Skin-Effect Model for Round Wires in PEEC

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Abstract—In this paper we consider the skin- and radiation effects for a thin wire segment. Circuit models are being constructed which are suitable for the modeling of thin wires like the wire bonds. We use the internal-external inductance model for this purpose of including the skin-effect in the partial inductance. These models make it possible to include thin, round conductors in a PEEC model at a much lower cost in compute time. The ultimate goal is to construct a circuit model which can be applied both in the time and frequency domain to signal and power integrity problems.

I. INTRODUCTION

The inclusion of the skin-effect for thin round conductors in the Partial Element Equivalent Circuit (PEEC) technique is an important addition to the approach. Quite a few interesting signal and power integrity electromagnetic compatibility problems can be solved like the package modeling for an integrated circuit with wire bonds. In this paper, we concentrate on the frequency dependent partial self-inductance for a wire segment. It is clear from previous work [1], [2] that the proximity effect can be ignored if the distance between the wires is equal to the wire diameter or larger. This is usually the case for wire bond type structures. Then the partial mutual inductance between the wire segments can be computed with center filaments models. However, the most important aspect is the proper computation of the self-term which we consider in this paper.

Originally, skin-effect model have been of importance for power engineering problems. Hence, interesting models have been available for more than fifty years, e.g. [3]. Recently we considered the skin-effect for planar PEEC geometries e.g., [4]. We found that the widely used internal-external inductance model approximation worked well for relatively thin structures but with reduced accuracy for non-thin geometries. Hence, the model can be applied to the thin wire structures of interest.

Our approach is based on the conventional round conductor model which again has a long history [5], [6]. To come up with an internal-external inductance partial inductance for the model, we need to connect the partial inductance for a shell \( L_{p_{shell}} \) in series to skin-effect model. Our basic skin-effect model is similar to the ones used for transmission lines [7], [8].

II. ROUND WIRE MACROMODEL

Wire-bonds represent a key example where relatively distant, thin wires are a key part of the geometry. The most efficient way to include them in a PEEC model is to treat them as having a circular symmetrical current penetration. This assumption is accurate enough if the wires are more than a diameter apart. Also, as shown in Fig. 1, we represent the wires as partially straight segments. We consider a section of wire shown in Fig 1 where we assume that the length \(|z_2 - z_1| > d\) and the current is assumed to be strictly in the \(\hat{z}\) direction. Of course, the model can be rotated in any arbitrary direction. The formula for a partial inductance is given by [9], [10]

\[
L_{p_{ml}} = \frac{1}{a_m a_l} \int_{v_m} \int_{v_l} \hat{t}_m \cdot \hat{t}_l \frac{d v_m}{R_{s,ml}} d v_l
\]

(1)

where \(\hat{t} = \hat{z}\) are the unit vectors tangential to the currents and where \(R_{s,ml}\) is the distance metric between the source and observation points. For a round bar, the partial inductance can be approximated as [5], [6]

\[
L_{p_{11}} = \frac{\mu_0}{2\pi} \ell \left[ \ln \left( \frac{\ell}{a} + \sqrt{\left( \frac{\ell}{a} \right)^2 + 1} \right) - \sqrt{1 + \left( \frac{a}{\ell} \right)^2 + \frac{a}{\ell}} \right]
\]

(2)

where \(a\) is the wire radius and \(\ell\) is the length. We originally assumed that this is an approximation to the low frequency inductance for a cylindrical conductor. However, the value we obtained form it seems to be closer to the value for a tube, at least for the aspect ratios we are considering in this paper.

Another limiting case for the model is given at the high frequency limit. The infinite frequency limit is the partial inductance of a zero thickness cylindrical tube with a radius \(a\). The partial inductance (1) for the cylinder reduces to

\[
L_{p_{11}} = \frac{\mu_0}{4\pi^2} \int_{\phi=0}^{\pi} \int_{z=0}^{\ell} \int_{z'=0}^{\ell} dz dz' d\phi
\]

\[
\frac{d v_m}{\sqrt{4a^2 \sin^2(\phi/2) + (z-z')^2}}
\]

