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<tr>
<th><strong>Title</strong></th>
<th>On the Mahler measure of matrix pencils</th>
</tr>
</thead>
<tbody>
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On the Mahler Measure of Matrix Pencils

Graziano Chesi

Abstract—It is well-known that determining the Mahler measure is important in networked control systems. Indeed, this measure allows one to derive stabilizability conditions in such systems. This paper investigates the Mahler measure in networked control systems linearly affected by a single uncertain parameter constrained into an interval, i.e. systems described by a matrix pencil. It is shown that conditions for establishing an upper bound of the largest Mahler measure over the matrix pencil can be formulated through linear matrix inequalities (LMIs). In particular, two LMI conditions are proposed, one based on the construction of a parameter-dependent Lyapunov function, and another based on eigenvalue analysis through the determinants of augmented matrices. The proposed LMI conditions have the advantage to be exact, i.e. they are sufficient for any size of the LMIs and they are also necessary for a certain size of the LMIs which is known a priori.

I. INTRODUCTION

A way to quantify the unstable in discrete-time linear systems exploits the Mahler measure [16], which is the absolute product of the unstable eigenvalues of a matrix, see also [20]. This measure plays a key role in networked control systems. Indeed, an important issue in this area is stabilization with information constraint in the input channel, see e.g. [1], [2], [12], [18]. This information constraint can be modeled in several ways including data-rate constraint [3], [17], quantization [9], and signal-to-noise ratio [4]. As it has been shown in the literature, solutions for this issue can be obtained in terms of the Mahler measure of the system, see e.g. [10], [13].

As it is well-known, the model of a networked control system is very often affected by uncertainty, for instance representing physical quantities that cannot be measured exactly or that are subject to changes. As a consequence, one has to consider a family of admissible models of the networked control system parametrized by the uncertainty. Clearly, the Mahler measure becomes a function of the uncertainty as well, and the target is to determine, among all the admissible models, the worst-case Mahler measure. Such worst-case value is the largest Mahler measure, since the larger this measure is, the more restrictive are the conditions for stabilizability.

A typical way of modeling an uncertain system in the literature consists of introducing a polytopic system, i.e. a system where the uncertainty, generally represented by a vector, is constrained into a polytope, and the coefficients system are linear (or, possibly, nonlinear) functions of the uncertainty. See e.g. [7], [8], [11], [15], [19], [22] and references therein among many contributions. In [6], the computation of the largest Mahler measure of a polytopic system has been considered, providing a linear matrix inequality (LMI) condition based on the construction of a Lyapunov function. Although the sufficiency of this condition is achieved for any degree of this function, the degree required for achieving necessity is unknown.

This paper aims to cope with this problem in the case of networked control systems affected by a single uncertain parameter constrained into an interval, i.e. systems described by a matrix pencil. It is shown that conditions for establishing an upper bound of the largest Mahler measure over the matrix pencil can be formulated through LMIs. In particular, two LMI conditions are proposed, one based on the construction of a parameter-dependent Lyapunov function, and another based on eigenvalue analysis through the determinants of augmented matrices. The proposed LMI conditions have the advantage to be exact, i.e. they are sufficient for any size of the LMIs and they are also necessary for a certain size of the LMIs which is known a priori. Some numerical examples illustrate the proposed results.

The paper is organized as follows. Section II introduces the problem formulation and some preliminaries on bivariate matrix forms. Section III describes the proposed results. Section IV presents some illustrative examples. Lastly, Section V concludes the paper with some final remarks.

II. PRELIMINARIES

A. Problem Formulation

Notation:
- $\mathbb{N}$: space of natural numbers;
- $\mathbb{R}$: space of real numbers;
- $\mathbb{C}$: space of complex numbers;
- $0_n$: $n \times 1$ null vector;
- $I$: identity matrix (of size specified by the context);
- $A'$: transpose of matrix $A$;
- $A > 0$, $A \geq 0$: symmetric positive definite and symmetric positive semidefinite matrix $A$;
- $j$: imaginary unit, i.e. $j = \sqrt{-1}$;
- $\Re(a)$, $\Im(a)$: real and imaginary parts of $a \in \mathbb{C}$, i.e. $a = \Re(a) + j\Im(a)$;
- $|a|$: magnitude of $a \in \mathbb{C}$, i.e. $|a| = \sqrt{\Re(a)^2 + \Im(a)^2}$;
- $a^2$, where $a = (a_1, \ldots, a_n)'$: $(a_1^2, \ldots, a_n^2)'$.

