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Integral representation of renormalized self-intersection local times

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Abstract

In this paper we apply Clark-Ocone formula to deduce an explicit integral representation for the renormalized self-intersection local time of the $d$-dimensional fractional Brownian motion with Hurst parameter $H \in (0, 1)$. As a consequence, we derive the existence of some exponential moments for this random variable.

1 Introduction

The purpose of this paper is to apply Clark-Ocone’s formula to the renormalized self-intersection local time of the $d$-dimensional fractional Brownian motion. As a consequence, we derive the existence of some exponential moments for this local time.

A well-known result in Itô’s stochastic calculus asserts that any square integrable random variable in the filtration generated by a $d$-dimensional Brownian motion $W = \{W_t, t \geq 0\}$ can be expressed as the sum of its expectation plus the stochastic integral of a square integrable adapted process:

$$F = E(F) + \sum_{i=1}^{d} \int_0^\infty u^i(t)dW^i_t.$$ 

The process $u$ is determined by $F$, except on sets of measure zero. In this context, Clark-Ocone formula provides an explicit representation of $u$ in

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terms of the derivative operator in the sense of Malliavin calculus. More precisely, if \( F \) belongs to the Sobolev space \( D^{1,2} \), then \( u^i(t) = E(D^i F | \mathcal{F}_t) \), where \( D^i \) denotes the derivative with respect to the \( i \)th component of the Brownian motion and \( \{ \mathcal{F}_t, t \geq 0 \} \) is the filtration generated by the Brownian motion. Extensions of this formula have been developed by Üstünel in [17], and by Karatzas, Ocone and Li in [12]. Clark-Ocone formula has proved to be a useful tool in finding hedging portfolios in mathematical finance (see, for instance, [11]).

The fractional Brownian motion on \( \mathbb{R}^d \) with Hurst parameter \( H \in (0, 1) \) is a \( d \)-dimensional Gaussian process \( B^H = \{ B^H_t, t \geq 0 \} \) with zero mean and covariance function given by

\[
E(B^H_i B^H_j) = \frac{\delta_{ij}}{2} (t^{2H} + s^{2H} - |t - s|^{2H}), \quad (1.1)
\]

where \( i, j = 1, \ldots, d, s, t \geq 0, \) and

\[
\delta_{ij} = \begin{cases} 
1 & \text{if } i = j \\
0 & \text{if } i \neq j 
\end{cases}
\]

is the Kronecker symbol. Assume \( d \geq 2 \). The self-intersection local time of \( B^H \) is formally defined as

\[
L = \int_0^T \int_0^t \delta_0(B^H_t - B^H_s) ds,
\]

where \( \delta_0 \) is the Dirac delta function. It measures the amount of time that the process spends intersecting itself on the time interval \([0, T]\). Rigorously, \( L \) is defined as the limit in \( L^2 \), if it exists, of \( L_\varepsilon = \int_0^T \int_0^t p_\varepsilon(B^H_t - B^H_s) dsdt \), as \( \varepsilon \) tends to zero, where \( p_\varepsilon \) denotes the heat kernel.

For \( H = \frac{1}{2} \), the process \( B^H \) is a classical Brownian motion and its self-intersection local time has been studied by many authors (see Albeverio et al. [1], Calais and Yor [1], He et al. [6], Hu [7], Imkeller et al. [10], Varadhan [18], Yor [20], and the references therein). In this case, if \( d = 2 \), Varadhan [18] has proved that \( L_\varepsilon \) does not converge in \( L^2 \), but it can be renormalized so that \( L_\varepsilon - E(L_\varepsilon) \) converges in \( L^2 \) as \( \varepsilon \) tends to zero to a random variable that we denote by \( \tilde{L} \). This result has been extended by Rosen [16] to the case \( H \in (\frac{1}{2}, \frac{3}{4}) \) (still when \( d = 2 \)), and by Hu and Nualart in [9], where they have obtained the following complete result on the existence of the self-intersection local time of the fractional Brownian motion:

(i) The self-intersection local time \( L \) exists if and only if \( Hd < 1 \).
(ii) If $Hd \geq 1$, the renormalized self-intersection local time $\tilde{L}$ exists if and only if $Hd < \frac{3}{2}$.

An important question is the existence of moments and exponential moments for the (renormalized) self-intersection local time. Along this direction, Le Gall [13] proved that for the planar Brownian motion, there is a critical exponent $\lambda_0$, such that $E\left(\exp \lambda \tilde{L}\right) < \infty$ for all $\lambda < \lambda_0$, and $E\left(\exp \lambda \tilde{L}\right) = \infty$ if $\lambda > \lambda_0$. Using the theory of large deviations, Bass and Chen proved in [2] that the critical exponent $\lambda_0$ coincides with $A^{-4}$, where $A$ is the best constant in the Gagliardo-Nirenberg inequality.

Clark-Ocone formula seems to be a suitable tool to analyze the renormalized self-intersection local time, because in this formula we do not take into account the expectation of the random variable. The fractional Brownian motion can be expressed as the stochastic integral

$$B^H_t = \int_0^t K_H(t,s)dW_s$$

of a square integrable kernel $K_H(t,s)$ with respect to an underlying Brownian motion $W$. In this way the renormalized self-intersection local time $\tilde{L}$ is a functional of the Brownian motion $W$, and we can obtain an explicit integral representation $\tilde{L}$, in the general case $Hd < \frac{3}{2}$. This formula allows us to obtain some exponential moments for the renormalized self-intersection local time, using the method of moments.

The paper is organized as follows. In Section 2 we present some preliminaries on Malliavin calculus and Clark-Ocone formula. Section 3 is devoted to derive estimates for the moments of the self-intersection local time in the case of a general $d$-dimensional Gaussian process, using the method of moments. In the case of the fractional Brownian motion, this provides the existence of exponential moments in the case $Hd < 1$. Section 4 contains the main result, which is the integral representation of the renormalized self-intersection local time of the fractional Brownian motion in the case $H < \min\left(\frac{d}{2d}, \frac{d}{d+1}\right)$. As an application we show that $E\left(\exp \frac{1}{p} \tilde{L}\right) < \infty$ if $p < \frac{1}{2} \left[\left(\frac{1}{2} + H\right) \left(\frac{d}{2} - \frac{1}{d+1}\right)\right]^{-1}$. A crucial tool is the local nondeterminism property introduced by Berman in [3] and developed by many authors (see Xiao [19] and the references therein).
2 Preliminaries on Malliavin calculus and
Clark-Ocone formula

We need some preliminaries on the Malliavin calculus for the $d$-dimensional Brownian motion $W = \{W_t, t \geq 0\}$. We refer to Malliavin [14] and Nualart [15] for a more detailed presentation of this theory.

We assume that $W$ is defined in a complete probability space $(\Omega, \mathcal{F}, P)$, and the $\sigma$-field $\mathcal{F}$ is generated by $W$. Let us denote by $H$ the Hilbert space $L^2(\mathbb{R}_+; \mathbb{R}^d)$, and for any function $h \in H$ we set

$$W(h) = \sum_{i=1}^{d} \int_{0}^{\infty} h^i(t) dW^i_t.$$ 

Let $\mathcal{S}$ be the class of smooth and cylindrical random variables of the form

$$F = f(W(h_1), \ldots, W(h_n)),$$

where $n \geq 1$, $h_1, \ldots, h_n \in H$, and $f$ is an infinitely differentiable function such that together with all its partial derivatives has at most polynomial growth order. The derivative operator of the random variable $F$ is defined as

$$D^i_t F = \sum_{j=1}^{n} \frac{\partial f}{\partial x_j}(W(h_1), \ldots, W(h_n)) h^i_j(t),$$

where $i = 1, \ldots, d$ and $t \geq 0$. In this way, we interpret $DF$ as a random variable with values in the Hilbert space $H$. The derivative is a closable operator on $L^2(\Omega)$ with values in $L^2(\Omega; H)$. We denote by $\mathbb{D}^{1,2}$ the Hilbert spaced defined as the completion of $\mathcal{S}$ with respect to the scalar product

$$\langle F, G \rangle_{1,2} = E(FG) + E\left(\sum_{i=1}^{d} \int_{0}^{\infty} D^i_t F D^i_t G dt\right).$$

The divergence operator $\delta$ is the adjoint of the derivative operator $D$. The operator $\delta$ is an unbounded operator from $L^2(\Omega; H)$ into $L^2(\Omega)$, and is determined by the duality relationship

$$E(\delta(u) F) = E(\langle u, DF \rangle_H),$$

for any $u$ in the domain of $\delta$, and $F$ in $\mathbb{D}^{1,2}$. Gaveau and Trauber [5] proved that $\delta$ is an extension of the classical Itô integral in the sense that any $d$-dimensional square integrable adapted process belongs to the domain of $\delta$, 

\[\text{4}\]
and \( \delta(u) \) coincides with the Itô integral of \( u \):

\[
\delta(u) = \sum_{i=1}^{d} \int_{0}^{\infty} u^i(t) dW^i_t.
\]

It is well-known that any random variable \( F \in L^2(\Omega) \), possesses a stochastic integral representation of the form

\[
F = E(F) + \sum_{i=1}^{d} \int_{0}^{\infty} u^i(t) dW^i_t,
\]

for some \( d \)-dimensional square integrable adapted process \( u \). Clark-Ocone formula says that if \( F \in D^{1,2} \), then

\[
F = E(F) + \sum_{i=1}^{d} \int_{0}^{\infty} E(D_t^i F | \mathcal{F}_t) dW^i_t.
\]

(2.1)

3 Exponential integrability of the self-intersection local time

Suppose that \( W = \{W_t, t \geq 0\} \) is a \( d \)-dimensional standard Brownian motion, defined in a complete probability space \( (\Omega, \mathcal{F}, P) \). Suppose that \( \mathcal{F} \) is generated by \( W \). We denote by \( \{\mathcal{F}_t, t \geq 0\} \) the filtration generated by \( W \) and the sets of probability zero. Consider a \( d \)-dimensional Gaussian process of the form

\[
B_t = \int_{0}^{t} K(t,s) dW_s,
\]

(3.1)

where \( K(t,s) \) is a measurable kernel satisfying \( \int_{0}^{t} K(t,s)^2 ds < \infty \) for all \( t \geq 0 \). We will assume that \( K(t,s) = 0 \) if \( s > t \).

