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Robust Domain of Attraction: Computing and Controlling Estimates with Non-Polynomial Lyapunov Functions

Graziano Chesi

Abstract—This paper addresses the estimation and control of the robust domain of attraction (RDA) of equilibrium points through rational Lyapunov functions (LFs). Specifically, continuous-time uncertain polynomial systems are considered. The uncertainty is represented by a vector that affects polynomially the system and is constrained in a semialgebraic set. The estimation problem consists of computing the largest estimate of the RDA (LERDA) provided by a given rational LF. The control problem consists of computing a polynomial static output controller of given degree for maximizing such a LERDA. It is shown that lower bounds of the LERDA in the estimation problem, or the maximum achievable LERDA in the control problem, can be obtained by solving either an eigenvalue problem or a generalized eigenvalue problem with smaller dimension. The conservatism of these lower bounds can be reduced by increasing the degree of some multipliers introduced in the construction of the optimization problems. Some numerical examples illustrate the use of the proposed results.

I. INTRODUCTION

Studying the RDA of equilibrium points is a key problem in uncertain nonlinear control systems. In fact, the RDA is the set of initial conditions for which the state of the system asymptotically converges to the equilibrium point under consideration for all admissible uncertainties. Hence, when dealing with uncertain nonlinear control systems, it is not sufficient to establish that the desired equilibrium point is robustly locally asymptotically stable, but one has also to make sure that the initial condition lies inside the RDA of such an equilibrium.

It is well-known that studying the RDA is a nontrivial task. Indeed, for the case of (certain) nonlinear control systems, numerous methods have been proposed for studying the domain of attraction of equilibrium points, in general looking for inner estimates with simple shape, see e.g. [1], [2] where classic methods such as Zubov equation and La Salle theorem are discussed, and recent works such as [3], [4] based e.g. on the computation of reachable sets and logical composition of LFs. A common way of dealing with estimates of the domain of attraction in nonlinear control systems is based on linear matrix inequality (LMI) optimizations and polynomial LFs (possibly composite), see e.g. [5]–[9] and references therein. Some of these techniques have been extended to deal with estimates of the RDA, see e.g. [10]–[13]. Clearly, it would be useful to enlarge the class of LFs that can be used with these methods, and doing this it would be also useful to derive efficient ways of obtaining the estimates of the RDA and to provide conditions for establishing the optimality of these estimates.

This paper provides a contribution in this direction, addressing the problem of estimation and control of the RDA of equilibrium points through LMI-based techniques and rational LFs. Specifically, continuous-time uncertain polynomial systems are considered. The uncertainty is represented by a vector that affects polynomially the system and is constrained in a semialgebraic set. The estimation problem consists of computing the LERDA provided by a given rational LF. The control problem consists of computing a polynomial static output controller of given degree for maximizing such a LERDA. It is shown that lower bounds of the LERDA in the estimation problem, or the maximum achievable LERDA in the control problem, can be obtained by solving either an eigenvalue problem, which is a convex optimization problem with LMIs, or a generalized eigenvalue problem with smaller dimension, which is a quasi-convex optimization problem with LMIs and a special class of bilinear matrix inequalities (BMIs). These optimization problems are mainly obtained by exploiting sums of squares of polynomials (SOS) and by introducing suitable decompositions of the Gram matrices. The conservatism of these lower bounds can be reduced by increasing the degree of some multipliers introduced in the construction of the optimization problems. Some numerical examples illustrate the use of the proposed results. This paper extends to the case of uncertain nonlinear control systems our results in [14] where the system is assumed to be exactly known, i.e. not affected by uncertainty.

The paper is organized as follows. Section II introduces some preliminaries. Section III describes the proposed strategy. Section IV presents some illustrative examples. Lastly, Section V concludes the paper with some final remarks.

II. PRELIMINARIES

In this section we introduce the problem formulation and some preliminaries about positive polynomials.