Let us consider the uncertain networked control system described by

$$x(t + 1) = A(p)x(t)$$

(1)
where \( t \in \mathbb{N} \) is the discrete time, \( x(t) \in \mathbb{R}^n \) is the state, and \( p \in \mathbb{R} \) is an uncertain parameter constrained as
\[
p \in [p_{-}, p_{+}]
\]
and \( A : \mathbb{R} \rightarrow \mathbb{R}^{n \times n} \) is an affine linear matrix function expressed as
\[
A(p) = A_{0} + pA_{1}.
\]
The set of matrices defined by
\[
\mathcal{A} = \{ A(p), \ p \in [p_{-}, p_{+}] \}
\]
is the matrix pencil of system (1)–(2).

Let us introduce the Mahler measure. Let \( U \in \mathbb{R}^{n \times n} \). The Mahler measure of \( U \) is defined as
\[
M(U) = \prod_{i=1}^{n} \max\{1, |\lambda_i(U)|\}
\]
where \( \lambda_1(U), \ldots, \lambda_n(U) \in \mathbb{C} \) are the eigenvalues of \( U \).

**Problem.** The problem that we consider in this paper consists of determining the largest Mahler measure of the matrix pencil \( \mathcal{A} \), i.e.
\[
\mu = \sup_{U \in \mathcal{A}} M(U).
\]

**B. Bivariate Matrix Forms**

In this section we introduce a key tool that will be exploited in the sequel to derive the proposed conditions. A function \( V(s) \in \mathbb{R}^{u \times u} \) is a bivariate matrix form if:
1) \( s \in \mathbb{R}^2 \);
2) \( V(s) \) is a polynomial function;
3) there exists \( d \in \mathbb{N} \) such that \( V(\xi s) = \xi^d V(s) \) for all \( \xi \in \mathbb{R} \) and for all \( s \in \mathbb{R}^2 \).

Hence, a bivariate matrix form is a matrix whose entries are forms (i.e., homogeneous polynomials) of the same degree in two scalar variables. Such a degree is the integer \( d \) that verifies the third property above.

Next, let \( V(s) = V(s)^{T} \in \mathbb{R}^{u \times u} \) be a symmetric bivariate matrix form of degree \( 2m \). Let \( b(s, m) \in \mathbb{R}^{\sigma(m)} \) be a vector containing all monomials of degree equal to \( m \) in \( s \), where \( \sigma(m) \) is the number of such monomials given by
\[
\sigma(m) = m + 1.
\]

Then, \( V(s) \) can be written as
\[
V(s) = (b(s, m) \otimes I) (W + L(\alpha))(b(s, m) \otimes I)
\]
where the identity matrix \( I \) has size \( u \times u \), \( W \in \mathbb{R}^{u \sigma(m) \times u \sigma(m)} \) is a matrix satisfying
\[
V(s) = (b(s, m) \otimes I) W (b(s, m) \otimes I),
\]
\[
L(\alpha) = L(\alpha) \in \mathbb{R}^{u \sigma(m) \times u \sigma(m)},
\]
and \( \alpha \in \mathbb{R}^{(m, u)} \) is a vector of free parameters, where
\[
\omega(m, u) = \frac{1}{2} u((m + 1)(u(m + 1) + 1) - (u + 1)(2m + 1)).
\]

The representation (8) is known as square matricial representation (SMR) of matrix forms and extends the Gram matrix method used to represent forms to the matricial case. This representation is useful for establishing whether \( V(s) \) is a sum of squares (SOS) of matrix forms, i.e. if there exist matrix forms \( V_1(s), V_2(s), \ldots \) such that
\[
V(s) = \sum_{i} V_i(s)^{T}V_i(s).
\]

Indeed, \( V(s) \) is SOS if and only if there exists \( \alpha \) satisfying the LMI
\[
W + L(\alpha) - aI \succeq 0.
\]

In particular, one can quantify how SOS is \( V(s) \) by introducing the SOS index
\[
\zeta(V) = \sup_{a > 0} \frac{a}{W + L(\alpha) - aI \succeq 0}
\]
where the identity matrix \( I \) has size \( u\sigma(m) \times u\sigma(m) \). Indeed, \( V(S) \) is SOS if and only if \( \zeta(V) \geq 0 \). See e.g. [5] and references therein for details.