Fix a time interval \([0,T]\). We will make use of the following property on the kernel \( K(t,s) \):

\textbf{(H1)} For any \( s,t \in [0,T], s < t \) we have

\[
\int_{s}^{t} K(t,\theta)^2 d\theta \geq k_1(t-s)^{2H}
\]

(3.2)

for some constants \( k_1 > 0 \), and \( H \in (0,1) \).

Notice that \( \text{Var} (B_t^i | \mathcal{F}_s) = \int_{s}^{t} K(t,\theta)^2 d\theta \), so condition \textbf{(H1)} is equivalent to say that \( \text{Var} (B_t^i | \mathcal{F}_s) \geq k_1(t-s)^{2H} \), for each component \( i = 1,\ldots,d \). This property is satisfied, for instance, in the following two examples:
Example 1 Suppose that $K(t, s) = (t - s)^{H - \frac{1}{2}}$. Then, we have equality in (3.2) with $k_1 = \frac{1}{2H}$.

Example 2 Condition (H1) is satisfied by the kernel of the fractional Brownian motion, as a consequence of the local nondeterminism property (see (4.1) below).

We will denote by $C$ a generic constant depending on $T$, the dimension $d$, and the constants appearing in the hypothesis such as $H$ and $k_1$.

The self-intersection local time of the process $B$ in the time interval $[0, T]$, denoted by $L$, is defined as the limit in $L^2$ as $\varepsilon$ tends to zero of

$$L_\varepsilon = \int_0^T \int_0^t p_\varepsilon(B_t - B_s) ds,$$

where $p_\varepsilon$ denotes the heat kernel

$$p_\varepsilon(x) = (2\pi\varepsilon)^{-\frac{d}{2}} \exp \left( -\frac{|x|^2}{2\varepsilon} \right).$$

The next theorem asserts that $L$ exists if $Hd < 1$, and it has exponential moments of order $\frac{1}{Hd}$.

Theorem 1 Suppose that $Hd < 1$. Then, the self-intersection local time $L$ exists as the limit in $L^2$ of $L_\varepsilon$, as $\varepsilon$ tends to zero, and for all integers $n \geq 1$ we have

$$E(L^n) \leq C^n (n!)^{Hd},$$

for some constant $C$. As a consequence,

$$E(e^{L^p}) < \infty,$$

for any $p < \frac{1}{Hd}$, and there exists a constant $\lambda_0 > 0$ such that $E(e^{\lambda L^{\frac{1}{Hd}}}) < \infty$ for all $\lambda < \lambda_0$.

Proof. From the equality

$$p_\varepsilon(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \exp \left( i \langle \xi, x \rangle - \frac{\varepsilon|\xi|^2}{2} \right) d\xi$$

and the definition of $L_\varepsilon$, we obtain

$$L_\varepsilon = \frac{1}{(2\pi)^d} \int_0^T \int_0^t \int_{\mathbb{R}^d} \exp \left( i \langle \xi, B_t - B_s \rangle - \frac{\varepsilon|\xi|^2}{2} \right) d\xi ds dt.$$
This expression allows us to compute the moments of $L_\varepsilon$. Fix an integer $n \geq 1$. Denote by $T_n$ the set $\{0 < s < t < T\}^n$. Then

$$E(L_\varepsilon^n) = \frac{1}{(2\pi)^{nd}} \int_{T_n} \int_{\mathbb{R}^{nd}} E\left[\exp \left( i \langle \xi_1, B_{t_1} - B_{s_1} \rangle + \cdots + i \langle \xi_n, B_{t_n} - B_{s_n} \rangle \right) \right]$$

$$\times \exp \left( -\frac{\varepsilon}{2} \sum_{j=1}^n |\xi_j|^2 \right) d\xi_1 \cdots d\xi_n ds dt, \quad (3.4)$$

where $s = (s_1, \ldots, s_n)$ and $t = (t_1, \ldots, t_n)$. Notice that

$$\int_{\mathbb{R}^{nd}} E\left[\exp \left( i \langle \xi_1, B_{t_1} - B_{s_1} \rangle + \cdots + i \langle \xi_n, B_{t_n} - B_{s_n} \rangle \right) \right]$$

$$\times e^{-\frac{\varepsilon}{2} \sum_{j=1}^n |\xi_j|^2} d\xi_1 \cdots d\xi_n$$

$$= \int_{\mathbb{R}^{nd}} \exp \left( -\frac{1}{2} E \left( \langle \xi_1, B_{t_1} - B_{s_1} \rangle + \cdots + \langle \xi_n, B_{t_n} - B_{s_n} \rangle \rangle \right)^2 \right)$$

$$\times e^{-\frac{\varepsilon}{2} \sum_{j=1}^n |\xi_j|^2} d\xi_1 \cdots d\xi_n$$

$$= \left( \int_{\mathbb{R}^n} \exp \left( -\frac{1}{2} \xi^T Q \xi \right) e^{-\frac{\varepsilon}{2} \xi^T \xi} d\xi \right)^d, \quad (3.5)$$

where $Q$ is the covariance matrix of the $n$-dimensional random vector $(B_{t_1}^1 - B_{s_1}^1, \ldots, B_{t_n}^1 - B_{s_n}^1)$. Substituting (3.5) into (3.4) yields

$$E(L_\varepsilon^n) = \frac{1}{(2\pi)^{nd}} \int_{T_n} \left( \int_{\mathbb{R}^n} \exp \left( -\frac{1}{2} \xi^T Q \xi \right) e^{-\frac{\varepsilon}{2} \xi^T \xi} d\xi \right)^d ds dt,$$

and $E(L_\varepsilon^n)$ converges as $\varepsilon$ tends to zero to

$$\alpha_n = \frac{1}{(2\pi)^{nd}} \int_{T_n} \left( \int_{\mathbb{R}^n} \exp \left( -\frac{1}{2} \xi^T Q \xi \right) d\xi \right)^d ds dt$$

$$= \frac{1}{(2\pi)^{nd}} \int_{T_n} (\det Q)^{-\frac{d}{2}} ds dt,$$

provided $\alpha_n$ is finite.

If $\alpha_2 < \infty$, then in the same way as before we obtain

$$\lim_{\varepsilon, \delta \downarrow 0} E(L_\varepsilon L_\delta) = \alpha_2,$$

which implies that $L_\varepsilon$ converges in $L^2$ as $\varepsilon$ tends to zero. Furthermore, if $\alpha_n$ is finite for all $n \geq 1$, then we deduce the convergence in $L^p$ for any $p \geq 2$.

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of $L_\varepsilon$ as $\varepsilon$ tends to zero. The limit, denoted by $L$, will be, by definition, the self-intersection local time of the process $B$ in the time interval $[0,T]$. To complete the proof of the theorem it suffices to show that $\alpha_n$ is bounded by $C^n (nt)^{Hd}$, for some constant $C$.