A. Problem Formulation

The notation adopted throughout the paper is as follows:

- \( \mathbb{R} \): space of real numbers;
- \( 0_n, 0_{n \times m} \): origins of \( \mathbb{R}^n, \mathbb{R}^{n \times m} \);
- \( \mathbb{R}_0^n, \mathbb{R}_0^{n \times m}, \mathbb{R}^n \setminus \{0_n\}, \mathbb{R}^{n \times m} \setminus \{0_{n \times m}\} \);
- \( I_n \): identity matrix \( n \times n \);
- \( A' \): transpose of matrix \( A \);
- \( A > 0 \ (A \geq 0) \): symmetric positive definite (semidefinite) matrix \( A \);
• $a > 0$ ($a \geq 0$): entry-wise positive (nonnegative) vector $a$;
• $A \otimes B$: Kronecker product of matrices $A$ and $B$;
• $\partial p, \partial q$: degree of polynomial $p(x)$, degree of polynomial $q(x,y)$ in $x$;
• s.t.: subject to.

Let us consider the continuous-time uncertain polynomial system

$$\begin{align*}
\dot{x}(t) &= f(x(t), \theta) + G(x(t), \theta)u(t) \\
y(t) &= h(x(t), \theta) \\
x(0) &= x_{init} \\
\theta &\in \Theta
\end{align*}$$

(1)

where $x \in \mathbb{R}^n$ is the state, $x_{init} \in \mathbb{R}^n$ is the initial condition, $u \in \mathbb{R}^{n_u}$ is the input, $y \in \mathbb{R}^{n_y}$ is the output, and $\theta \in \mathbb{R}^{n_\theta}$ is the time-invariant uncertainty. The functions $f(x, \theta)$, $G(x, \theta)$ and $h(x, \theta)$ are polynomial of suitable size. The uncertainty $\theta$ is constrained in the semialgebraic set $\Theta$ defined as

$$\Theta = \{ \theta \in \mathbb{R}^{n_\theta} : a(\theta) \geq 0, \ b(\theta) = 0 \}$$

(2)

where $a(\theta) \in \mathbb{R}^{n_a}$ and $b(\theta) \in \mathbb{R}^{n_b}$ are polynomial functions.

We consider that the origin is the common equilibrium point of interest. In the case of autonomous system (i.e., $u(t) = 0_{n_u}$) this means that

$$f(0_n, \theta) = 0_n \ \forall \theta \in \Theta.$$  

(3)

The RDA of the origin is the set of initial conditions for which the state asymptotically converges to the origin, and it is indicated by

$$\mathcal{R} = \left\{ x_{init} \in \mathbb{R}^n : \lim_{t \to +\infty} x(t) = 0_n \ \forall \theta \in \Theta \right\}.$$  

(4)

In the sequel the dependence on the time $t$ will be omitted for ease of notation.

In this paper we consider the estimation and control of the RDA of the origin via rational LFs, i.e. LFs of the form

$$v(x) = \frac{v_{num}(x)}{v_{den}(x)}.$$  

(5)

where $v_{num}(x)$ and $v_{den}(x)$ are polynomials. Throughout the paper we assume that $v_{num}(x)$ and $v_{den}(x)$ are respectively positive definite and positive, and that $v(x)$ is radially unbounded, i.e.

$$v_{num}(x) > 0 \ \forall x \in \mathbb{R}_0^n \text{ and } v_{num}(0_n) = 0$$

$$v_{den}(x) > 0 \ \forall x \in \mathbb{R}_0^n$$

$$\lim_{\|x\| \to \infty} v(x) = \infty.$$  

(6)

To this end, we introduce the generic sublevel set of $v(x)$ as

$$\mathcal{V}(c) = \{ x \in \mathbb{R}^n : v(x) \leq c \}.$$  

(7)

where $c \in \mathbb{R}$.

The problems considered in this paper are as follows.

• Estimation problem: for the autonomous system (i.e., $u(t) = 0_{n_u}$), to obtain the LERDA provided by the LF $v(x)$, i.e. the set $\mathcal{V}(\gamma)$ where $\gamma$ is the solution of the optimization problem

$$\begin{align*}
\gamma &= \sup_{c} e \\
&\text{s.t. } \mathcal{V}(c) \subseteq \mathcal{R} \ \forall \theta \in \Theta.
\end{align*}$$

(8)

• Control problem: for the controlled system, to design a polynomial static output controller enlarging the RDA, specifically a feedback law of the form

$$u(t) = k(y(t))$$

(9)

where $k(y) \in \mathbb{R}^{n_y}$ is a polynomial function to determine. The controller $k(y)$ has to satisfy the condition

$$f(0_n, \theta) + G(0_n, \theta)k(h(0_n, \theta)) = 0_n \ \forall \theta \in \Theta,$$  