**III. PROPOSED RESULTS**

Let us start by rewriting the system (1)–(2) with the equivalent expression
\[
x(t + 1) = \bar{A}(s)x(t)
\]
where \( s \in \mathbb{R}^2 \) is constrained as
\[
s \in \mathcal{S}
\]
where \( \mathcal{S} \) is the bidimensional simplex
\[
\mathcal{S} = \{ s \in \mathbb{R}^2 : s_1 + s_2 = 1, \ s_1 \geq 0, \ s_2 \geq 0 \}
\]
and \( \bar{A} : \mathbb{R}^{s} \rightarrow \mathbb{R}^{n \times n} \) is a linear bivariate matrix form that can be built as follows. Let us introduce the function
\[
\theta(s) = s_1 p_- + s_2 p_+.
\]
It follows that
\[
p \in [p_{-}, p_{+}] \iff \exists s \in \mathcal{S} : \theta(s) = p.
\]

Hence, \( \bar{A}(s) \) can be defined as
\[
\bar{A}(s) = A(\theta(s)) = s_1 (A_0 + p_- A_1) + s_2 (A_0 + p_+ A_1).
\]

Next, we recall a result that provides an equivalent reformulation of the Mahler measure through the largest eigenvalue of a family of matrices. Specifically, for a matrix \( U \in \mathbb{R}^{n \times n} \) and for \( k = 1, \ldots, n \), let us define
\[
c_k = \frac{n!}{(n-k)!k!}
\]
and let $\Pi_k(U) \in \mathbb{R}^{c_k \times c_k}$ be a matrix whose $(i,j)$-th entry is defined as

$$\Pi_k(U)_{i,j} = \det(Y_k(U, i, j))$$

(22)

where $Y_k(U, i, j) \in \mathbb{R}^{k \times k}$ is the submatrix of $U$ built with the rows indexed by $y(i)$ and the columns indexed by $y(j)$, where $y(l)$ is the $l$-th $k$-tuple built with increasing integers in $[1, n]$. For instance, for $n = 3$ one has

$k = 1: \ y(1) = 1, y(2) = 2, y(3) = 3$

$k = 2: \ y(1) = (1, 2), y(2) = (1, 3), y(3) = (2, 3)$

$k = 3: \ y(1) = (1, 2, 3)$.

**Theorem 1 (6):** Let $U \in \mathbb{R}^{n \times n}$. For $k = 1, \ldots, n$ let us define

$$f_k(U) = \max_{\lambda \in \text{spec}(\Pi_k(U))} |\lambda|.$$  

(23)

Then,

$$M(U) = \max_{k=1,\ldots,n} \max \{1, f_k(U)\}.$$  

(24)

Theorem 1 provides an expression of the Mahler measure of a matrix $U$ based on the spectrum of the matrices $\Pi_1(U), \ldots, \Pi_n(U)$, specifically it states that the Mahler measure of $U$ is the maximum between 1 and the largest absolute eigenvalue of these matrices.

We can exploit Theorem 1 to determine the largest Mahler measure of the matrix pencil $A$, i.e. $\mu$ in (6). Indeed, for $k = 1, \ldots, n$ let us define

$$\tilde{B}_k(s) = \Pi_k(\tilde{A}(s)).$$

(25)

We have that $\tilde{B}_k(s)$ is a bivariate matrix form of degree $k$.