We can write

$$\alpha_n = \frac{n!}{(2\pi)^{\frac{nd}{2}}} \int_{T_n \cap \{t_1 < \cdots < t_n\}} (\det Q)^{-\frac{d}{2}} ds dt.$$ 

For each $i = 1, \ldots, n$ we denote by $\tau_i$ the point in the set $\{s_i, s_i+1, \ldots, s_n, t_i-1\}$ which is closer to $t_i$ from the left. Then, by (H1) and the fact that $s_i < t_i$, $i = 1, \ldots, n$, we obtain, using Lemma 5 in the Appendix,

$$\det Q = \text{Var}(B_{t_1}^{1} - B_{s_1}^{1}) \text{Var}(B_{t_2}^{1} - B_{s_2}^{1} | B_{t_1}^{1} - B_{s_1}^{1}) \times \cdots \times \text{Var}(B_{t_n}^{1} - B_{s_n}^{1} | B_{t_{n-1}}^{1} - B_{s_{n-1}}^{1})$$

$$\geq \text{Var}(B_{t_1}^{1} | B_{s_1}^{1}) \text{Var}(B_{t_2}^{1} | B_{s_1}^{1}, B_{s_2}^{1}) \times \cdots \times \text{Var}(B_{t_n}^{1} | B_{s_1}^{1}, B_{s_2}^{1}, \ldots, B_{t_{n-1}}^{1}, B_{s_{n-1}}^{1})$$

$$\geq \text{Var}(B_{t_1}^{1} | \mathcal{F}_{\tau_1}) \text{Var}(B_{t_2}^{1} | \mathcal{F}_{\tau_2}) \cdots \text{Var}(B_{t_n}^{1} | \mathcal{F}_{\tau_n})$$

$$\geq k_1^n (t_1 - \tau_1)^{2H} (t_2 - \tau_2)^{2H} \cdots (t_n - \tau_n)^{2H}.$$ 

As a consequence,

$$\alpha_n \leq \frac{n!}{(2\pi)^{\frac{nd}{2}}} k_1^{-\frac{nd}{2}} \int_{T_n \cap \{t_1 < \cdots < t_n\}} \prod_{i=1}^{n} (t_i - \tau_i)^{-Hd} ds dt.$$ 

If we fix the points $t_1 < \cdots < t_n$, there are $3 \times 5 \times \cdots \times (2n-1) = (2n-1)!$ possible ways to place the points $s_1, \ldots, s_n$. In fact, $s_1$ must be in $(0, t_1)$. For $s_2$ we have three choices: $(0, s_1)$, $(s_1, t_1)$ and $(t_1, t_2)$. By a recursive argument it is clear that we have $(2i-1)!$ possible choices for $s_i$, given $s_1, \ldots, s_{i-1}$. In this way, up to a set of measure zero, we can decompose the set $T_n \cap \{t_1 < \cdots < t_n\}$ into the union of $(2n-1)!$ disjoint subsets. The integral of $\prod_{i=1}^{n} (t_i - \tau_i)^{-Hd}$ on each one of these subset can be expressed as

$$\Phi_{\sigma} = \int_{\{0 < z_1 < \cdots < z_{2n} < T\}} \prod_{i=1}^{n} (z_{\sigma(i)} - z_{\sigma(i)-1})^{-Hd} dz,$$

where $\sigma(1) < \cdots < \sigma(n)$ are $n$ elements in $\{1, 2, \ldots, 2n\}$, and $z = (z_1, \ldots, z_{2n})$. Making the change of variables $y_i = z_i - z_{i-1}$, $i = 1, \ldots, 2n$ (with the con-
vention \( z_0 = 0 \) we obtain

\[
\Phi_\sigma = \int_{\{0<y_1+\cdots+y_{2n}<T\}} \prod_{i=1}^{n} y_i^{-Hd}dy \leq \frac{T^n}{n!} \int_{\{0<y_1+\cdots+y_{n}<T\}} \prod_{i=1}^{n} y_i^{-Hd}dy
\]

\[
= \frac{1}{n!}T^{n(2-Hd)+Hd} \frac{\Gamma(1-Hd)^{n-1}}{\Gamma(n(1-Hd)+Hd+1)}.
\]

Therefore

\[
\alpha_n \leq \frac{k_1^{-\frac{n}{2}}(2n-1)!!T^{n(2-Hd)+Hd}\Gamma(1-Hd)^{n-1}}{(2\pi)^{\frac{n}{2}}n!\Gamma(n(1-Hd)+Hd+1)}
\]

\[
= C_1C_2 \frac{(2n-1)!!}{\Gamma(n(1-Hd)+Hd+1)},
\]

with \( C_1 = T^{Hd}\Gamma(1-Hd)^{-1} \), and \( C_2 = k_1^{-\frac{d}{2}}\Gamma(1-Hd)T^{2-Hd}. \) Taking into account that \((2n-1)!! \leq 2^{n-1}n!\), and that

\[
\Gamma(n(1-Hd)+Hd+1) \geq C_2 n!(1-Hd),
\]

for some constant \(C\), we obtain the desired estimate. \( \blacksquare \)

If \(Hd \geq 1\), the above result is no longer true. In that case the expectation of \(L_\varepsilon\) blows up as \(\varepsilon\) tends to zero. In fact, if we denote \(\sigma^2(s,t) = \text{Var}(B^1_t - B^1_s)\), for \(s < t\), then

\[
E(L_\varepsilon) = \int_0^T \int_0^t p_{\varepsilon^2}(s,t) dsdt = (2\pi)^{-\frac{d}{2}} \int_0^T \int_0^t (\varepsilon + \sigma^2(s,t))^{-\frac{d}{2}} dsdt,
\]

which converges to

\[
(2\pi)^{-\frac{d}{2}} \int_0^T \int_0^t \sigma^2(s,t)^{-\frac{d}{2}} dsdt \geq (2\pi)^{-\frac{d}{2}} k_1^{-\frac{d}{2}} \int_0^T \int_0^t (t-s)^{-Hd} dsdt = \infty.
\]

In this case, one can study the existence of the renormalized self-intersection local time defined as the limit as \(\varepsilon\) tends to zero of \(L_\varepsilon - E(L_\varepsilon)\). In the next section we discuss the existence and exponential moments of the renormalized self-intersection local time, using Clark-Ocone formula, in the case of the fractional Brownian motion.
4 Renormalized self-intersection local time of the fBm

The fractional Brownian motion on \( \mathbb{R}^d \) with Hurst parameter \( H \in (0, 1) \) is a \( d \)-dimensional Gaussian process \( B^H = \{ B^H_t, t \geq 0 \} \) with zero mean and covariance function given by (1.1). We will assume that \( d \geq 2 \).

It is well-known that \( B^H \) possesses the following integral representation

\[
B^H_t = \int_0^t K^H(t, s) dW_s,
\]

where \( W = \{ W_t, t \geq 0 \} \) is a \( d \)-dimensional Brownian motion, and \( K^H(s, t) \) is the square integrable kernel given by

\[
K^H(t, s) = C_{H,1} s^{\frac{1}{2} - H} \int_s^t (u - s)^{H - \frac{3}{2}} u^{H - \frac{1}{2}} du,
\]

if \( \frac{3}{2} - H > 0 \), and by

\[
K^H(t, s) = C_{H,2} \left[ \left( \frac{t}{s} \right)^{H - \frac{1}{2}} (t - s)^{H - \frac{1}{2}} - (H - \frac{1}{2}) s^{\frac{1}{2} - H} \int_s^t u^{H - \frac{3}{2}} (u - s)^{H - \frac{1}{2}} du \right],
\]

if \( \frac{3}{2} - H < 0 \), for any \( s < t \), where the constants are \( C_{H,1} = \left[ \frac{H(2H - 1)}{B(2H, H - \frac{1}{2})} \right]^{\frac{1}{2}} \), and \( C_{H,2} = \left[ \frac{2H}{(1 - 2H)(1 - 2H, H + \frac{1}{2})} \right]^{\frac{1}{2}} \), where \( B(\alpha, \beta) \) denotes the beta function.

The processes \( B^H \) and \( W \) generate the same filtration, that is, \( \mathcal{F}_t = \sigma\{ W_s, 0 \leq s \leq t \} = \sigma\{ B^H_s, 0 \leq s \leq t \} \).

The fractional Brownian motion satisfies the following local nondeterminism property:

**(LND)** There exists a constant \( k_2 > 0 \), depending only on \( H \) and \( T \), such that for any \( t \in [0, T] \), \( 0 < r < t \wedge (T - t) \) and for \( i = 1, \ldots, d \),

\[
\text{Var}(B^H_t, B^H_s : |s - t| \geq r) \geq k_2 r^{2H}.
\]

(4.1)

Consider the approximated self-intersection local time \( L_\varepsilon \) introduced in (3.3). From the general result proved in Section 2 it follows that if \( Hd < 1 \), then \( L_\varepsilon \) converges in \( L^2 \) to the self-intersection local time \( L \), and the random variable \( L \) has exponential moments. If \( Hd \geq 1 \), this result is no longer true, and one considers the renormalization of the self-intersection local time, introduced by Varadhan.
The purpose of this section is to apply the Clark-Ocone formula to provide a stochastic integral representation for the renormalized self-intersection local time $\tilde{L}$. As a consequence, we will prove the existence of some exponential moments for the random variable $\tilde{L}$.