(10)

in order to maintain a common equilibrium point at the origin. Moreover, we express $k(y)$ as

$$k(y) = \begin{pmatrix}
k_{11} + k_{12}y_1 + k_{13}y_2 + \ldots + k_{1n_\theta} + k_{21} + k_{22}y_1 + k_{23}y_2 + \ldots + \\
\vdots
\end{pmatrix}$$

(11)

and we consider that the coefficients of $k(y)$ are possibly constrained within some given bounds, i.e.

$$k_{ij} \in [k_{ij}^-, k_{ij}^+] \ \forall i = 1, \ldots, n_\theta \ \forall j = 1, 2, \ldots$$

(12)

for some $k_{ij}^-, k_{ij}^+ \in \mathbb{R}$. The problem amounts to determining an admissible controller $k(y)$ of chosen degree that maximizes the estimate $\mathcal{V}(c)$, i.e.

$$\begin{align*}
\gamma &= \sup_{c,k} e \\
&\text{s.t. } \mathcal{V}(c) \subseteq \mathcal{R} \ \forall \theta \in \Theta \\
&\text{\text{(10)--(12) hold} } \\
&\partial k = d
\end{align*}$$

(13)

where $d$ is the chosen degree of $k(y)$.

B. SOS Polynomials

Here we briefly review how SOS polynomials (in the general case and in some specific cases) can be investigated through LMIs. See e.g. [15], [13] and references therein for more details.

Let $p(x)$ be a polynomial with $x \in \mathbb{R}^n$. We can express $p(x)$ as

$$p(x) = b_{pol}(x, m)'(P + L(\alpha))b_{pol}(x, m)$$

(14)

where

$$m = \begin{pmatrix}
\partial p \\
\frac{\partial^2 p}{2}
\end{pmatrix},$$

(15)

$b_{pol}(x, m)$ (called power vector) is a vector containing all monomials of degree less than or equal to $m$ in $x$, $P$ is a symmetric matrix, $L(\alpha)$ is any linear parametrization of the set

$$\mathcal{L}_{pol} = \{ L = L' : b_{pol}(x, m)'Lb_{pol}(x, m) = 0 \}.$$  

(16)
and \( \alpha \) is a free vector. This representation is known as Gram matrix method and square matrix representation (SMR). We denote the matrices in (14) as
\[
P = SMR_{pol}(p), \quad P + L(\alpha) = CSMR_{pol}(p).
\]
(17)

As an example, let us consider \( n = 1 \) and \( p(x) = 1 - 3x + 2x^4 \). We can choose
\[
b_{pol}(x, m) = \begin{pmatrix} 1 \\ x \\ x^2 \end{pmatrix}, \quad P + L(\alpha) = \begin{pmatrix} 1 & -1.5 & -\alpha \\ -1.5 & 2\alpha & 0 \\ -\alpha & 0 & 2 \end{pmatrix}.
\]

A polynomial \( p(x) \) is said SOS if there exist polynomials \( p_1(x), p_2(x), \ldots \) such that
\[
p(x) = \sum_i p_i(x)^2.
\]
(18)

By using the expression (14), one can obtain a sufficient and necessary condition for establishing whether \( p(x) \) is SOS through LMIs: \( p(x) \) is SOS if and only if there exists \( \alpha \) such that
\[
P + L(\alpha) \succeq 0.
\]
(19)

Condition (19) is an LMI feasibility test, which amounts to solving a convex optimization problem.

