received probability is

$$\gamma_{\text{min}} = \left(\sqrt{\frac{(q_a - q)}{h} - \delta}\right)^2$$

(60)

It is known that $\delta + \sqrt{\gamma} > \sqrt{\frac{(q_a - q)}{h}}$ since condition (37) is satisfied. Therefore, it is easy to obtain (59) and (60).

V. AN ILLUSTRATIVE EXAMPLE

In this section, a simulation example is given to illustrate the effectiveness of the proposed MHE method for networked systems with quantized measurements and packet dropouts. Consider the following discrete-time system with quantization and packet dropouts

$$x_{k+1} = \begin{bmatrix} 0 & 0.95 \\ -1 & -1 \end{bmatrix} x_k + w_k$$

$$z_k = \begin{bmatrix} 1 & 0 \end{bmatrix} x_k$$

$$y_k = \gamma_k q(z_k) + v_k$$

where $w_k$ and $v_k$ are the bounded noise sequences with $w_{\text{max}} = 0.02$ and $v_{\text{max}} = 0.02$, respectively. $\gamma_k$ is a Bernoulli distributed random variable with $\text{Prob}\{\gamma_k = 1\} = \bar{\gamma}$, which means that the packet dropout probability is $1 - \bar{\gamma}$.

We choose the moving horizon size $N = 10$. It can be easily obtained that $\alpha = 1.5964$ and $h = 3.0432$. It is assumed that the quantization density and the packet received probability are 0.6 and 0.9, respectively. Then, the matrix $Q$ can be chosen as a diagonal matrix with $q = 3.2$ and $q = 1$ by Proposition 1. It is easy to verify that the exponential decay rate $\alpha = 0.9508$, which ensures convergence of the estimation error sequence. The main objective is to find the optimal estimator $\hat{x}_k^\text{opt}$ as given in (32) and the predictor $\hat{x}_k^\text{opt}$ as given in (10). Then, by applying Theorem 1, the optimal estimator is obtained and the tracking performance is shown in Fig. 2, where the solid curve denotes the true values of the states and the asterisk one denotes the estimated states. It can be seen from Fig. 2 that the designed estimator can effectively estimate the states of the system. The measurement signal and its quantized values are shown in Fig. 3. We can see the significant difference between the measurements and estimator inputs from Fig. 3. Thus, it can be concluded that quantization errors and packet dropouts may degrade the estimation performance.

For comparison, let us consider the performance index given by the root mean square error (RMSE) and the asymptotic root
mean square error (ARMSE) [27], [28]

\[
RMSE(k) = \left( \sum_{l=1}^{L} \left\| r_{k,l} \right\|^2 \right)^{\frac{1}{2}}
\]

\[
ARMSE = \sum_{k=K-S}^{K-1} \frac{1}{S+1} \left( \sum_{l=1}^{L} \left\| r_{k,l} \right\|^2 \right)^{\frac{1}{2}}
\]

where \( L \) is the number of simulation runs, \( K \) is the simulation horizon, \( S \) is an evaluation batch size, and \( \left\| r_{k,l} \right\| \) is the norm of the state estimation error at time step \( k \) in the \( l \)th simulation run. The values of \( L \), \( K \) and \( S \) are 500, 100, 50, respectively. Fig. 4 shows the curves of the RMSEs simulated by 500-time Monte Carlo tests for optimal estimator (32) and predictor (10) with \( \rho = 0.6 \) and \( \bar{\gamma} = 0.9 \). From Fig. 4, it can be seen that the estimator has a better accuracy than the predictor.

In what follows, we will show the relation between the quantization density and the estimation performance. First, we consider the case where there is no packet loss, which means \( \bar{\gamma} = 1 \). In order to ensure the convergence of the estimation error sequence, condition (37) should be satisfied. By choosing a suitable diagonal matrix \( Q \) with \( \bar{q} = 3.2 \) and \( q = 1 \), we can obtain the minimum sector bound \( \delta_{\text{min}} \) by applying Proposition 2, which is \( \delta_{\text{min}} = 0.1619 \). Then, the allowed maximum quantization density is \( \rho_{\text{max}} = 0.7213 \). We choose the different quantization density \( \rho \) as \{0.1, 0.3, 0.5, 0.7\}. Fig. 5 shows the curves of the RMSEs simulated by 500-time Monte Carlo tests for the estimator performance with respect to different value of \( \rho \). It can be seen from Fig. 5 that a larger \( \rho \) (smaller \( \bar{\gamma} \)) leads to a better estimation performance.