(3)

by recognizing that the symmetry can be used to reduce the four-fold integral to a three fold integral. We were able to analytically solve the integrals for the case of interest where the length \(\ell\) of the wire is longer than the diameter \(d = 2a\). The result for the tube conductor is given by

\[
L_{p_{11}} = \frac{\ell}{4\pi^2} \left[ \left( \frac{k^2}{\Theta_0} + \frac{k^4}{\Theta_2} + \frac{1}{\Theta_0} \right) \pi^3 
+ \left( \frac{1}{18} - \frac{k^2}{24} \right) \pi
+ (-2 \log(\ell) + 6 \log(2) + 2 + 2 \log(a) - 4 \log(k\pi)) \frac{1}{\pi}
+ \frac{8a}{\ell^2} \right]
\]

(4)

Evaluating (4) resulted in 0.54995 nH which is the high frequency limit for \(a = 0.05\) mm and \(\ell = 1\) mm while the approximate formula (2) lead to 0.54775 nH. Hence, the...
approximate formula does not represent a low frequency result for the full cylinder, since its value is supposed to be larger than the high frequency result.

For the skin-effect, we use the external/internal inductance model approximation where the external partial inductance consists of the partial tube inductance. It is important to take the skin-effect of such wires into account in the model without excessive compute time since a model for a package can include a multitude of wires. Also, we prefer circuit models since they apply in both the frequency and time domain without any change.

If we take (6) and use \( J = \sigma E \) to get
\[
\nabla \times J = -\mu \sigma \frac{\partial H}{\partial t}.
\]
where with the current \( J \) in the \( z \)-direction and \( H \) is in the \( \phi \) direction we get
\[
\frac{\partial J_z}{\partial r} = -\mu \sigma \frac{H_\phi(r)}{\partial t} \tag{10}
\]
From this and (8) we finally get a first order differential equation
\[
\frac{\partial J_z}{\partial r} = -\mu \sigma \frac{\partial I_\phi(r)}{\partial t} \tag{11}
\]
The goal of this work is to come up with an efficient circuit model for the internal cylindrical conductor from the outside layers to the inside ones where the vertical current is impeded to go to the inside layers by the inductances. We can transform (11) by multiplying both sides by the section length \( l \) and diving by \( \sigma \) and by approximating the spacial derivative for the first section as
\[
\frac{\Delta z}{\sigma} \frac{J_1 - J_2}{r_1 - r_2} = \frac{-\mu \Delta z}{2\pi r_2} \frac{\partial I_1}{\partial t} \tag{12}
\]
Finally, by multiplying by \( r_1 - r_2 \) and by replacing \( J_1 = (I_p - I_1)/a_1 \) and \( J_2 = (I_1 - I_2)/a_1 \) we get
\[
\frac{\Delta z}{\sigma} \frac{I_p - I_1}{a_1} - \frac{\Delta z}{\sigma} \frac{I_1 - I_2}{a_1} = -\mu \Delta z \frac{(r_1 - r_2)}{2\pi r_2} \frac{\partial I_1}{\partial t} \tag{13}
\]
where \( I_p \) is the current from the external partial inductance \( \Delta z = |z_e - z_s| \). We define
\[
I_2 = \frac{\mu |z_e - z_s|(r_1 - r_2)}{2\pi r_2} \tag{14}
\]
is the differential induction inductance. Also, in the left hand side of the equation we recognize the resistors of the form \( R_1 = |z_e - z_s|/\sigma a_1 \) where the area of the first section is \( a_1 = \pi(r_1^2 - r_2^2) \) etc. Of course, for the first section \( r_1 = a \). The currents \( I_1, I_2, \cdot \cdot \cdot \) are pertaining the appropriate loops. With this we can recognize the loop form of the circuit equation corresponding to (13) to be
\[
R_1 I_p - R_1 I_1 + L_2 \frac{\partial I_1}{\partial t} + R_2 I_1 - R_2 I_2 = 0 \tag{15}
\]
Since all the other loops are of the same form we can construct the equivalent circuit in Fig.6. Note that the partial inductance of the outer zero thickness shell (2) has been added to the model. We emphasize that, since this model is a self-term, it can be applied to any orientation for a cylindrical conductor in the global coordinate system.

B. Analytic cylinder impedance comparison computation

We start out with the comparison model for our solution which is the conventional approach for the round wire skin-effect.