**Theorem 2** Suppose that $H < \min \left( \frac{3}{2d}, \frac{2}{d+1} \right)$. Then the renormalized self-intersection local time of the $d$-dimensional fractional Brownian motion $B^H$ exists in $L^2$ and it has the following integral representation

$$\tilde{L} = -\sum_{i=1}^{d} \int_0^T \left( \int_0^T \int_0^t \frac{A_{r,t,s}^i}{\sigma_{r,s,t}^2} \right) p_{r,s,t} \left( A_{r,t,s}^i \right) [K_H(t,r) - K_H(s,r)] \frac{dsdt}{\sigma_{r,s,t}^2} \right) dW^i_r,$$

where

$$A_{r,t,s} = E(B^H_t - B^H_s | F_r)$$

and

$$\sigma_{r,s,t}^2 = \text{Var}(B^{H,i}_t - B^{H,i}_s | F_r).$$

**Proof.** The proof will be done in several steps.

**Step 1** We are going to apply Clark-Ocone formula to the random variable $L_\varepsilon$. It is clear that $L_\varepsilon$ belongs to $\mathbb{D}^{1,2}$, and its derivative can be computed as follows

$$D^i_rL_\varepsilon = \int_0^T \int_0^t \frac{\partial p_\varepsilon}{\partial x_i} (B^H_t - B^H_s) D^j_r (B^{H,j}_t - B^{H,j}_s) dsdt,$$

where $r \in [0,T]$, and $i = 1, \ldots, d$. Using

$$D^i_r (B^{H,i}_t - B^{H,i}_s) = [K_H(t,r) - K_H(s,r)] 1_{[0,t]}(r),$$

we obtain

$$D^i_rL_\varepsilon = \int_r^T \int_0^t \frac{\partial p_\varepsilon}{\partial x_i} (B^H_t - B^H_s) [K_H(t,r) - K_H(s,r)] dsdt. \quad (4.3)$$

The next step is to compute the conditional expectation $E(D^i_rL_\varepsilon | F_r)$. The conditional law of $B^H_t - B^H_s$ given $F_r$ is normal with mean $A_{r,t,s}$ and covariance matrix $\sigma_{r,s,t}^2 I_d$, where $I_d$ is the $d$-dimensional identity matrix. Hence,
the conditional expectation \( E \left( \frac{\partial p_\varepsilon}{\partial x_i} \left( B^H_t - B^H_s \right) \right) \mid \mathcal{F}_r \) is given by

\[
E \left( \frac{\partial p_\varepsilon}{\partial x_i} \left( B^H_t - B^H_s \right) \right) \mid \mathcal{F}_r = \int_{\mathbb{R}^d} \frac{\partial p_\varepsilon}{\partial x_i} (y) p_{\varepsilon^2 + \sigma_{r,s,t}^2} (y - A_{r,t,s}) dy
\]

\[
= \frac{\partial p_{\varepsilon + \sigma_{r,s,t}^2}}{\partial x_i} (A_{r,t,s})
\]

\[
= -\frac{A_{r,t,s}}{\varepsilon + \sigma_{r,s,t}^2} p_{\varepsilon + \sigma_{r,s,t}^2} (A_{r,t,s}).
\]

As a consequence, from (4.3) we obtain

\[
E \left( D_i L_\varepsilon \right) \mid \mathcal{F}_r = -\int_0^T \int_0^t A_{r,t,s} p_{\varepsilon + \sigma_{r,s,t}^2} (A_{r,t,s}) [K_H(t,r) - K_H(s,r)] dsdt,
\]

and this leads to the following integral representation for \( L_\varepsilon - E(L_\varepsilon) \)

\[
L_\varepsilon - E(L_\varepsilon)
= -\sum_{i=1}^d \int_0^T \left( \int_0^T \int_0^t A_{r,t,s} p_{\varepsilon + \sigma_{r,s,t}^2} (A_{r,t,s}) [K_H(t,r) - K_H(s,r)] dsdt \right) dW^i_r.
\]

**Step 2** In order to pass to the limit as \( \varepsilon \) tends to zero we proceed as follows. Set

\[
\Sigma_r^i (r,t,s) = \frac{A_{r,t,s}}{\varepsilon + \sigma_{r,s,t}^2} p_{\varepsilon + \sigma_{r,s,t}^2} (A_{r,t,s}) [K_H(t,r) - K_H(s,r)] . \tag{4.4}
\]

Clearly, \( \Sigma_r^i (r,t,s) \) converges pointwise as \( \varepsilon \) tends to zero to

\[
\Sigma_r^i (r,t,s) = \frac{A_{r,t,s}}{\sigma_{r,s,t}^2} p_{\sigma_{r,s,t}^2} (A_{r,t,s}) [K_H(t,r) - K_H(s,r)].
\]

In order to establish the convergence of the integrals in the variables \( s \) and \( t \), we will first decompose the interval \([0,t]\) into the disjoint union of \([r,t]\) and \([0,r]\). In this way we obtain

\[
L_\varepsilon - E(L_\varepsilon) = L_\varepsilon^{(1)} + L_\varepsilon^{(2)},
\]

where

\[
L_\varepsilon^{(1)} = -\sum_{i=1}^d \int_0^T \left( \int_0^T \int_0^t \Sigma_r^i (r,t,s) dsdt \right) dW^i_r,
\]

\[
L_\varepsilon^{(2)} = -\sum_{i=1}^d \int_0^T \left( \int_0^T \int_0^t \Sigma_r^i (r,t,s) dsdt \right) dW^i_r.
\]
and
\[ L_{\epsilon}^{(2)} = -\sum_{i=1}^{d} \int_{0}^{T} \left( \int_{r}^{T} \int_{r}^{r} \Sigma_{\epsilon}^{i}(r, t, s) ds dt \right) dW_{r}^{i}. \]

**Step 3** We claim that the random field \( \Sigma_{\epsilon}^{i}(r, t, s) \) is uniformly bounded on the set \( 0 < r < s < t \) by an integrable function not depending on \( \epsilon \). In fact, using the local nondeterminism property \((\text{LND})\), and Lemma 5 in the Appendix, we obtain the following lower bound for the conditional variance \( \sigma_{r,s,t}^{2} = \text{Var}(B_{t}^{H,i} - B_{s}^{H,i}|\mathcal{F}_{r}) \):
\[
\sigma_{r,s,t}^{2} \geq \text{Var}(B_{t}^{H,i} - B_{s}^{H,i}|\mathcal{F}_{s}) = \text{Var}(B_{t}^{H,i}|\mathcal{F}_{s}) \geq k^{2}(t-s)^{2H}. \tag{4.5}
\]
We can get rid off the factor \( A_{r,t,s}^{i} \) in the expression (4.4) of \( \Sigma_{\epsilon}^{i}(r, t, s) \) using the inequality
\[
p_{t}(x) \leq C\frac{t^{-d/2 + 1/2} - \frac{|x|^{2}}{t}}{|x|} \leq C\frac{t^{-d/2 + 1/2}}{|x|}, \tag{4.6}
\]
for some constant \( C > 0 \). In this way we obtain, using (4.5) and (4.6)
\[
|\Sigma_{\epsilon}^{i}(r, t, s)| \leq C (t-s)^{-Hd-H}|K_{H}(t, r) - K_{H}(s, r)|, \tag{4.7}
\]
for some constant \( C > 0 \), and by Lemma 7 in the Appendix we obtain that
\[
\int_{r}^{T} \int_{r}^{t} (t-s)^{-Hd-H}|K_{H}(t, r) - K_{H}(s, r)| ds dt \leq C(r^{1/2 - H} \lor 1). \tag{4.8}
\]
By dominated convergence we deduce the convergence of the integrals
\[
\lim_{\epsilon \to 0} \int_{r}^{T} \int_{r}^{t} \Sigma_{\epsilon}^{i}(r, t, s) ds dt = \int_{r}^{T} \int_{r}^{t} \Sigma^{i}(r, t, s) ds dt
\]
for all \((r, \omega) \in [0, T] \times \Omega\), and a second application of the dominated convergence theorem yields that \( \int_{r}^{T} \int_{r}^{t} \Sigma_{\epsilon}^{i}(r, t, s) ds dt \) converges in \( L^{2}([0, T] \times \Omega) \) to \( \int_{r}^{T} \int_{r}^{t} \Sigma^{i}(r, t, s) ds dt \). This implies the convergence of \( L_{\epsilon}^{(1)} \) to
\[
\sum_{i=1}^{d} \int_{0}^{T} \left( \int_{r}^{T} \int_{r}^{t} \Sigma^{i}(r, t, s) ds dt \right) dW_{r}^{i}
\]
in \( L^{2}(\Omega) \) as \( \epsilon \) tends to zero.