If \( p(x) \) is a locally quadratic polynomial, i.e. a polynomial without monomials of degree zero and one, a more compact power vector can be used in (14), specifically a power vector without the constant monomial. We refer to such a power vector as \( b_{\text{lin}}(x, m) \). Moreover, we denote the corresponding matrices as
\[
P = SMR_{qua}(p), \quad P + L(\alpha) = CSMR_{qua}(p).
\]
(20)

Parameter-dependent polynomials can be similarly expressed by using the SMR. Indeed, if \( p(x, \theta) \) is a polynomial with coefficients depending polynomially on \( \theta \in \mathbb{R}^{n_{\theta}} \), one can write
\[
p(x, \theta) = (b_{pol}(\theta, m_{\theta}) \otimes b_{pol}(x, m))' (P + L(\alpha)) \cdot (b_{pol}(\theta, m_{\theta}) \otimes b_{pol}(x, m))
\]
(21)

where
\[
m_{\theta} = \left[ \frac{\partial \theta p}{2} \right],
\]
(22)

and \( L(\alpha) \) is any linear parametrization of the set
\[
\mathcal{L}_{pol} = \{ L = L^\prime : (b_{pol}(\theta, m_{\theta}) \otimes b_{pol}(x, m))' L \cdot (b_{pol}(\theta, m_{\theta}) \otimes b_{pol}(x, m) = 0 \}.
\]
(23)

We denote the matrices in (21) as
\[
P = SMR_{polpol}(p), \quad P + L(\alpha) = CSMR_{polpol}(p).
\]
(24)

Lastly, if \( p(x, \theta) \) is locally quadratic for all \( \theta \), a more compact representation can be used by replacing \( b_{pol}(\theta, m_{\theta}) \otimes b_{pol}(x, m) \) with \( b_{pol}(\theta, m_{\theta}) \otimes b_{\text{lin}}(x, m) \). We denote the corresponding matrices as
\[
P = SMR_{qua}(p), \quad P + L(\alpha) = CSMR_{qua}(p).
\]
(25)

### III. Proposed Results

First of all, let us express the controller \( k(y) \) as
\[
k(y) = K b_{pol}(y, d)
\]
(26)

where \( K \) is a constant real matrix of suitable size and \( d \) is the degree of \( k(y) \). Let us observe that \( K = 0 \) in the estimation problem, while \( K \) has to satisfy (10)–(12) in the control problem. Hence, we denote the set of admissible matrices \( K \) with
\[
K = \begin{cases} 0 & \text{if “estimation problem”} \\
\{ K : (10)–(12) \text{ hold} \} & \text{if “control problem”}. \end{cases}
\]
(27)

Since (10)–(12) impose linear equations and inequalities on the entries on \( K \), it follows that \( K \) is either a single number or a convex polytope.

Next, let us obtain the closed-loop description of (1) in the presence of the controller \( u = k(y) \) as
\[
\begin{cases} \dot{x} = f(x, \theta) + G(x, \theta) K b_{pol}(h(x, \theta), d) \\
x(0) = x_{\text{init}}. \end{cases}
\]
(28)

#### A. Estimates of Fixed Size

The following result provides a condition for establishing whether a sublevel set is an inner estimate of the RDA of the origin (either in the absence or in the presence of a controller) by testing the positivity of some polynomials.

**Theorem 1:** Let \( v : \mathbb{R}^n \to \mathbb{R} \) be a rational function satisfying (5)–(6) and let \( c \in \mathbb{R} \) be positive. Suppose that there exists \( K \in K \) and polynomial functions \( q(x, \theta) \in \mathbb{R}^2 \), \( r(x, \theta) \in \mathbb{R}^{n_{\theta}} \) and \( s(x, \theta) \in \mathbb{R}^{n_{\theta}} \) such that
\[
\begin{cases} p(x, \theta) > 0 \\
q(x, \theta) > 0 \\
r(x, \theta) \geq 0 \end{cases} \quad \forall x \in \mathbb{R}_c^n \forall \theta \in \mathbb{R}_{n_{\theta}}
\]
(29)

where
\[
p(x, \theta) = -q(x, \theta)' \begin{pmatrix} w(x, \theta) \\
\nu_{\text{den}}(x) - \nu_{\text{num}}(x) \end{pmatrix}
\]
(30)

and
\[
w(x, \theta) = \begin{pmatrix} \nu_{\text{den}}(x) \nabla \nu_{\text{num}}(x) - \nu_{\text{num}}(x) \nabla \nu_{\text{den}}(x) \end{pmatrix}' \cdot (f(x, \theta) + G(x, \theta) K b_{pol}(h(x, \theta), d)).
\]
(31)

Then, \( v(x) \) is a common LF for the origin, and \( \mathcal{V}(c) \subseteq \mathcal{R} \).