We use a curl operator on both sides of (6) and replace the current \( J = \sigma E \) in the frequency domain we get
\[
\nabla \times \nabla \times J = -j \omega \mu \sigma \nabla \times H \tag{16}
\]
We use the identity \( \nabla \times \nabla \times J = \nabla (\nabla \cdot J) - \nabla^2 J \) to simplify (16) to
\[
\nabla^2 J = \nabla \cdot \nabla J_z \hat{z} = j 2 \pi r H_\phi(r, t) \tag{17}
\]

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since \( J = J_z \hat{z} \). We need to evaluate the left hand side using the form for cylindrical coordinates to get

\[
\nabla \cdot \nabla J_z \hat{z} = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial J_z(r)}{\partial r} \right)
\]

(18)

\[
\nabla \times \mathbf{H} = \hat{z} \frac{1}{r} \frac{\partial}{\partial r} H_\phi = \hat{z} J_z
\]

(19)

The last two equations after differentiation combined yield the differential equation

\[
\frac{\partial^2 J_z(r)}{\partial r^2} + \frac{1}{r} \frac{\partial J_z(r)}{\partial r} - j\omega \mu \sigma J_z(r) = 0
\]

(20)

The solution to this differential equation is well known \[12\], \[13\], \[11\]. Finally, the solution to the differential equation is given by the zeroth order Bessel function of first kind \( J_0 \), or

\[
J_z(r) = \alpha J_0(kr)
\]

(21)

where \( k = \sqrt{-j\omega \mu \sigma} \). We can find \( \alpha \) by matching the current density \( J_z \) at the surface of the at \( r = a \). Hence, \( \alpha = J_z(a)/J_0(ka) \). Finally, the solution is

\[
J_z(r) = \frac{J_z(a)}{J_0(ka)} J_0(kr)
\]

(22)

We need to finally find the internal impedance of the wire cylinder which is given by

\[
Z_i = \frac{E_z(a)|z_e - z_s|}{2\pi a H_\phi(a)}.
\]

(23)

If we use (6) for the z-directed current and the \( \phi \) directed magnetic field for the cylindrical coordinates reduces to

\[
\frac{\partial E_z(r)}{\partial r} = j\omega \mu H_\phi(r)
\]

(24)

Using (21) we can express the electric field as

\[
E_z(r) = \frac{J_z(r)}{\sigma} = \frac{\alpha J_0(kr)}{\sigma}
\]

(25)

at \( r = a \). This yields the numerator for (23). The denominator can be found from (24) by solving for \( H_\phi(r) \) to get

\[
2\pi a H_\phi(a) = \frac{2\pi \alpha \mu \partial J_0(kr)}{j\omega \mu \sigma} \frac{\partial}{\partial r}
\]

(26)

at \( r = a \). If we insert the numerator and denominator results into (23) we obtain

\[
Z_i = \frac{j\omega \mu J_0(ka)(z_e - z_s)}{2\pi a} \frac{\partial J_0(kr)}{\partial r}
\]

(27)

where the derivative denominator is evaluated at \( r = a \).

Using the relationship for the derivative of the zeroth order Bessel function, we have

\[
\frac{\partial J_0(kr)}{\partial r} = -k J_1(kr)
\]

(28)

where \( J_1(kr) \) is the Bessel function of the first kind and first order. If we replace the denominator in (27) with (28) and some algebra we get

\[
Z_i = R_i(\omega) + j\omega L_i(\omega) = \sqrt{\frac{j\omega \mu}{\sigma}} \frac{z_e - z_s}{2\pi a} J_0(ka) J_1(ka)
\]

(29)

which is the differential internal inductance. We want to compare this with the results from the circuit solution in the last Section.

III. INCLUDING HIGH FREQUENCY RETARDATION IN THE MODEL

We encounter two different regimes in PEEC model applications depending on the problem size and the spectrum of the input signal(s). For a quasi-static regime, the size of the cells is limited by the requirement to provide sufficiently fine meshes such that the current flow is represented with sufficient accuracy. Hence, geometries like bends may require rather small cells in some areas.

In contrast, for high frequencies, the additional requirements are that the cell size may be further reduced due to the highest frequency \( f_{\text{max}} \) or the shortest wavelength \( \lambda_{\text{min}} \) involved in the problem. For these cases, we have to limit the largest cell dimension to \( \lambda_{\text{min}}/20 \). In this case, we would like to take the retardation internal to the cells into account since this adds additional damping behavior to the partial element. Conventionally, the external retardation is taken into account \[9\] and other techniques are used in PEEC. For example, for antenna structures \[14\] combines the inductive and capacitive retardation. For very high frequencies the damping behavior and phase of the PEEC model is improved if we also include the retardation even if the length of the cylinder is restricted in the conventional way to \( \lambda/20 \). The partial inductance (1) with the retardation is for the parallel currents in the axial direction

\[
L_{p,\ell} = \frac{1}{\omega a \ell} \frac{\mu}{4\pi} \int \int e^{-j\beta R} \frac{1}{R} \ 0 \ dv_m \ dv_{\ell}
\]