**Step 4** Consider now the case \( s < r < t \). In this case the integral of the term \( \Sigma_{\epsilon}^{i}(r, t, s) \) is not necessarily bounded, and in order to show the convergence of \( L_{\epsilon}^{(2)} \) we will prove uniform bounds in \( \epsilon \) for the expectation
\[ E \left( \int_{r}^{T} \int_{r}^{t} \left| \Sigma_{\varepsilon}^{i}(r, t, s) \right|^{p} ds dt \right), \] for some \( p > 1 \). We can write for \( s < r < t \), using the first inequality in (4.6)

\[
\left| \Sigma_{\varepsilon}^{i}(r, t, s) \right| \leq \frac{|A_{r, t, s}|}{(\varepsilon + \sigma_{r,s}^{2})} \exp \left( -\frac{|A_{r, t, s}|^{2}}{2(\varepsilon + \sigma_{r,s}^{2})} \right) |K_{H}(t, r)|
\]

for some constant \( C > 0 \). If \( s < r < t \), using the local nondeterminism property (LND) we obtain the following lower bound for the conditional variance \( \sigma_{r,s,t}^{2} \):

\[
\sigma_{r,s,t}^{2} = \text{Var}(B_{H,i}^{H,i} - B_{H,i}^{H,i} | F_{r}) = \text{Var}(B_{H,i}^{H,i} - B_{H,i}^{H,i} | F_{r}) \geq k_{2}(t - r)^{2H}. \tag{4.10}
\]

On the other hand, if \( s < r < t \)

\[
\sigma_{r,s,t}^{2} = \text{Var}(B_{H,i}^{H,i} - B_{H,i}^{H,i} | F_{r}) = \text{Var}(B_{H,i}^{H,i} - B_{H,i}^{H,i} | F_{r}) \leq \text{Var}(B_{H,i}^{H,i} - B_{H,i}^{H,i}) = (t - r)^{2H}. \tag{4.11}
\]

Also we will make use of the estimate (see [8])

\[
|K_{H}(t, r)| \leq k_{3}(t - r)^{H - \frac{1}{2} - H}. \tag{4.12}
\]

Substituting the estimates (4.10), (4.11) and (4.12) into (4.9) yields

\[
\left| \Sigma_{\varepsilon}^{i}(r, t, s) \right| \leq C |t - r|^{-H} \Psi_{\varepsilon}(r, t, s), \tag{4.13}
\]

for some constant \( C \), where

\[
\Psi_{\varepsilon}(r, t, s) = \left( \varepsilon + k_{2}(t - r)^{2H} \right)^{-\frac{d+1}{2}} (t - r)^{-\frac{1}{2}} \exp \left( -\frac{|A_{r, t, s}|^{2}}{4(\varepsilon + (t - r)^{2H})} \right). \tag{4.14}
\]

Notice that if \( Hd < \frac{1}{2} \), then \( |\Sigma_{\varepsilon}^{i}(r, t, s)| \) is uniformly bounded by the integrable function \( C_{r}^{\frac{3}{2} - H} (t - r)^{-Hd - \frac{1}{2}} \), and we can conclude as in Step 3. For this reason, we can assume that \( Hd \geq \frac{1}{2} \).

We claim that for some \( p > 1 \), we have

\[
\sup_{\varepsilon > 0} E \left( \int_{r}^{T} \int_{0}^{r} \Psi_{\varepsilon}^{p}(r, t, s) ds dt \right) < \infty. \tag{4.15}
\]

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To show this estimate we first derive a lower bound for the expectation of 

\[ |A_{r,t,s}^1|^2 = \left[ E(B_{r,t,s}^{H,1} - B_{r,t,s}^{H,1} | \mathcal{F}_r) \right]^2. \]

The main idea is to add and subtract the term \( B_{r,t,s}^{H,1} \), and then neglect the expectation \( E \left( \left( E(B_{r,t,s}^{H,1} | \mathcal{F}_r) - B_{r,t,s}^{H,1} \right)^2 \right) \).

This argument will be used later to find a lower bound for the covariance matrix of the vector \( \left( E(B_{t_i}^{H,1} - B_{s_i}^{H,1} | \mathcal{F}_r), 1 \leq i \leq n \right) \).

\[
E (|A_{r,t,s}^1|^2) = E \left( \left( E(B_{r,t,s}^{H,1} | \mathcal{F}_r) - B_{r,t,s}^{H,1} \right)^2 \right) + 2E \left( \left( E(B_{r,t,s}^{H,1} | \mathcal{F}_r) - B_{r,t,s}^{H,1} \right) \left( B_{r,t,s}^{H,1} - B_{s,t,s}^{H,1} \right) \right) + E \left( (B_{r,t,s}^{H,1} - B_{s,t,s}^{H,1})^2 \right) \\
\geq 2E \left( \left( B_{r,t,s}^{H,1} - B_{s,t,s}^{H,1} \right) \left( B_{r,t,s}^{H,1} - B_{s,t,s}^{H,1} \right) \right) + E \left( (B_{r,t,s}^{H,1} - B_{s,t,s}^{H,1})^2 \right) \\
= E \left( (B_{r,t,s}^{H,1} - B_{s,t,s}^{H,1})^2 \right) - E \left( \left( B_{r,t,s}^{H,1} - B_{r,t,s}^{H,1} \right)^2 \right) \\
= (t-s)^{2H} - (t-r)^{2H}.
\]

As a consequence, we obtain, assuming \( p < 2 \)

\[
E \left( \exp \left( -\frac{p |A_{r,t,s}^1|^2}{4(\varepsilon + (t-r)^{2H})} \right) \right) \leq C(\varepsilon + (t-r)^{2H}) \frac{4}{7} (t-r)^{2H} \alpha (t-s)^{-2H\beta}, \quad (4.16)
\]
where $\alpha + \beta = \frac{d}{2}$. Substituting (4.16) into (4.14) yields
\[
E \left( \int_r^T \int_0^r \Psi_\varepsilon(r,t,s) ds dt \right) \leq C \int_r^T \int_0^r \left( \varepsilon + (t-r)^2 H \right)^{-\frac{d+1}{2} + \frac{d}{2} - \alpha} \times (t-r)^{H-\frac{1}{2}} (t-s)^{-\frac{\beta 2 H}{2}} ds dt
\leq C \int_r^T \int_0^r (t-r)^{-p H - \frac{d}{2} + 2 H \beta} (t-s)^{-2 H \beta} ds dt.
\]
If $Hd > 1$, we can choose $\beta$ such that $2 H \beta > 1$, and integrating in the variable $s$, the above integral is bounded by
\[
C \int_r^T (t-r)^{-p H - \frac{d}{2} + 1} dt,
\]
which is finite if $p > 1$ satisfies $\left( Hd + \frac{1}{2} \right) p < 2$ (this is possible because $Hd + \frac{1}{2} < 2$). If $Hd \leq 1$, we can choose $\beta$ such that $2 H \beta = Hd - \delta$, for any $\delta > 0$, and we obtain the bound
\[
C \int_r^T (t-r)^{-p H - \frac{d}{2} + H d - \delta} dt,
\]
which is again finite if $p > 1$ is close to one, and $\delta > 0$ is small enough.

As a consequence, from (4.13) and (4.15), for any fixed $r \in [0,T]$, the family of functions $\{ \Sigma_\varepsilon(r,t,s), \varepsilon > 0 \}$, is uniformly integrable in $[r,T] \times [0,r]$, so it converges in $L^1([r,T] \times [0,r]) \times \Omega$ to $\Sigma(r,t,s)$, for $i = 1, \ldots, d$.
This implies the convergence of the integrals
\[
\lim_{\varepsilon \to 0} \int_r^T \int_0^r \Sigma_\varepsilon(r,t,s) ds dt = \int_r^T \int_0^r \Sigma(r,t,s) ds dt,
\]
for each fixed $r \in [0,T]$ in $L^1(\Omega)$.

Finally, we claim that this convergence also holds in $L^2([0,T] \times \Omega)$, and this implies the convergence of $L^2_{\varepsilon}$ to
\[
- \sum_{i=1}^d \int_0^T \left( \int_r^T \int_0^r \Sigma^i(r,t,s) ds dt \right) dW^i_r
\]
in $L^2(\Omega)$ as $\varepsilon$ tends to zero. To show the convergence in $L^2([0,T] \times \Omega)$ of the integrals
\[
Y^i_\varepsilon(r) = \int_r^T \int_0^r \Sigma^i_\varepsilon(r,t,s) ds dt
\]
it suffices to prove that

\[
\sup_{\varepsilon>0} \int_0^T E \left( |Y_{\varepsilon}^i(r)|^p \right) \, dr < \infty \quad (4.17)
\]

for all \( i = 1, \ldots, d \) and for some \( p > 2 \). The proof of (4.17) will be the last step in the proof of this theorem.

**Step 5** Suppose first that \( Hd < 1 \). Then, from (4.13) we obtain

\[
\int_0^T E \left( |Y_{\varepsilon}^i(r)|^p \right) \, dr \leq C \int_0^T E \left[ \left( \int_r^T \int_0^r \Psi_{\varepsilon}(r, t, s) \, ds \, dt \right)^p \right] \, r^{p(\frac{4}{d} - H)} \, dr.
\]

Using (4.14) and Minkowski’s inequality yields

\[
\left\| \int_r^T \int_0^r \Psi_{\varepsilon}(r, t, s) \, ds \, dt \right\|_p \leq \int_r^T \int_0^r \left( \varepsilon + h_2 (t - r)^{2H} \right)^{-\frac{d+1}{d}} (t - r)^{H - \frac{1}{2}} \times \left\| \exp \left( -\frac{|A_{r, t, s}|^2}{4(\varepsilon + (t-r)^{2H})} \right) \right\|_p \, ds \, dt, \quad (4.18)
\]

and from (4.16), choosing \( \beta = \frac{d}{2} \), we get

\[
\left\| \exp \left( -\frac{|A_{r, t, s}|^2}{4(\varepsilon + (t-r)^{2H})} \right) \right\|_p \leq C(\varepsilon + (t-r)^{2H})^\frac{d}{2r} (t-s)^{-\frac{Hd}{p}}. \quad (4.19)
\]

Substituting (4.19) into (4.18) yields

\[
\left\| \int_r^T \int_0^r \Psi_{\varepsilon}(r, t, s) \, ds \, dt \right\|_p \leq C \int_r^T (t - r)^{-Hd - \frac{1}{2} + \frac{Hd}{p}} \, dr,
\]

which is finite if we choose \( p > 2 \) such that \( p < \frac{2Hd}{2Hd - 1} \). Finally, if \( p \left( \frac{1}{2} - H \right) > -1 \) we complete the proof of (4.17) in the case \( Hd < 1 \).