In order to derive the first LMI condition proposed in this paper, let us define the function

$$G_k(s) = w^2(s^1 + s^2)^{2k} F_k(s) - \tilde{B}_k(s)' F_k(s) \tilde{B}_k(s)$$

(26)

where $w \in \mathbb{R}$ and $F_k(s) \in \mathbb{R}^{c_k \times c_k}$ is a bivariate matrix form. The following result provides an exact LMI condition for establishing an upper bound of $\mu$ through the use of parameter-dependent Lyapunov functions.

**Theorem 2:** Consider $w \in [1, \infty)$. Then,

$$\mu \leq w$$

(27)

if and only if, for all $k = 1, \ldots, n$, there exists a bivariate matrix form $F_k(s) \in \mathbb{R}^{c_k \times c_k}$ of degree $m_k$ such that

$$\begin{cases} 
\zeta(F_k^*) > 0 \\
\zeta(G_k^*) > 0
\end{cases}$$

(28)

where

$$F_k^* = F_k(s^2)$$

$$G_k^* = G_k(s^2)$$

(29)

and where $m_k$ satisfies

$$m_k \leq \bar{m}_k$$

(30)

with

$$\bar{m}_k = kc_k(c_k + 1) - 1.$$  

(31)

**Proof.** “$\Rightarrow$” Suppose that there exists a bivariate matrix form $F_k(s) \in \mathbb{R}^{c_k \times c_k}$ of degree $m_k$ such that (28) holds. This means that

$$\begin{cases} 
F_k(s^2) > 0 \\
G_k(s^2) > 0
\end{cases}$$

(29)

From [5] it follows that

$$\begin{cases} 
F_k(s) > 0 \\
G_k(s) > 0
\end{cases}$$

(30)

for all $s$ where $\tilde{x}(t)$ is not uniquely defined. Hence, there exists a Lyapunov function

$$\hat{V}(\tilde{x}(t)) = \hat{x}(t)' F_k(s) \hat{x}(t)$$

proving asymptotical stability of the system

$$\tilde{x}(t + 1) = \tilde{B}_k(s) \tilde{x}(t)$$

for all $s \in \mathcal{S}$, where $\tilde{x}(t) \in \mathbb{R}^{c_k}$. Therefore,

$$w \geq \max_{s \in \mathcal{S}, k=1,\ldots,n} |\lambda| = f_k(\tilde{A}(s))$$

and from (24) one concludes that

$$\mu = \sup_{s \in \mathcal{S}} M(\tilde{A}(s))$$

$$\leq \sup_{s \in \mathcal{S}, k=1,\ldots,n} \max \{1, f_k(\tilde{A}(s))\}$$

$$\leq w$$

since $w \geq 1$.

“$\Leftarrow$” Suppose that (27) holds. Proceeding as in the previous part of the proof one has that

$$\sup_{s \in \mathcal{S}, k=1,\ldots,n} f_k(\tilde{A}(s)) \leq w.$$

From (23) this implies that

$$\frac{\tilde{B}_k(s)}{w} \text{ is Schur } \forall s \in \mathcal{S} \forall k = 1, \ldots, n.$$

Clearly, $\frac{\tilde{B}_k(s)}{w}$ is Schur if and only if the discrete-time Lyapunov equation

$$(s^1 + s^2)^{2k} F_k(s) - \tilde{B}_k(s)' F_k(s) \frac{\tilde{B}_k(s)}{w} = I$$

has a unique solution $\tilde{x}_k(s)$ that satisfies $\tilde{x}_k(s) > 0$ for all $s \in \mathcal{S}$ and for all $k = 1, \ldots, n$, where the identity matrix $I$ has size $c_k \times c_k$. The discrete-time Lyapunov equation above can be rewritten as

$$\hat{A}_k(s) \hat{x}_k(s) = \hat{b}_k(s)$$

where $\hat{x}_k(s)$ is a vector with all the independent entries of $F_k(s)$, whose number is $c_k(c_k + 1)/2$, and $\hat{A}_k(s), \hat{b}_k(s)$ are, respectively, a square matrix and a vector of suitable size. Since the solution for $F_k(s)$ is unique, it follows that $\hat{A}_k(s)$ is nonsingular for all $s \in \mathcal{S}$ and for all $k = 1, \ldots, n$, in particular one can choose $\hat{A}_k(s)$ such that