**Proof.** Suppose that (29) holds, and let \( x \in \mathcal{V}(c) \setminus \{0\} \) and \( \theta \in \Theta \). Then, from the first inequality it follows that
\[
0 < -q_1(x, \theta) w(x, \theta) - q_2(x, \theta)(\nu_{\text{den}}(x) - \nu_{\text{num}}(x)) - r(x, \theta)' a(\theta) - s(x, \theta)' b(\theta)
\]
\[
\leq -q_1(x, \theta) w(x, \theta)
\]
since \( q_2(x, \theta) > 0 \) from the second inequality in (29), \( r(x, \theta)' a(\theta) \geq 0, s(x, \theta)' b(\theta) = 0 \), and \( v(x) \leq c \). Moreover, since \( q_1(x, \theta) > 0 \) from the second inequality in (29), this implies that
\[
0 > w(x, \theta) = \frac{\dot{v}(x, \theta)}{\nu_{\text{den}}(x)^2}.
\]
Hence, it follows that \( \dot{v}(x, \theta) < 0 \), i.e. \( v(x) \) is a LF for the origin (since it proves local asymptotical stability of this equilibrium point) and \( \mathcal{V}(\cdot) \subseteq \mathcal{R} \). \( \square \)

Theorem 1 provides a condition for establishing whether \( \mathcal{V}(\cdot) \) is included in the RDA, either in the case of uncontrolled system or in the case of controlled one. This condition is based on the introduction of the auxiliary polynomial functions \( q(x, \theta), r(x, \theta) \) and \( s(x, \theta) \), which act as multipliers. Let us observe that this condition does not require a priori knowledge of the fact whether \( v(x) \) is a common LF for the origin: indeed, it is easy to see that (29) cannot be satisfied for any positive \( c \) if \( v(x) \) is not a common LF for the origin.

By requiring that \( p(x, \theta), q_i(x, \theta) \) and \( r_i(x, \theta) \) are SOS polynomials, condition (29) exploits Stengle’s Positivstellensatz and can be checked through an LMI feasibility test. To this end, let us observe that

(29) holds

\[
p(x, \theta), q_2(x, \theta) \) and \( r_i(x, \theta), i = 1, \ldots, n_a, \) are locally quadratic polynomials in \( x \) for all \( \theta \in \mathbb{R}^{n_\theta} \)

since \( w(0, \theta) = 0 \). Hence, let us define

\[
Q_1 = \text{SMR}_{\text{polpol}}(q_1)
\]

\[
Q_2 = \text{SMR}_{\text{quapol}}(q_2)
\]

\[
R_i = \text{SMR}_{\text{quapol}}(r_i) \quad \forall i = 1, \ldots, n_a
\]

\[
\tilde{s}_i = s_i(x, \theta) = s_i'(b_{\text{pol}}(\theta, \theta^b s_i) \otimes b_{\text{lin}}(x, \theta^b s_i))
\]

and from these quantities let us define

\[
P(c, K, Q, R, \tilde{s}, \alpha) = \frac{\text{CSM}R_{\text{quapol}}(p)}{Q} = \text{diag}(Q_1, Q_2)
\]

\[
R = \text{diag}(R_1, \ldots, R_{n_a})
\]

\[
\tilde{s} = (\tilde{s}_1, \ldots, \tilde{s}_{n_a})'
\]

(32)

Theorem 2: Let \( v : \mathbb{R}^n \to \mathbb{R} \) be a rational function satisfying (5)–(6), and let \( c \in \mathbb{R} \) be positive. Define the quantities in (32)–(33). Suppose that there exist \( K \in \mathcal{K}, Q, R, \tilde{s} \) and \( \alpha \) such that the following LMIs hold:

\[
\begin{align*}
P(c, K, Q, R, \tilde{s}) + L(\alpha) &> 0 \\
Q &> 0 \\
R &> 0
\end{align*}
\]

(34)

Then, \( v(x) \) is a common LF for the origin, and \( \mathcal{V}(\cdot) \subseteq \mathcal{R} \).