In the case \( Hd \geq 1 \) we cannot apply the previous arguments, and the proof of (4.17) follows from the moment estimates given in Proposition 3.

**Remark 1** Theorem 2 also provides an alternative proof of the existence of the self-intersection local time in the case \( H \in \left[ \frac{1}{2}, \min\left( \frac{3}{2d}, \frac{2}{d+1} \right) \right) \), which was proved by Hu and Nualart in [9] in the general case \( Hd < \frac{3}{2} \). Notice that for \( d \geq 3 \), the condition \( H \in \left[ \frac{1}{2}, \min\left( \frac{3}{2d}, \frac{2}{d+1} \right) \right) \) is equivalent to \( 1 \leq Hd < \frac{3}{2} \), and for \( d = 2 \) we require \( H < \frac{2}{3} \), instead of the more general condition \( H < \frac{3}{4} \). 

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that guarantees the existence of the renormalized local time (see [16] and [9]).

The next Proposition contains the basic estimates on the moments of the quadratic variation of the stochastic integral appearing in the representation of the renormalized self-intersection local time.

**Proposition 3** Assume \(1 \leq Hd < \frac{3}{2}\). Set

\[
\Lambda_\varepsilon(r) = \int_r^T \int_0^r \Psi_\varepsilon(r, t, s) ds dt,
\]

where \(\Psi_\varepsilon(r, t, s)\) has been defined in (4.14). Then, for any integer \(n \geq 1\),

\[
E(\Lambda_\varepsilon^n(r)) \leq C^n (n!)^{\gamma},
\]

for some constant \(C > 0\), where

\[
\gamma > \left( \frac{1}{2} + H \right) \left( d - \frac{1}{2H} \right).
\]

**Proof.** Set \(g_\varepsilon(t-r) = (\varepsilon + k_2 (t-r)^{2H})^{-d+1 \over 2} (t-r)^{H-\frac{1}{2}}\). We have

\[
E(\Lambda_\varepsilon^n(r)) = E \left[ \left( \int_r^T \int_0^r g_\varepsilon(t-r) \exp \left( -\frac{|A_{r,s,t-i}|^2}{4(\varepsilon + (t_i-r)^{2H})} \right) ds dt \right)^n \right] = n! \int_{[r,T]}^n \int_{S_n} \prod_{i=1}^n g_\varepsilon(t_i-r) \times \left( E \left( \exp \left( -\sum_{i=1}^n \frac{|A_{r,s_i,t_i}|^2}{4(\varepsilon + (t_i-r)^{2H})} \right) \right) \right) dS_i dt_i,
\]

where \(S_n = \{0 < s_1 < \cdots < s_n < r\}\), \(s = (s_1, \ldots, s_n)\) and \(t = (t_1, \ldots, t_n)\).

We denote by \(Q\) the covariance matrix of the vector

\[
\left( E(B_{t_1}^{H,1} - B_{s_1}^{H,1} | F_r), \ldots, E(B_{t_n}^{H,1} - B_{s_n}^{H,1} | F_r) \right).
\]

Then, a well-known formula for Gaussian random variables implies that

\[
E \left[ \exp \left( -\sum_{i=1}^n \frac{|A_{r,s_i,t_i}|^2}{4(\varepsilon + (t_i-r)^{2H})} \right) \right] = \det \left( I + \frac{1}{2} Q D^{-1} \right)^{-\frac{1}{2}} = 2^{n \over 2} \prod_{i=1}^n \sqrt{a_i} \det (2D + Q)^{-\frac{1}{2}}. \tag{4.21}
\]
where $D$ denotes the $n \times n$ diagonal matrix with entries $a_i = \varepsilon + (t_i - r)^{2H}$.

As in the computation of $E(|A_{r,t,s}^1|^2)$, adding and substracting the term $B_r^{H,1}$ yields

$$Q_{ij} = E \left( E(B_{t_i}^{H,1} - B_{s_i}^{H,1}|F_r)E(B_{t_j}^{H,1} - B_{s_j}^{H,1}|F_r) \right)$$

$$= E \left( E(B_{t_i}^{H,1} - B_{s_i}^{H,1}|F_r)E(B_{t_j}^{H,1} - B_{s_j}^{H,1}|F_r) \right)$$

$$+ E \left( (B_{t_i}^{H,1} - B_{s_i}^{H,1})(B_{t_j}^{H,1} - B_{s_j}^{H,1}) \right) + E \left( (B_{t_i}^{H,1} - B_{r}^{H,1})(B_{r}^{H,1} - B_{s_j}^{H,1}) \right)$$

$$+ E \left( (B_{r}^{H,1} - B_{s_i})(B_{r}^{H,1} - B_{s_j}^{H,1}) \right)$$

$$= E \left( E(B_{t_i}^{H,1} - B_{r}^{H,1}|F_r)E(B_{t_j}^{H,1} - B_{r}^{H,1}|F_r) \right)$$

$$- E \left( (B_{t_i}^{H,1} - B_{r}^{H,1})(B_{t_j}^{H,1} - B_{r}^{H,1}) \right) + E \left( (B_{t_i}^{H,1} - B_{s_i}^{H,1})(B_{t_j}^{H,1} - B_{s_j}^{H,1}) \right).$$

Hence, we obtain

$$Q = R - N + M,$$

where

$$R_{ij} = E \left( E(B_{t_i}^{H,1} - B_{r}^{H,1}|F_r)E(B_{t_j}^{H,1} - B_{r}^{H,1}|F_r) \right),$$

$$M_{ij} = E \left( (B_{t_i}^{H,1} - B_{s_i}^{H,1})(B_{t_j}^{H,1} - B_{s_j}^{H,1}) \right),$$

$$N_{ij} = E \left( (B_{t_i}^{H,1} - B_{r}^{H,1})(B_{t_j}^{H,1} - B_{s_j}^{H,1}) \right).$$

All these matrices are nonnegative definite. The main idea will be to get rid off the matrix $R$, and control the matrix $N$ by its diagonal elements which are

$$N_{ii} = (t_i - r)^{2H}.$$ 

Indeed, the matrix $N$ is nonnegative definite and, hence, it satisfies the inequality

$$N \leq nD_N,$$

where $D_N$ is a diagonal matrix whose entries are $N_{ii}$. Therefore,

$$Q \geq -N + M \geq -nD_N + M,$$

and for any $1 \leq \delta < 2$, we can write

$$\det(2D + Q) \geq \det(2D + 2 \frac{2 - \delta}{n}Q) \leq \det(2D - (2 - \delta)D_N + 2 \frac{2 - \delta}{n}M).$$

(4.23)
The entries of the diagonal matrix $D_1 = 2D - (2 - \delta)D_N$ are the positive numbers
$$2\varepsilon + \delta(t_i - r)^{2H} > 0.$$ From (4.20), (4.21) and (4.23) we obtain
$$E(\Lambda^\alpha_\varepsilon(r)) \leq 2^{n^2}n! \int_{[r,T]^n} \int_{S_n} \prod_{i=1}^{n} (g_\varepsilon(t_i - r)a_i^d) \times \det(D_1 + \frac{2 - \delta}{n} M)^{-\frac{d}{2}} dsdt.$$ We have
$$\det(D_1 + \frac{2 - \delta}{n} M)^{-\frac{d}{2}} \leq \left(\frac{n}{2 - \delta}\right)^{n^2} (\det D_1)^{-\alpha} (\det M)^{-\beta},$$ where $\alpha + \beta = \frac{d}{2}$. Hence,
$$E(\Lambda^\alpha_\varepsilon(r)) \leq \left(\frac{n}{2 - \delta}\right)^{n^2} 2^{n^2}n! \int_{[r,T]^n} \int_{S_n} \prod_{i=1}^{n} (g_\varepsilon(t_i - r)a_i^d (2\varepsilon + \delta(t_i - r)^{2H})^{-\alpha}) \times (\det M)^{-\beta} dsdt.$$ Then,
$$g_\varepsilon(t_i - r)a_i^d (2\varepsilon + 2(t_i - r)^{2H})^{-\alpha} \leq C(t_i - r)^{-\frac{1}{2} - 2H\alpha},$$ for some constant $C > 0$. Thus
$$E(\Lambda^\alpha_\varepsilon(r)) \leq C n^{3n}n! \int_{[r,T]^n} \int_{S_n} \prod_{i=1}^{n} (t_i - r)^{-\frac{1}{2} - 2H\alpha} (\det M)^{-\beta} dsdt,$$ (4.24) for some constant $C > 0$. Applying Lemma 5 in the Appendix and the local nondeterminism property of the fractional Brownian motion we obtain
$$\det M = \Var(B_{tn} - B_{sn}) \Var(B_{tn-1} - B_{sn-1} | B_{tn} - B_{sn}) \times \cdots \times \Var(B_{t1} - B_{s1} | B_{t2} - B_{s2}, \ldots, B_{tn} - B_{sn}) = (t_n - s_n)^{2H} \Var(B_{sn-1} | B_{tn-1}, B_{tn}, B_{sn}) \times \cdots \times \Var(B_{s1} | B_{t1}, \ldots, B_{tn}, B_{s1}, \ldots, B_{sn-1}) \geq k_2^{n-1} (r - s_n)^{2H} ((s_n - s_{n-1}) \wedge s_{n-1})^{2H} \cdots ((s_2 - s_1) \wedge s_1)^{2H}. (4.25)$$
Substituting (4.25) into (4.24), and choosing \( \alpha \) such that \( \alpha < \frac{1}{4H} \) (this is possible because \( Hd \geq 1 \)) yields

\[
E (\Lambda^n_\varepsilon(r)) \leq C^n n^{\beta n} n! \int_{S^n} \left[ (r - s_n) ((s_n - s_{n-1}) \wedge s_{n-1}) \cdots ((s_2 - s_1) \wedge s_1) \right]^{-23H} ds.
\]