$$\det(\hat{A}_k(s)) > 0 \forall s \in \mathcal{S} \forall k = 1, \ldots, n.$$

Hence,

$$\tilde{x}_k(s) = \hat{A}_k(s)^{-1} \hat{b}_k(s)$$

(31)
which implies that
\[ F_k(s) = \frac{\hat{F}_k(s)}{\det(\hat{A}_k(s))} \]
where \( \hat{F}_k(s) \) is a bivariate matrix form of degree \( \bar{m}_k \) satisfying
\[ \hat{F}_k(s) > 0 \quad \forall s \in S \quad \forall k = 1, \ldots, n. \]
Now, if we replace \( F_k(s) \) with \( \hat{F}_k(s) \) in (26), one obtains the new matrix \( \hat{G}_k(s) \) given by
\[
\hat{G}_k(s) = w^2(s_1 + s_2)^{2k}\hat{F}_k(s) - B_k(s)\hat{F}_k(s)\hat{B}_k(s) = w^2 \det(\hat{A}_k(s)) I
\]
where the identity matrix \( I \) has size \( c_k \times c_k \). This means that there exists a bivariate matrix form \( F_k(s) = \hat{F}_k(s) \) of degree \( \bar{m}_k \) such that
\[
\begin{aligned}
F_k(s) > 0 & \quad \forall s \in S \quad \forall k = 1, \ldots, n \\
G_k(s) > 0 & \quad \forall s \in S \quad \forall k = 1, \ldots, n
\end{aligned}
\]
or, equivalently from [5],
\[
\begin{aligned}
F_k^*(s) > 0 & \quad \forall s \in \mathbb{R}^2 \quad \forall k = 1, \ldots, n \\
G_k^*(s) > 0 & \quad \forall s \in \mathbb{R}^2 \quad \forall k = 1, \ldots, n
\end{aligned}
\]
By denoting with \( d_F \) and \( d_G \) the degrees of \( F_k^*(s) \) and \( G_k^*(s) \), this implies that there exists \( \varepsilon > 0 \) such that
\[
\begin{aligned}
F_k^*(s) - \varepsilon(s_1 + s_2)^{d_F} & \quad \text{are SOS} \quad \forall k = 1, \ldots, n \\
G_k^*(s) - \varepsilon(s_1 + s_2)^{d_G} & \quad \text{are SOS} \quad \forall k = 1, \ldots, n
\end{aligned}
\]
since a bivariate matrix form is positive semidefinite if and only if it SOS [5]. This means that \( F_k^*(s) \) and \( G_k^*(s) \) admit positive definite SMR matrices, and hence their SOS index is positive, i.e. (28) holds. \( \square \)

Theorem 2 provides a sufficient and necessary LMI condition for establishing whether a given scalar is an upper bound of \( \mu \). Let us observe that this condition consists of checking feasibility of the two LMIs obtained by imposing that the SOS index of \( F_k^*(s) \) and \( G_k^*(s) \) are positive. In particular, such LMIs are given by
\[
\left\{ \begin{array}{cc}
W_F + L(\alpha_F) & > 0 \\
W_G + L(\alpha_G) & > 0
\end{array} \right. \tag{32}
\]
where \( W_F + L(\alpha_F) \) and \( W_G + L(\alpha_G) \) are SMR matrices of \( F_k^*(s) \) and \( G_k^*(s) \) according to Section II-B. Hence, (28) holds if and only if there exist \( \alpha_F \) and \( \alpha_G \) fulfilling (32).

In the LMI condition of Theorem 2, one searches for a bivariate matrix form \( F_k(s) \) of degree \( m_k \) which defines a parameter-dependent Lyapunov function candidate. According to the theorem, such a degree is bounded by \( \bar{m}_k \), i.e. (27) holds if and only if the LMI condition can be satisfied with a bivariate matrix form of degree \( \bar{m}_k \). Clearly, depending on the system, a lower degree might be sufficient to prove (27) by fulfilling the LMI condition.

In order to define the best upper bound of \( \mu \) provided by Theorem 2, let us introduce, for \( k = 1, \ldots, n \), the quantity
\[
w_k^*(m_k) = \inf_{w \in [1, \infty)} w \\
\text{s.t. } E_k(s) \text{ of degree } m_k \tag{33}
\]
such that (28) holds.