(30)

Since the length of the wire segment is \( \lambda/20 \) or shorter at the largest frequency, we can expand the exponential in a series, or

\[
e^{-j\beta R} = 1 - j\beta R + \frac{\beta^2 R^2}{2} - \cdots
\]

(31)
We can view the third terms in (31) as an error term by using the first two terms in (30) to get
\[ L_{p11}^R = L_{p11} - \frac{j \beta \mu}{4\pi} |z_e - z_s|^2. \]  
(32)

Since the inductive impedance is given by \( j \omega L_{p11} \) and \( \beta = \omega/v \) where \( v \) is the velocity in the material, we have
\[ j \omega L_{p11}^R = j \omega L_{p11} + \frac{\omega^2 \mu}{4\pi v}|z_e - z_s|^2. \]  
(33)

where, \( L_{p11} \) is the quasi-static partial inductance. Clearly, the second part is a frequency dependent local radiation resistance given by
\[ R_{rad}(\omega) = \frac{\omega^2 \mu}{4\pi v}|z_e - z_s|^2. \]  
(34)

The impact of the local radiation resistance is shown in the next Section. Also, a modified resistance model is presented.

A. Mutual inductance coupling

A natural next question is the radiation resistance associated with a mutual inductance coupling term. This issue is particularly important for very closely located coupling elements. For this reason, we consider one of the special cases for rectangular coordinates where the wires are in line. The geometry for this case is shown in Fig. 3. Since for the thin wires, thickness is less of an issue, we consider the case of zero thickness wires so that we can find an analytical solution. Then, the integral for the partial inductance corresponds to the geometry in Fig. 3. We again assume that both wires are less than \( \lambda/20 \) in length for the highest frequency considered.

\[ L_{p12}^M = \frac{\mu e^{-j\beta b_s}}{4\pi} \int_{x_1=0}^{x_1} \int_{x_2=0}^{(x_2-b_s)} e^{-j\beta(x_2-x_1)} \frac{dx_1}{(b_2 + x_2 - x_1)} \]  
(35)

A very interesting issue is the difference between the partial mutual inductance for the case where the delay is taken into account only external to the integral as opposed to the case where the retardation is internal to the cell also. Therefore, this can be viewed as a correction factor for the external delayed partial inductance which is \( e^{-j\beta b_s} L_{p12}^M \). As a next step, we use
\[ L_{p12}^R = e^{-j\beta b_s} L_{p12} - e^{-j\beta b_s}(L_{p12} - L_{p12}^R e^{j\beta b_s}) \]  
(36)

With this, the integral to be solved is
\[ L_{p12}^M = \frac{\mu}{4\pi} \int_{x_1=0}^{x_1} \int_{x_2=0}^{(x_2-b_s)} \frac{1 - e^{-j\beta(x_2-x_1)}}{(b_2 + x_2 - x_1)} dx_1 dx_2 \]  
(37)

where we remove the common retardation factor \( e^{j\beta b_s} \) from all the integrals. We re-iterate that the segment size is such that each cell length, \( e_2 - b_1 \) and \( e_1 \) are less than \( \lambda/20 \). This allows us to evaluate the integrals analytically as
\[ Re(L_{p12}^M) = \frac{\mu e_1 (e_2 - b_2)}{96 \pi b_2^3} \left[ (7 b_2^4 + 10 e_1 b_2^2 - 17 b_2^3 e_2 + 13 b_2^2 e_2^3 + 6 e_1 b_2^3 + 12 e_2 b_2^3 + 3 e_1^3 b_2 - 6 e_1^2 b_2 e_2 - 18 e_1 b_2^2 e_2 - 3 b_2 e_2^3) \beta^2 \right] \]  
(38)

and the imaginary part is
\[ Im(L_{p12}^M) = \frac{\mu e_1 (e_2 - b_2)}{96 \pi b_2^3} \left[ (-12 e_1 b_2^3 + 6 e_2^3 - 26 b_2^3 + 36 e_1 e_2 b_2 - 6 e_1^3 - 26 e_2^3 - 56 e_2 b_2^2 - 20 e_1^2 b_2 + 36 e_1 b_2^3) \beta + (2 e_1 b_2^3 + 3 e_2 b_2^2 - 3 b_2 e_2 + e_1^3 b_2 - b_2^3 e_2 - 3 b_2^2 e_2 - 2 e_1 b_2^2 e_2 + b_2^5 + 2 e_1 b_2^2 e_2 - 4 e_1 b_2 e_2) \beta^3 \right] \]  
(39)

Finally, the result for the entire coupling factor will be
\[ L_{p12}^R = e^{-j\beta b_s}[L_{p12} - Re(L_{p12}^M) - j Im(L_{p12}^M)] \]  
(40)

which results in an additional contribution to the quasi-static partial inductance computation.