Finally, by Lemma 8 in the Appendix we obtain

\[
E (\Lambda^n_\varepsilon(r)) \leq \frac{C^n n^{\beta n} n!}{\Gamma(n(1 - 2H\beta) + 1)}.
\]

Notice that \( \beta = \frac{d}{2} - \alpha \geq \frac{d}{2} - \frac{1}{4H} \). And hence,

\[
E (\Lambda^n_\varepsilon(r)) \leq C^n (n!)^{2H\beta},
\]

where

\[
\beta(1 + 2H) > \frac{d}{2} - \frac{1}{4H} + Hd - \frac{1}{2} = \left( \frac{1}{2} + H \right) \left( d - \frac{1}{2H} \right).
\]

This concludes the proof. ■

Using the above proposition we can deduce the following integrability results for the renormalized self-intersection local time.

**Theorem 4** Assume \( \frac{1}{d} \leq H < \min \left( \frac{2}{d}, \frac{3}{d^2} \right) \). For any integer \( p < \frac{1}{2} \left( \left( \frac{1}{2} + H \right) \left( d - \frac{1}{2H} \right) \right)^{-1} \) we have

\[
E(\exp |\tilde{L}|^p) < \infty.
\]

**Proof.** Taking into account Lemma 8 in the Appendix, it suffices to show that

\[
E \left( \exp \left( \langle \tilde{L} \rangle^p \right) \right) < \infty,
\]

where

\[
\langle \tilde{L} \rangle = \sum_{i=1}^{d} \int_{0}^{T} \left( \int_{T}^{t} \int_{0}^{t} \Sigma^i(r, t, s) ds dt \right)^2 dr.
\]

As in the proof of Theorem 2 we make the decomposition

\[
\int_{T}^{T} \int_{0}^{t} \Sigma^i(r, t, s) ds dt = \int_{T}^{T} \int_{r}^{t} \Sigma^i(r, t, s) ds dt + \int_{T}^{T} \int_{r}^{T} \Sigma^i(r, t, s) ds dt + \int_{T}^{T} \int_{r}^{T} \Sigma^i(r, t, s) ds dt.
\]

From (4.7) and (4.8) we know that

\[
\left| \int_{r}^{T} \int_{r}^{t} \Sigma^i(r, t, s) ds dt \right| \leq C(r\frac{1}{2} - H \vee 1).
\]

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Therefore, applying Fatou’s lemma and the estimate (4.13) yields

\[
E(\exp(\langle \tilde{L} \rangle^p)) \leq CE \left( \exp \left( \left| \sum_{i=1}^{d} \int_0^T \left( \int_0^T \int_0^r \Sigma^i(r, t, s) dsdt \right)^2 dr \right|^p \right) \right)
\]

\[
\leq C \liminf_{\epsilon \downarrow 0} E \left( \exp \left( \left| \sum_{i=1}^{d} \int_0^T \left( \int_0^T \int_0^r \Sigma^i_\epsilon(r, t, s) dsdt \right)^2 dr \right|^p \right) \right)
\]

\[
\leq C \liminf_{\epsilon \downarrow 0} E \left( \exp \left( C \int_0^T r^{1-2H} \left( \int_0^r \int_0^T \psi_\epsilon(r, t, s) dsdt \right)^2 dr \right)^p \right).
\]

Applying Hölder and Jensen inequalities we obtain

\[
E(\exp(\langle \tilde{L} \rangle^p)) \leq C \liminf_{\epsilon \downarrow 0} \int_0^T r^{1-2H} E \left( \exp \left( C \left( \int_0^T \int_0^r \psi_\epsilon(r, t, s) dsdt \right)^{2p} \right) \right) dr.
\]

Finally,

\[
E \left( \exp \left( C \left( \int_0^T \int_0^r \psi_\epsilon(r, t, s) dsdt \right)^{2p} \right) \right) = \sum_{n=1}^{\infty} \frac{C^n}{n!} E \left( \left( \int_0^T \int_0^r \psi_\epsilon(r, t, s) dsdt \right)^{2np} \right)
\]

\[
\leq \sum_{n=1}^{\infty} \frac{C^n}{n!} ([2np] + 1)!^{\gamma},
\]

and it suffices to apply Proposition 3 to conclude the proof.

**Remark 2** The exponent \( p_0 = \frac{1}{2} \left[ \frac{1}{2} + H \right] \left( d - \frac{1}{2H} \right)^{-1} \) is not optimal. For instance, if \( Hd = 1 \), then \( p_0 = 1 \), and we know that for \( Hd < 1 \), then \( p_0 = \frac{d}{1-H} \). In particular, if \( H = \frac{1}{2} \) and \( d = 2 \) we obtain \( p_0 = \frac{1}{2} \), and we know that in this case the critical exponent is \( p_0 = 1 \). The lack of optimality is due to the factor \( n \) in the estimation of the positive definite matrix \( N \) by its diagonal elements given in (4.22). Without this factor \( n \) we would get the critical exponent \( \frac{1}{2d-1} \), but our method does not allow to get this value.
Remark 3 In the case of the planar Brownian motion $B = \{B_t, t \geq 0\}$ (that is, $d = 2$, and $H = \frac{1}{2}$), formula (4.2) yields
\[
\tilde{L} = -\frac{1}{2\pi} \sum_{i=1}^{2} \int_0^T \left( \int_r^T \int_0^r \frac{B_t^i - B_s^i}{(t-r)^2} \exp \left( -\frac{|B_t - B_s|^2}{2(t-r)} \right) dsdt \right) dB_t^i.
\]

The quadratic variation of this stochastic integral is
\[
\langle \tilde{L} \rangle = \frac{1}{4\pi^2} \sum_{i=1}^{2} \int_0^T \left( \int_r^T \int_0^r \frac{B_t^i - B_s^i}{(t-r)^2} \exp \left( -\frac{|B_t - B_s|^2}{2(t-r)} \right) dsdt \right)^2 dr
\]
\[
\leq \frac{1}{4\pi^2} \int_0^T \left( \int_r^T \int_0^r \frac{|B_t - B_s|}{(t-r)^2} \exp \left( -\frac{|B_t - B_s|^2}{2(t-r)} \right) dsdt \right)^2 dr
\]
\[
= \frac{1}{\pi^2} \int_0^T \left( \int_r^T \frac{1}{|B_t - B_s|} \exp \left( -\frac{|B_t - B_s|^2}{2(T-r)} \right) ds \right)^2 dr
\]
\[
\leq \frac{1}{\pi^2} \int_0^T \left( \int_0^r \frac{ds}{|B_t - B_s|} \right)^2 dr.
\]

From Itô’s calculus we know that
\[
\int_0^r \frac{ds}{|B_t - B_s|} = \frac{1}{d-1} (X_r - b_r),
\]
where $X_r$ has the law of the modulus of a $d$-dimensional Brownian motion at time $r$ (Bessel process), and $b_r$ has a normal $N(0, r)$ law. We can write
\[
\exp \left( \lambda \langle \tilde{L} \rangle \right) \leq \frac{1}{T} \int_0^T \exp \left( \frac{T\lambda}{\pi^2} \left( \int_0^r \frac{ds}{|B_t - B_s|} \right)^2 \right) dr,
\]
which clearly imply the existence of some $\lambda_0$ such that $E \left( \exp \left( \lambda \langle \tilde{L} \rangle \right) \right) < \infty$ for all $\lambda < \lambda_0$. From Lemma 6 we get that there exists $\beta_0$ such that $E \left( \exp \left( \beta \langle \tilde{L} \rangle \right) \right) < \infty$ for all $\beta < \beta_0$. This method does not allows us to obtain the critical exponent, just the existence of exponential moments.

Remark 4 The above results remain true if we replace the fractional Brownian motion with Hurst parameter $H$, by an arbitrary centered Gaussian process of the form (3.1) satisfying the local nondeterminism property (LND) and following properties:

(C1) For any $s, t \in [0,T], s < t$, there exist constants $k_3$ and $k_4$ such that
\[ k_3(t-s)^{2H} \leq E(|B_t^i - B_s^i|^2) \leq k_4(t-s)^{2H}. \]
(C2) The kernel $K(t,s)$ satisfies the estimates

$$|K(t,s)| \leq k_5(t-s)^{H-\frac{1}{2}s^{\frac{1}{2}}-H},$$

for all $s < t$, and

$$\int_r^T \int_r^t (t-s)^{-Hd-H} |K(t,r) - K(s,r)| ds dt \leq \psi(r),$$

where $\int_0^T \psi(r)^2 dr < \infty$.