The quantity \( w_k^*(m_k) \) can be computed through a line-search on \( w \) where the LMI condition (28) is checked for any fixed \( w \), for instance via a bisection algorithm. We also define
\[
w_k^* = \sup_{m_k \geq 0} w_k^*(m_k). \tag{34}\]
Clearly, since the condition of Theorem 2 is sufficient and necessary for \( m_k = \bar{m}_k \), one has that
\[
w_k^* = w_k^*(\bar{m}_k). \tag{35}\]
The best upper bound of \( \mu \) provided by Theorem 2 for degrees \( m_1, \ldots, m_n \), chosen according to
\[
m_k = m \quad \forall k = 1, \ldots, n \tag{36}\]
for some \( m \in \mathbb{N} \) is hence given by
\[
\phi(m) = \max_{k=1,\ldots,n} w_k^*(m_k). \tag{37}\]
Clearly, from Theorem 2 it follows that the upper bound \( \phi(m) \) is tight for \( m = \bar{m} \) where
\[
\bar{m} = \max_{k=1,\ldots,n} \bar{m}_k, \tag{38}\]
i.e.
\[
\mu = \phi(\bar{m}) = \max_{k=1,\ldots,n} w_k^*. \tag{39}\]

Next, we derive the second LMI condition proposed in this paper for computing \( \mu \). For \( k = 1, \ldots, n \) and \( w \in [1, \infty) \) let us define
\[
h_{k,i}(s) = \det(H_{k,i}(s)) \quad \forall i = 1, 2, 3 \tag{40}\]
where
\[
\begin{align*}
H_{k,1}(s) &= (s_1 + s_2)^{d_1}I - w^{-1}B_k(s) \\
H_{k,2}(s) &= (s_1 + s_2)^{d_2}I - w^{-1}B_k(s) \\
H_{k,3}(s) &= (s_1 + s_2)^{d_3}I - w^{-2}\Omega_2(B_k(s))
\end{align*} \tag{41}\]
and
\[
\begin{cases}
d_1 = d_2 = k \\
d_3 = 2k.
\end{cases} \tag{42}\]
The following result provides the second exact LMI condition proposed in this paper for establishing an upper bound of \( \mu \).

**Theorem 3.** Consider \( w \in [1, \infty) \), and let \( p_0 \) be an arbitrary chosen scalar in \([p_-, p_+]\). Then, (27) holds if and only if
\[
M(A(p_0)) \leq w \tag{43}\]
and
\[
\zeta(h_{k,i}^*)(0) > 0 \quad \forall k = 1, \ldots, n \quad \forall i = 1, 2, 3 \tag{44}\]
where
\[
h_{k,i}^*(s) = h_{k,i}(s^2). \tag{45}\]

**Proof.** “\( \Rightarrow \)” Suppose that (43)–(44) hold. From (44) and the definition of SOS index one gets
\[
h_{k,i}^*(s) > 0 \quad \forall s \in \mathbb{R}^2 \backslash \{0_2\} \quad \forall k = 1, \ldots, n \quad \forall i = 1, 2, 3. \tag{46}\]
From [5] this implies that
\[
h_{k,i}(s) > 0 \quad \forall s \in S \quad \forall k = 1, \ldots, n \quad \forall i = 1, 2, 3. \tag{47}\]
For $i = 1$, this condition implies that
\[ \{1\} \not\in \text{spec} \left( w^{-1} \bar{B}_k(s) \right) \ \forall s \in S \ \forall k = 1, \ldots, n, \]
while, for $i = 2$,
\[ \{-1\} \not\in \text{spec} \left( w^{-1} \bar{B}_k(s) \right) \ \forall s \in S \ \forall k = 1, \ldots, n. \]
For $i = 3$, this condition implies that
\[ \{e^{\pm \omega}\} \not\in \text{spec} \left( w^{-1} \bar{B}_k(s) \right) \ \forall \omega \in (0, \pi) \ \forall s \in S \ \forall k = 1, \ldots, n, \]
since [14]
\[ h_{k,3}(s) = \prod_{t=1,\ldots,c_0-1 \atop r=t+1,\ldots,c_k} \lambda_t(w^{-1} \bar{B}_k(s)) \lambda_r(w^{-1} \bar{B}_k(s)). \]
Consequently, no eigenvalue of $w^{-1} \bar{B}_k(s)$ lies on the unit complex circumference for all $s \in S$ and for all $k = 1, \ldots, n$. Moreover, (43) implies that all the eigenvalues of $w^{-1} \bar{B}_k(s_0)$ strictly lie inside the unit complex circumference for all $k = 1, \ldots, n$, where $s_0 \in S$, since from Theorem 1 one has that
\[ M(A(p_0)) \leq w \]
\[ f_k(A(s_0)) \leq w \ \forall k = 1, \ldots, n \]
\[ |\lambda| \leq w \ \forall \lambda \in \text{spec}(\bar{B}_k(s_0)) \ \forall k = 1, \ldots, n. \]
Hence, it follows that all the eigenvalues of $w^{-1} \bar{B}_k(s)$ strictly lie inside the unit complex circumference for all $s \in S$ and for all $k = 1, \ldots, n$ due to the continuity of the eigenvalues with respect to $s$, and hence (27) holds.

