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An Enhanced Mixed-Form Fast Multipole Algorithm Using Rotation Methods

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Abstract—In this paper, the broadband accuracy of the mixed-form fast multipole algorithm (MF-FMA) is improved by applying rotation techniques. Two kinds of rotation methods are employed respectively. One is the coordinate system rotation technique. The other one is the pseudospectral projection based on fast Fourier transform (FFT). Either of them has certain advantages depending on the number of multipoles. Through rotation, translation matrices become very sparse, which enables us to save the storage as well as the CPU time. Hence, we can increase the number of multipoles in the low frequency regime to shift up the transition region of MF-FMA. The overall accuracy is thereby improved significantly.

I. INTRODUCTION

The fast multipole algorithm (FMA) has been developed with the $O(N)$ computational complexity for static problems where circuit physics dominates, and with the $O(N \cdot \log N)$ computational complexity for dynamic problems where wave physics dominates [1] [2]. The low frequency FMA (LF-FMA) and multilevel FMA (MLFMA) manifest two very different forms, i.e. multipole expansions and plane wave expansions, although both of them are marked with the term “FMA”. LF-FMA and MLFMA can operate in their respective regimes efficiently, but are not applicable in the other regime. In 2005, a novel mixed-form fast multipole algorithm (MF-FMA) was proposed to provide a complete solution for wide-band applications [3]. MF-FMA contains transformers between multipoles and plane waves. Hence, it combines both expansions for the Gaunt coefficient in terms of the Wigner $\bar{D}$ function.

II. THEORY

The MF-FMA equation can be expressed as [3]

$$\bar{\alpha}(r_{lj})|_{L \times L'} = \begin{bmatrix} \beta(r_{lj})|_{L \times L'} & \beta(r_{lj})|_{L \times L_1} & \beta(r_{lj})|_{L_1 \times L_2} \\ \end{bmatrix} \cdot \begin{bmatrix} D \end{bmatrix}_{L_2 \times L_1} \cdot \begin{bmatrix} \alpha \beta \gamma \end{bmatrix}_L \cdot \begin{bmatrix} \alpha \beta \gamma \end{bmatrix}_{L_1}$$

(1)

where $\bar{\alpha}$ is the translation matrix from point $i$ to $j$, $\bar{\beta}$ represents vectors of radiation and evaluation patterns, translation matrices of aggregation and disaggregation. $L_i$ refers to the number of multipoles used in LF-FMA at level $i$ and $S_i$ is the sample number of propagating waves over a unit sphere at level $i$ for diagonal translations in MLFMA. $T$ is the translator in MLFMA. The matrix $[\beta]$ is the interpolation matrix and $[D]$ is the transformer from multipoles to plane waves.

Now, let’s consider $\bar{\beta}^T_{L \times L}$ translation matrix. One sample element of $\bar{\beta}^T_{L \times L}$ when the translation direction is along $z$-axis is

$$\bar{\beta}^T_{L \times L}(r \hat{z}) = \begin{cases} \sum_{l', n'} 4\pi i^{l'+l''}Y_{l''}^m(0,0)j_{l''}(kr''), & m' = m \\ 0, & m' \neq m \end{cases}$$

(2)

where $Y_{l,m}$ is the spherical harmonic function, $A_{L,L',L''}$ is the Gaunt coefficient in terms of the Wigner $3 - j$ symbol, $L = (l, m)$, and $L' = (l', m')$. The coordinate system rotation technique used in angular momentum can be directly applied here. It can be denoted as [6]

$$\bar{D}(\alpha, \beta, \gamma) = \bar{D}(0,0, \gamma) \cdot \bar{D}(0, \beta, 0) \cdot \bar{D}(\alpha, 0, 0),$$

(3)

where $\alpha$, $\beta$ and $\gamma$ are known as Euler angles. To point $z$-axis to the translation direction by rotation, we choose $\alpha = \phi,$
\[ \beta = \theta \quad \text{and} \quad \gamma = 0 \quad (\mathbf{r} = (r, \theta, \phi)). \] Hence, we have
\[ \tilde{\beta}_{L' \times L}(\mathbf{r}) = \tilde{D}(\alpha, -\beta, 0) \cdot \tilde{\beta}_{L' \times L}(r \hat{z}) \cdot \tilde{D}(\alpha, \beta, 0). \] (4)

The other method is the pseudospectral projection method [7]. When \( \beta \) translation matrix is multiplied to the radiation vector, it can be written as the summation of spherical harmonic expansions as
\[
\tilde{\beta} = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} M^m_l Y_l^m(\theta, \phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} M^m_l Y_l^m(\theta', \phi'),
\] (5)

where \((\theta, \phi)\) and \((\theta', \phi')\) denote a point in the original and the new coordinate systems respectively with corresponding expansion coefficients, i.e. \(M^m_l\) and \(M^{m'}_l\). Then equispaced sampling nodes in \(\phi'\) component of the new coordinate are used to calculate the matrix-vector products of these sampled points in the old coordinate. After that, FFT is used to calculate the new coefficients \(M^{m'}_l\).

III. NUMERICAL RESULTS

Numerical tests are used to verify whether rotation techniques can be applied to improve MF-FMA. Here, we consider the worst case, in which the source point and the field point are the diagonal vertexes of one “buffer” box. Three levels and rotation methods are employed in both aggregation and disaggregation translations. In Fig.1, the left one shows the error versus the number of multipoles corresponding to different box sizes. The right one shows the error versus box sizes corresponding to different numbers of multipoles. From these two sub-figures, we can find that, the transition point of box sizes can be shifted up to \(10\lambda\) with higher accuracy, while in the original work, the transition point was choose around \(0.2\lambda\) with lower accuracy. The results of two kinds rotation techniques are almost same in this test. When the number of multipoles is not very large, the rotation matrix method is a little faster than the pseudospectral projection method. However, if significantly large number of multipoles is used, the latter has advantages. For MF-FMA, Fig.2 gives the error versus the truncation number of translators in MLFMA when the number of multipoles in LF-FMA is 15. The error limit of each curve is due to the MLFMA translator’s error bound. By selecting proper truncation numbers and using more multipoles, we are able to improve the accuracy of MF-FMA at the transition region.

IV. CONCLUSION

In this paper, the transition region error performance of MF-FMA is improved by using rotation methods that allow more multipoles to be calculated more efficiently. As a result, the transition region between LF-FMA and MLFMA is shifted up in MF-FMA. The error is better controlled and thereby the broadband performance of MF-FMA is enhanced.

REFERENCES