Proof: Suppose that (34) holds. Let us observe that the first inequality in (34) implies that \( p(x, \theta) > 0 \) for all \( x \in \mathbb{R}^n_0 \) and \( \theta \in \mathbb{R}^{n_\theta} \) since \( b_{\text{lin}}(x, \cdot) \) is nonzero whenever \( x \) is nonzero. Similarly, one has that \( q_i(x, \theta) > 0 \) for all \( x \in \mathbb{R}^n \) and \( \theta \in \mathbb{R}^{n_\theta} \), and \( q_2(x) > 0 \) for all \( x \in \mathbb{R}^n_0 \) and \( \theta \in \mathbb{R}^{n_\theta} \). This implies that (29) holds, and from Theorem 1 we conclude the proof. \( \square \)

Theorem 2 shows how (29) can be converted into an LMI feasibility test. Let us observe that the constraint \( \text{trace}(Q_1) = 1 \) normalizes the variables involved in the test: in fact,

(34) holds for some \( Q, R, \tilde{s} \) and \( \alpha \)

(35) holds for \( qQ, qR, q\tilde{s} \) and \( q\alpha \) for all \( q > 0 \).

Let us consider the selection of the degrees of \( q_i(x, \theta) \), \( r_i(x, \theta) \) and \( s_i(x, \theta) \). A possibility is to choose them in order to maximize the degrees of freedom in (34) for fixed degrees of \( p(x, \theta) \) in \( x \) and \( \theta \). This is equivalent to requiring that the degrees of \( q_1(x, \theta)w(x, \theta), q_2(x, \theta)(v_{\text{den}}(x) - v_{\text{num}}(x)), r_i(x, \theta)a_i(\theta) \) and \( s_i(x, \theta)b_i(\theta) \), rounded to the smallest following even integers, are equal. This can be achieved by choosing (even) degrees for \( q_1(x, \theta) \), and setting the degrees of \( q_2(x, \theta), r_i(x, \theta) \) and \( s_i(x, \theta) \) based on this choice.

B. LERDA

Theorem 2 can be exploited to either estimate or control the LERDA, i.e. to solve problems (8) and (13). Indeed, from Theorem 2 one can define a natural lower bound of \( \gamma \) in the estimation problem or in the control problem as

\[
\hat{\gamma} = \sup_{c, K, Q, R, \tilde{s}, \alpha} c \left( P(c, K, Q, R, \tilde{s}) + L(\alpha) > 0 \right) \quad \text{s.t.} \quad \begin{cases}
P(c, K, Q, R, \tilde{s}) + L(\alpha) > 0 \\
Q > 0 \\
R > 0 \\
K \in \mathcal{K}
\end{cases} \quad \text{trace}(Q_1) = 1.
\]

(35)

Let us observe that the computation of this lower bound is not straightforward because the first constraint in (35) is a BMI (due to the product of \( c \) and \( K \) with \( Q \)), which may lead to nonconvex optimization problems.

A way to cope with this problem is to fix \( Q_2 \), since in such a case the constraints in (35) are LMIs. This provides the lower bound

\[
\hat{\gamma}_1 = \sup_{c, K, Q_1, R, \tilde{s}, \alpha} c \left( P(c, K, Q_1, R, \tilde{s}) + L(\alpha) > 0 \right) \quad \text{s.t.} \quad \begin{cases}
P(c, K, Q_1, R, \tilde{s}) + L(\alpha) > 0 \\
Q_1 > 0 \\
R > 0 \\
K \in \mathcal{K}
\end{cases} \quad \text{trace}(Q_1) = 1.
\]

(36)

Problem (36) is an eigenvalue problem since \( P(c, K, Q_1, R, \tilde{s}) + L(\alpha) \) depends affine linearly on the decision variables [16].

Another way to address (35) is to fix \( Q_1 \). This provides the lower bound

\[
\hat{\gamma}_2 = \sup_{c, K, Q_2, R, \tilde{s}, \alpha} c \left( P(c, K, Q_2, R, \tilde{s}) + L(\alpha) > 0 \right) \quad \text{s.t.} \quad \begin{cases}
P(c, K, Q_2, R, \tilde{s}) + L(\alpha) > 0 \\
Q_2 > 0 \\
R > 0 \\
K \in \mathcal{K}
\end{cases} \quad \text{trace}(Q_1) = 1.
\]

(37)

Problem (37) still contains a BMI in the first constraint due to the product of \( c \) with \( Q_2 \). However, it is worth observing
that the matrix size and number of scalar variables in (37) are typically smaller than those of (36); in fact, \(q_1(x)\) multiplies \(w(x)\) in \(p(x)\), while \(q_2(x)\) multiplies \(cv_{\text{den}}(x) - v_{\text{num}}(x)\), and the degree of \(w(x)\) is always greater than the degree of \(cv_{\text{den}}(x) - v_{\text{num}}(x)\).