"⇒" Suppose that (27) holds. This means that (43) holds for any $p_0 \in [p_-, p_+]$, and from Theorem 1 that
\[ f_k(A(s)) \leq w \ \forall k = 1, \ldots, n \ \forall s \in S \]
or, equivalently,
\[ |\lambda| \leq w \ \forall \lambda \in \text{spec}(\bar{B}_k(s)) \ \forall k = 1, \ldots, n \ \forall s \in S. \]
Hence, $w^{-1} \bar{B}_k(s)$ is Schur for all $s \in S$ and for all $k = 1, \ldots, n$. Proceeding as in the previous part of this proof, such a condition implies that
\[ h_{k,i}(s) > 0 \ \forall s \in S \ \forall k = 1, \ldots, n \ \forall i = 1, 2, 3 \]
or, equivalently from [5].
\[ h_{k,i}(s) > 0 \ \forall s \in \mathbb{R}^2 \setminus \{0_2\} \ \forall k = 1, \ldots, n \ \forall i = 1, 2, 3. \]
From the proof of Theorem 2, one has that a bivariate form is positive definite if and only if such a form admits a positive definite SMR matrix, i.e. if and only if its SOS index is positive. This means that the SOS index of $h_{k,i}(s)$ is positive for all $k = 1, \ldots, n$ and for all $i = 1, 2, 3$, i.e. (44) holds.

Theorem 3 provides an alternative sufficient and necessary LMI condition for establishing whether a given scalar is an upper bound of $\mu$. This condition does not exploit Lyapunov functions, and consists of checking feasibility of the LMIs obtained by imposing that the SOS index of $h_{k,i}(s)$ is positive for all $k = 1, \ldots, n$ and for all $i = 1, 2, 3$.

From Theorem 3 one can determine the largest Mahler measure of the matrix pencil $A$ as
\[ \mu = \inf_{w \in [1, \infty)} w \]
s.t. (43)–(44) hold.
Let us also observe that one can exploit Theorem 3 to compute the quantity $w_k^*$. Indeed,
\[ w_k^* = \inf_{w \in [1, \infty)} w \]
s.t. \[ \{ (43) \text{ holds} \} \]
and for all $i = 1, 2, 3$.

**IV. ILLUSTRATIVE EXAMPLES**

In this section we present some illustrative examples of the proposed results. The computations have been done in Matlab by using the toolbox SeDuMi [21].

**A. Example 1**

Let us consider the uncertain system
\[ \begin{align*}
\dot{x}(t+1) &= A(p)x(t) \\
A(p) &= A_0 + pA_1 \\
p &\in [-1, 1]
\end{align*} \]
where
\[ A_0 = \begin{pmatrix} 0.5 & 1.2 \\ -1.8 & -2.3 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix} \]
and the problem of determining the largest Mahler measure $\mu$ in (6).