Next, we consider the relationship between the packet dropout probability and the estimation performance with a given quantization density \( \rho = 0.6 \). By choosing a suitable diagonal matrix \( Q \) with \( \bar{q} = 3.2 \) and \( q = 1 \), we can obtain the minimum packet received probability \( \bar{\gamma}_{\text{min}} \) by applying Proposition 2, which is \( \bar{\gamma}_{\text{min}} = 0.3157 \). We choose the different packet received probability \( \bar{\gamma} \) as \{0.4, 0.6, 0.8, 1\}. Fig. 6 presents the curves of the RMSEs simulated by 500-times Monte Carlo tests for the estimator performance with respect to different value of \( \bar{\gamma} \). It can be seen from Fig. 6 that a larger \( \bar{\gamma} \) leads to a better estimation performance, indicating that packet dropout degrades the estimation performance, which is as expected for the considered estimation system.

From Figs. 5 and 6, we know that a larger \( \delta \) yields a worse estimation performance and a larger \( \bar{\gamma} \) leads to a better estimation performance. Now, we consider the ARMSE for the optimal estimator with different sector bound and packet dropout probability. Choosing a suitable diagonal matrix \( Q \) with \( \bar{q} = 3.2 \) and \( q = 1 \), we know that the pair of \( (\delta, \bar{\gamma}) \) must be satisfied with \( \delta + \sqrt{\bar{\gamma}} > \sqrt{(\bar{q}a - q)/h} = 1.1619 \) by applying Proposition 2. The relation between \( \delta, \bar{\gamma} \) and the ARMSEs of the estimator is shown in Fig. 7. If \( \delta + \sqrt{\bar{\gamma}} \leq 1.1619 \), then the ARMSE goes to infinity, which indicates that the upper bound of the estimation error diverges.

The stability properties of the optimal estimator have been proposed in Part IV. Then, we will show the relationship of the quantization density, the packet received probability and the upper bound of the estimation error. Choosing a suitable diagonal matrix \( Q \) with \( \bar{q} = 0.5 \) and \( q = 0.5 \), the initial values \( x_0 = [0.1 \ 0.1]^T \) and \( \bar{x}_n = 0 \), the positive matrix
By applying Theorem 2, the convergence of the upper bound of $e_k$ is shown in Fig. 8–Fig. 10. Fig. 8 shows the relationship between the quantization density and the upper bound of $e_k$, Fig. 9 shows the relationship between the value of $\varepsilon_{200}$, $\overline{q}$ and the quantization density, and Fig. 10 shows the relationship between the packet received probability and the upper bound of $e_k$, where $\varepsilon_{200}$ is the value of the upper bound of estimation error at sampling time 200. It can be seen from Fig. 8 that a larger quantization density $\rho$ leads to a smaller upper bound of the estimation error. From Fig. 9, we can see that the value of $\varepsilon_{200}$ is decreased with the increase of $\rho$ by given $\overline{q} = 0.5$ and $\overline{\gamma} = 1$. However, the value of $\varepsilon_{200}$ is increased with the increase of $\rho$ by given $\overline{q} = 2$ and $\overline{\gamma} = 1$. It is known from (37) that a larger $\rho$ leads to a larger $\alpha$ by given $Q$ and $\overline{\gamma}$, or even $\alpha > 1$. e.g., given $\overline{q} = 3.2$, $q = 0.5$ and $\overline{\gamma} = 1$, we can obtain that $\alpha = 1.2$ by calculating (37) if $\rho = 0.8$, which infers that the upper bound of the estimation error sequence is divergent. However, this does not mean that a larger quantization density leads to a worse estimation performance. It can be seen from Fig. 5 that a larger quantization density $\rho$ leads to a better estimation performance. From Fig. 10, we can see that a larger packet received probability $\overline{\gamma}$ leads to a smaller upper bound of the estimation error.

VI. CONCLUSION

In this paper, the moving horizon estimation problem has been addressed for a class of networked systems with quantized measurement, packet dropouts and bounded noise. A logarithmic quantizer has been employed to quantize the measured output and a Bernoulli distributed stochastic variables has been used to model the packet dropout phenomena. The moving horizon estimator has been designed by minimizing a stochastic cost function, which can deal with the estimation and prediction problems in the case of quantized errors and packet dropouts in a unified framework. Moreover, the convergence properties of the estimator have been studied, and the allowed maximum quantization density and maximum packet dropout probability have been given to ensure the convergence of the upper bound of the estimation error sequence. Finally, a numerical example was presented to demonstrate the effectiveness of the proposed method.
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