5 Appendix

In this Appendix we will first state and prove some elementary lemmas. The first one is well-known.

**Lemma 5** Suppose that $\mathcal{G}_1 \subset \mathcal{G}_2$ are two $\sigma$-fields contained in $\mathcal{F}$. Then, for any square integrable random variable $F$ we have

$$\text{Var}(F|\mathcal{G}_1) \geq \text{Var}(F|\mathcal{G}_2).$$

Let $M = \{M_t, t \geq 0\}$ be a continuous local martingale such that $M_0 = 0$. Then, the following maximal exponential inequality is well-known

$$P\left(\sup_{0 \leq t \leq T} |M_t| \geq \delta, \langle M \rangle_T < \rho \right) \leq 2 \exp\left(-\frac{\delta^2}{2\rho}\right).$$

As a consequence of this inequality we can obtain exponential moments for $M_T$ from exponential moments of the quadratic variation $\langle M \rangle_T$

**Lemma 6** Suppose that for some $\alpha > 0$ and $p \in (0,1]$ we have $E(e^{\alpha \langle M \rangle_T^p}) < \infty$. Then,

(i) if $p = 1$, for any $\lambda < \sqrt{\frac{\alpha}{2}}$, $E(e^{\lambda|M_T|}) < \infty$, and

(ii) if $p < 1$, $E(e^{\lambda|M_T|^p}) < \infty$ for all $\lambda > 0$. 

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Proof. Set \( X = |M_T|^p \). For any constant \( c > 0 \) we can write
\[
E(e^{\lambda X}) = \int_0^\infty P(X \geq y) e^{\lambda y} dy
\]
\[
= \int_0^\infty [P(X \geq y, \langle M \rangle_T^p < cy) + P(X \geq y, \langle M \rangle_T^p \geq cy)] e^{\lambda y} dy
\]
\[
\leq \int_0^\infty 2 \exp \left( -\frac{y^p}{2c^p} \right) e^{\lambda y} dy + \int_0^\infty P \left( \frac{\langle M \rangle_T^p}{c} \geq y \right) e^{\lambda y} dy
\]
\[
= \int_0^\infty 2 \lambda \exp \left( \lambda y - \frac{y^p}{2c^p} \right) dy + E(e^{\lambda \langle M \rangle_T^p}).
\]
Then it suffices to choose \( c = \frac{\lambda}{\lambda} \alpha \) to complete the proof. \( \blacksquare \)

The next two results are technical lemmas used in the paper.

Lemma 7 Suppose that \( H < \min \left( \frac{d}{d+1}, \frac{3}{2d} \right) \). Then, we have
\[
\int_r^T \int_r^t (t-s)^{-Hd-H} |K_H(t,r) - K_H(s,r)| ds dt \leq C \left( r^{\frac{1}{2} - H} \lor 1 \right),
\]
for some constant \( C \).

Proof. We know that
\[
\frac{\partial K_H}{\partial t}(t,s) = c_H \left( H - \frac{1}{2} \right) \left( \frac{t}{s} \right)^{H-\frac{1}{2}} (t-s)^{H-\frac{3}{2}}.
\]
Then
\[
I := \int_r^T \int_r^t (t-s)^{-Hd-H} |K_H(t,r) - K_H(s,r)| ds dt
\]
\[
\leq C \int_r^T \int_r^t \int_s^t (t-s)^{-Hd-H} \left( \frac{\theta}{r} \right)^{H-\frac{1}{2}} (\theta - r)^{H-\frac{3}{2}} d\theta ds dt.
\]
If \( H < \frac{1}{2} \), then, \( \left( \frac{\theta}{r} \right)^{H-\frac{1}{2}} \leq 1 \), and if \( H > \frac{1}{2} \), then \( \left( \frac{\theta}{r} \right)^{H-\frac{1}{2}} \leq Cr^{\frac{3}{2} - H} \). Hence, the above integral is bounded by
\[
C \left( r^{\frac{1}{2} - H} \lor 1 \right) \int_r^T \int_r^t \int_s^t (t-s)^{-Hd-H} (\theta - r)^{H-\frac{3}{2}} d\theta ds dt.
\]
From the decomposition

\[ \frac{3}{2} - H = \alpha + \beta, \]
\[ Hd + H = \gamma + \delta, \]

we obtain

\[ \int_r^T \int_r^t \int_s^t (t-s)^{-Hd-H} (\theta - r)^{H\frac{3}{2}} d\theta ds dt \]
\[ = \int_r^T \int_r^t \int_s^t (s-r)^{-\alpha} (\theta - s)^{-\beta} (t - \theta)^{-\gamma} d\theta ds dt. \]

Finally, it suffices to show the parameters \( \alpha, \beta, \gamma \) and \( \delta \) in such a way that \( \alpha < 1, \delta < 1 \) and \( \beta + \gamma < 1 \). This leads to the condition

\[ \frac{1}{2} + Hd < \min(1, \frac{3}{2} - H) + \min(1, Hd + H), \]

which is satisfied if \( H < \min\left(\frac{2}{d+1}, \frac{3}{2d}\right) \).

**Lemma 8** Let \( a < 1 \). Fix an interval \([0,T] \). For each integer \( n \geq 1 \) we have

\[ \int_{\Delta_n(T)} \left[ ((T - s_n) \land s_n) ((s_n - s_{n-1}) \land s_{n-1}) \cdots ((s_2 - s_1) \land s_1) \right]^{-a} ds \]
\[ \leq \frac{T^{n(1-a)}}{\Gamma(n(1-a) + 1)} C^n, \] (5.1)

where \( \Delta_n(T) = \{0 < s_1 < \cdots < s_n < T\} \)

**Proof.** We proceed by induction on \( n \). For \( n = 1 \) we can write

\[ \int_0^T ((T - s_1) \land s_1)^{-a} ds_1 = \int_0^{T_1} s_1^{-a} ds_1 + \int_{T_1}^T (T - s_1)^{-a} ds_1 \]
\[ = \frac{2}{1-a} \left( \frac{T}{2} \right)^{1-a}, \]

which implies (5.1) with \( C = \frac{\Gamma(2-a)}{1-a} 2^a \).
Suppose that the result holds for \( n - 1 \). Then,

\[
I_n = \int_{\Delta_n(T)} \left[ (T - s_n) \land s_n \right] \left( (s_n - s_{n-1}) \land s_{n-1} \right) \cdots \left( (s_2 - s_1) \land s_1 \right) ds
\]

\[
= \int_0^T \left( (T - s_n) \land s_n \right)^{-a} ds
\]

\[
\times \left( \int_{\Delta_{n-1}(s_n)} \left[ (s_n - s_{n-1}) \land s_{n-1} \right] \cdots \left( (s_2 - s_1) \land s_1 \right) ds_1 \cdots ds_{n-1} \right) ds_n.
\]

By the induction hypothesis we can write

\[
I_n \leq \frac{C^{n-1}}{\Gamma(n-a)} \int_0^T \left( (T - s_n) \land s_n \right)^{-a} s_n^{(n-1)(1-a)} ds_n
\]

\[
= \frac{C^{n-1}}{\Gamma((n-1)(1-a) + 1)} \times \left( \int_0^T s_n^{(n-1)(1-a) - a} ds_n + \int_T^{T_2} (T - s_n)^{-a} s_n^{(n-1)(1-a)} ds_n \right)
\]

\[
\leq \frac{C^{n-1}}{\Gamma(n(1-a) + a)} \times \left( \frac{1}{n(1-a)} \left( \frac{T}{2} \right)^{n(1-a)} + T^{n(1-a)} \int_0^1 (1 - x)^{-a} x^{(n-1)(1-a)} dx \right)
\]

\[
\leq \frac{T^{n(1-a)} C^{n-1}}{\Gamma(n(1-a) + a)} \left( \frac{1}{n(1-a)} + \frac{\Gamma(1-a) \Gamma((n-1)(1-a) + 1)}{\Gamma(n(1-a) + 1)} \right)
\]

\[
= T^{n(1-a)} C^{n-1} \left( \frac{1}{n(1-a) \Gamma(n(1-a) + a)} + \frac{\Gamma(1-a)}{\Gamma(n(1-a) + 1)} \right).
\]

Using the relation \( \Gamma(n + 1) = n \Gamma(n) \) we obtain

\[
n(1-a) \Gamma(n(1-a) + a) \geq n(1-a) \Gamma(n(1-a)) = \Gamma(n(1-a) + 1),
\]

and, as a consequence

\[
I_n \leq T^{n(1-a)} C^{n-1} (1 + \Gamma(1-a)) \frac{1}{\Gamma(n(1-a) + 1)}.
\]

and it suffices to take \( C \geq \max \left( \frac{\Gamma(2-a) 2^a}{1-a}, 1 + \Gamma(1-a) \right) \). \( \blacksquare \)
References