Next, we give an example for the additional contribution of the magnitude of \( L_{p12}^M \) in comparison to the quasi-static impedance \( \omega L_{p12} \). In this comparison, we ignore the impact of the external retardation since it represent the same multiplier in both cases. In the example, we chose \( f = 10 \) GHz, where each wire is \( \lambda/20 = 1.499 \) mm and a distance of \( s_2 - e_1 = \lambda = 29.979 \) mm. For this example, the partial quasi-static impedance is \( \omega L_{p12} = 4.48710^{-4} \) k\( \Omega \), and \( Im(L_{p12}^M) = -5.43810^{-4} \) k\( \Omega \) and \( Re(L_{p12}^M) = 5.45610^{-7} \) H. We can conclude that the contribution of the radiation resistance dominates again in comparison of the contribution to the change in inductance with frequency.

IV. NUMERICAL EXPERIMENTS

We know that the analytical formula (29) represents the correct value of the internal impedance. We use this to compare the results obtained with the circuit macromodel in Fig. 6.

Clearly, the accuracy of the circuit macromodel can be traded for speed by choosing the appropriate number of sections in the model. We choose an example, with \( k = 11 \) layers or resistances and where the thickness of the inner layer to layer thickness is increased by a factor \( f_k = 1.7 \). We can chose the first layer to have a thickness of \( \delta_{\text{min}}/2 \) where \( \delta_{\text{min}} \)
is the skin-depth at the highest frequency of interest [15]. We should note that for the internal-external inductance model we use here, the skin-depth for $f \to \infty$ is zero due to the zero thickness external inductance layer. Hence, we can increase the thickness of all internal layers and still get get the correct high frequency value.

Figures 4 and 5 show a comparison for the internal differential inductance using the analytical model (29) and the equivalent circuit model in Fig. 6.

An important aspect is the choice of non-uniform cells towards the surface where the factor $f_k$ will be increased depending on the number of cells. For this reason, we can construct a model with very few circuit elements like a MOR macromodel. It is well known that including the skin-effect is relatively expensive. Hence, the model at hand can be unusually efficient.

A further issue with the internal model is that it is connected in series to the external partial inductance for the conductive shell as is shown in Fig. 6. As is evident from Fig. 6, the external partial inductance dominates such that the internal inductance results in a smaller error in the total inductance.

For high frequencies, also the resistance will increase due to the radiation resistance. The equation (34) for the example used is shown in Fig. 7. It is obvious that the value is very small for low and mid frequencies in comparison to the other impedances. Therefore, it can be omitted for these cases. From the result in Fig. 8 we can clearly identity the frequencies where the radiation resistance is important compared to the skin-effect resistance. For the high frequency model, also the resistance will increase. We give results on the high frequency radiation resistance. The equation (34) for the example used is shown in Fig. 8. It is obvious that the value is very small for low and mid frequencies in comparison to the other admittances. Hence, it can be omitted for these cases. From the result in Fig. IV we show that including a simple resistance $R_{Rad}$ in the equivalent circuit can be used to also include the radiation resistance in the same model! We should note that we did not change the value of the other elements in the equivalent circuit. In Fig. 9 provides a good match with the model without $R_{Rad}$. Further plan show that the same scheme will also improves the passivity of the PEEC model.
V. CONCLUSIONS

The inclusion of thin round conductors with a skin-effect model is important for some SI/PI problems. The partial self inductance model we presented in this paper includes both the skin-effect for the cell as well as the radiation resistance. An important result is that a circuit model has been made for all the different models of the problem. The model also includes a formula for the partial inductance of a thin shell.

For the partial mutual inductance we considered the important case when the cells are in-line. Again, frequency dependent models are given for the case were we approximate the thin wires with filaments. Formulas are given for the contributions due to retardation inside the integrals as well as the external ones.

REFERENCES