A way to obtain the solution of (37) is via a one-parameter sweep on \(c\) (for instance, a bisection algorithm) where an LMI feasibility test is solved for each fixed value of \(c\).

A further way to obtain the solution of (37) is via a quasi-convex optimization problem. Indeed, for \(\mu \in \mathbb{R}\) let us define the polynomials

\[
p_1(x, \theta) = -q(x, \theta) \left( \frac{w(x, \theta)}{v_{\text{num}}(x)} \right) - r(x, \theta)'a(\theta)
\]

\[
p_2(x, \theta) = q_2(x, \theta) \bar{v}(x)
\]

\[
\bar{v}(x) = v_{\text{den}}(x) + \mu v_{\text{num}}(x)
\]

and the SMR matrices

\[
P_1(K, Q, R, \bar{s}) + L(\alpha) = \text{CSMR}_{\text{qua}}(p_1)
P_2(Q_2) = \text{CSMR}_{\text{qua}}(p_2)
\]

\[
\frac{V}{\text{SMR}_{\text{pol}}(\bar{v})}
\]

where \(P_2(Q_2)\) is built such that

\[
\frac{V}{\text{SMR}_{\text{pol}}(\bar{v})} > 0 \quad \text{and} \quad Q_2 > 0 \quad \Rightarrow \quad P_2(Q_2) > 0.
\]

**Theorem 3**: Let \(v : \mathbb{R}^n \rightarrow \mathbb{R}\) be a rational function satisfying (5)–(6), and let \(c, \mu \in \mathbb{R}\) be positive. Define the quantities in (39), and assume that \(\bar{V} > 0\). Then,

\[
\hat{\gamma}_2 = -\frac{z^*}{1 + \mu z^*}
\]

where \(z^*\) is the solution of

\[
z^* = \inf_{K, Q_2, R, x, \alpha, z} z
\]

\[
\begin{align*}
zP_2(Q_2) + P_1(K, Q, R, \bar{s}) + L(\alpha) & > 0 \\
Q_2 & > 0 \\
R & > 0 \\
1 + \mu z^* & > 0 \\
K & \in K.
\end{align*}
\]

**Proof.** Suppose that the constraints in (42) hold. Let us pre- and post-multiply the first inequality by \(b(\theta, \theta)\theta^p/2 \otimes b_{\text{lin}}(x, \theta^p/2)\)' and its transpose, respectively, where \(x \neq 0_n\). We get:

\[
0 < zP_2(Q_2) + P_1(K, Q, R, \bar{s}) + L(\alpha) - q_1(x, \theta)w(x)
\]

\[
+q_2(x, \theta)v_{\text{den}}(x) + \mu v_{\text{num}}(x) - r(x, \theta)'a(\theta) - s(x, \theta)'b(\theta)
\]

\[
= -q_1(x, \theta)w(x) - q_2(x, \theta)(-z \bar{v}(x) = -z v_{\text{num}}(x) - v_{\text{num}}(x) - r(x, \theta)'a(\theta) - s(x, \theta)'b(\theta)
\]

\[
= -q_1(x, \theta)w(x) - (1 + \mu z)q_2(x, \theta)
\]

\[
\cdot \left( \frac{z v_{\text{den}}(x) - v_{\text{num}}(x)}{1 + \mu z} - r(x, \theta)'a(\theta) - s(x, \theta)'b(\theta). \right)
\]

Hence, the first inequality in (37) coincides with the first inequality in (42) whenever \(q_2(x, \theta)\) and \(c\) are replaced by \(q_2(x, \theta) \rightarrow q_2(x, \theta)(1 + \mu z)
\]

\[
c \rightarrow \frac{-z}{1 + \mu z}.
\]

Since \(1 + \mu z\) is positive, it follows that the constraints in (37) are equivalent to those in (42), i.e. (41) holds. □

Theorem 3 states that the solution of (37) can be found by solving the optimization problem (42) which is a generalized eigenvalue problem [16].