This system can be rewritten as in (15) with
\[ A(s) = s_1A_1 + s_2A_2 \]
and
\[ \bar{A}_1 = \begin{pmatrix} -0.5 & 1.2 \\ -3.8 & -1.3 \end{pmatrix}, \quad \bar{A}_2 = \begin{pmatrix} 1.5 & 1.2 \\ 0.2 & -3.3 \end{pmatrix}. \]
The matrices $\bar{B}_k(s)$ are given by
\[ k = 1 \quad \rightarrow \quad \bar{B}_1(s) = A(s) \]
\[ k = 2 \quad \rightarrow \quad \bar{B}_2(s) = 5.21s_1^2 + 4.02s_1s_2 - 5.19s_2^2. \]

First, let us compute $\mu$ using Theorem 2. For $m = 0$ we find
\[ w_1^*(0) = 3.3472, \quad w_2^*(0) = 5.2100 \]
which implies that $\phi(0) = 5.2100$. This upper bound is tight, and the total number of LMI scalar variables in (28) is 14. Indeed, the tightness of $\phi(0)$ can be shown repeating the computations for $m = \tilde{m}$ according to Theorem 2, which is given by $\tilde{m} = 5$, hence establishing that
\[ \mu = \phi(0) = 5.2100. \]

Another way to show that $\phi(0)$ is tight consists of determining $\mu$ using Theorem 3. In particular, $\mu$ is provided by (46). The bivariate forms $h_{k,i}(s)$ have degree 2. We find
\[ w_1^* = 3.3472, \quad w_2^* = 5.2100 \]
which implies that $\mu = 5.2100$. The total number of LMI scalar variables in (44) is 5.
B. Example 2

Here we consider

\[
\begin{align*}
    x(t+1) &= A(p)x(t) \\
    A(p) &= A_0 + pA_1 \\
    p &\in [0, 1]
\end{align*}
\]

where

\[
A_0 = \begin{pmatrix}
    -1.7 & -1.4 & -0.3 \\
    3.4 & 2.9 & 0 \\
    1.2 & 0 & 0
\end{pmatrix}
\]

\[
A_1 = \begin{pmatrix}
    4.4 & 0.5 & 0 \\
    -2.6 & -4.2 & 0.5 \\
    0 & 0 & -0.5
\end{pmatrix}
\]

and the problem of determining the largest Mahler measure \( \mu \) in (6).

First, let us compute \( \mu \) using Theorem 2. For \( m = 0 \) we find

\[
w_1^*(0) = 3.8013, \quad w_2^*(0) = 3.3130, \quad w_3^*(0) = 1.0443
\]

which implies that \( \phi(0) = 3.8013 \). This upper bound is not tight. In fact, for \( m = 1 \) we find

\[
w_1^*(1) = 2.3206, \quad w_2^*(1) = 3.2104, \quad w_3^*(1) = 1.0443
\]

which implies that \( \phi(1) = 3.2104 \). This upper bound is tight, and the total number of LMI scalar variables in (28) is 191. Indeed, the tightness of \( \phi(0) \) can be shown repeating the computations for \( m = 1 \) according to Theorem 2, hence establishing that

\[
\mu = \phi(1) = 3.2104.
\]

Another way to show that \( \phi(1) \) is tight consists of determining \( \mu \) using Theorem 3. In particular, \( \mu \) is provided by (46). The bivariate forms \( h_{k,i}(s) \) have degree between 3 and 12. We find

\[
w_1^* = 2.3206, \quad w_2^* = 3.2104, \quad w_3^* = 1.0443
\]

which implies that \( \mu = 3.2104 \). The total number of LMI scalar variables in (44) is 76.

V. CONCLUSION

We have investigated the Mahler measure of matrix pencils. It has been shown that conditions for establishing an upper bound of the largest Mahler measure over the matrix pencil can be formulated through LMIs. In particular, two LMI conditions have been proposed, one based on the construction of a parameter-dependent Lyapunov function, and another based on eigenvalue analysis through the determinants of augmented matrices. The proposed LMI conditions have the advantage to be exact, i.e. they are sufficient for any size of the LMIs and they are also necessary for a certain size of the LMIs which is known a priori.

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