**IV. EXAMPLES**

In this section we present two illustrative examples of the proposed results. The SMR matrices are built with algorithms similar to those reported in [13].

**A. Example 1**

Let us consider the family of Van der Pol systems described by

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -(1 + \theta)x_1 - x_2(1 - \theta^2x_1^2)
\end{align*}
\]

where \(x = (x_1, x_2)\)' is the state and \(\theta\) is an uncertain parameter. We consider the problem of determining the LERDA of the origin provided by the rational function

\[
v(x) = \frac{5x_1^2 + 2x_1x_2 + 3x_2^2 + x_1^4 - x_2^2x_1^2 + x_2^4}{1 + x_1^2 + x_2^2},
\]

i.e., computing \(\gamma\) in (8). To this end, we compute the lower bound \(\hat{\gamma}_2\) in (41). We simply select the degrees of \(q_1(x, \theta)\) in \(x\) and \(\theta\) equal to zero in the choice of the degrees reported after Theorem 2, hence finding

\[
\hat{\gamma}_2 = 1.527.
\]

The number of LMI variables is 441. Figure 1 shows the curve \(\psi(x, \theta) = 0\) for some admissible values of \(\theta\), and the boundary of the LERDA.

**B. Example 2**

Let us consider the system described by

\[
\begin{align*}
\dot{x}_1 &= -x_1 + \theta x_2^3 \\
\dot{x}_2 &= -x_2 + (1 - \theta)x_1^2 + x_1u \\
y &= x_1 - x_2 \\
\theta &\in [-1, 1]
\end{align*}
\]

where \(x = (x_1, x_2)\)' is the state, \(u \in \mathbb{R}\) is the input, \(y \in \mathbb{R}\) is the output, and \(\theta\) is an uncertain parameter. We consider the problem of designing a polynomial output controller for enlarging the RDA of the origin by using the rational LF

\[
v(x) = \frac{x_1^2 + x_2^2 + x_1^4 + x_2^4}{1 - x_1 - x_2 + x_1^2 - x_1x_2 + x_2^2}.
\]

The control structure is chosen as

\[
\begin{align*}
u &= k(y) \\
k(y) &= k_0 + k_1y + k_2y^2 \\
k_0, k_1, k_2 &\in [-1, 1]
\end{align*}
\]
where $k_0, k_1, k_2$ are the coefficients to determine. Observe that the origin is an equilibrium point for all $k_0, k_1, k_2$.

To this end, we compute the lower bound $\gamma_2$ in (41). We simply select the degrees of $q_1(x, \theta)$ in $x$ and $\theta$ equal to zero in the choice of the degrees reported after Theorem 2. First, we consider that $k(y)$ is constant (this means that $k_1 = k_2 = 0$). We find

$$\hat{\gamma}_2 = 0.722, \quad k(y) = 0.478.$$  

We repeat the computation supposing that $k(y)$ is linear (i.e. $k_2 = 0$). We find

$$\hat{\gamma}_2 = 1.598, \quad k(y) = 0.119 - 0.885y.$$  

Lastly, we suppose that $k(y)$ is quadratic, hence finding

$$\hat{\gamma}_2 = 1.727, \quad k(y) = -0.280 - 0.893y + 0.128y^2.$$  

The number of LMI variables in the latter case is 444. Figure 2 shows the boundaries of the LERDA provided by the three found controllers, and the curve $\dot{v}(x, \theta) = 0$ corresponding to the quadratic controller for some admissible values of $\theta$.

V. CONCLUSION

This paper has proposed a strategy for the estimation and for the control of the RDA of equilibrium points of uncertain polynomial systems through LMI-based techniques and rational LFs. It has been shown that lower bounds of the LERDA in the estimation problem, or the maximum achievable LERDA in the control problem where a polynomial static output controller has to be designed, can be obtained by solving either an eigenvalue problem or a generalized eigenvalue problem with smaller dimension. The conservatism of these lower bounds can be reduced by increasing the degree of some multipliers introduced in the construction of the optimization problems. Future works will investigate the extension of the proposed strategy to other classes of uncertain nonlinear control systems.

REFERENCES


