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The Linear Complexity of Whiteman’s Generalized Cyclotomic Sequences of Period $p^{m+1}q^{n+1}$

Li Qin Hu, Qin Yue, and Minzhong Wang, Member, IEEE

Abstract—In this paper, we mainly get three results. First, let $p$, $q$ be distinct primes with $\gcd((p-1)p,(q-1)q) = \gcd(p-1,q-1) = 1$; we give a method to compute the linear complexity of Whiteman’s generalized cyclotomic sequences of period $p^{m+1}q^{n+1}$. Second, if $e = 4$, we compute the exact linear complexity of Whiteman’s generalized cyclotomic sequences. Third, if $e = 4$ and $a$ is a common primitive root of both $p$ and $q$, then $2 \in \langle a \rangle$, which is a subgroup of the multiplicative group $\mathbb{Z}_{pq}^*$, if and only if Whiteman’s generalized cyclotomic numbers of order 4 depend on the decomposition $e = 4$. We have two relations (see [9])

$$Z_{p^{m+1}} = \bigcup_{i=0}^{m+1} p^{i}Z_{p^{i+1}}^{*}, \quad Z_{q^{n+1}} = \bigcup_{i=0}^{n+1} q^{i}Z_{q^{i+1}}^{*}$$

where $p^{m+1}Z_{p^{m+1}}^{*} = \{0\}$ and $q^{n+1}Z_{q^{n+1}}^{*} = \{0\}$.

Now we investigate a factorization of $Z_N$. Let $d := \gcd(N,g)$ denote the multiplicative order of $g$ modulo $N$; then

$$d = \gcd(N,g) = \text{lcm}(\gcd(p^{m+1},g)), \quad \text{ord}_{g^{m+1}}(g) = \frac{(p-1)(q-1)p^{m+1}}{e}.$$ 

Then, the subgroup $D_2 = \langle g \rangle$ of the multiplicative group $\mathbb{Z}_N^{*}$ is of order $d$.

Let $y$ be an integer satisfying the simultaneous congruences

$$y \equiv g \pmod{p^{m+1}}, \quad y \equiv 1 \pmod{q^{n+1}}.$$ 

We define generalized cyclotomic classes analogous to [15]

$$D_k = \{g^k y^s : s = 0, 1, \ldots, d-1\}, \quad k = 0, 1, \ldots, e-1.$$ 

Then, we get

$$Z_N = \bigcup_{k=0}^{e-1} D_k.$$ 

Lemma 1.1:

$$Z_N = \bigcup_{i=0}^{m+1} \bigcup_{j=0}^{n+1} q^{i} Z_{q^{j+1}}^{*}$$

where the multiplicative order is performed in the ring $Z_N$ and $p^{m+1}q^{n+1}Z_{N}^{*} = \{0\}$.

Proof: It is clear from [5, Lemma 12].

If $i \leq m$, $j \leq n$, then

$$p^i q^j D_k = \{p^i q^j a \mid a \in D_k\}, \quad k = 0, \ldots, d-1.$$ 

Hence

$$Z_N = \bigcup_{i=0}^{m} \bigcup_{j=0}^{n} \bigcup_{k=0}^{e-1} q^{i} D_k \bigcup_{i=0}^{m} \bigcup_{j=0}^{n+1} Z_{q^{j+1}}^{*} \bigcup_{i=0}^{m+1} \bigcup_{j=0}^{n+1} q^{i} Z_{q^{j+1}}^{*}.$$ 

(1.4)

For convenience, we give a definition.
Definition 1.2: The assumptions are as above. Define subsets of $Z_N$

\[D_k^{i,j} = \begin{cases} p^i q^j D_k & \text{if } i \leq m, j \leq n, 0 \leq k \leq e - 1 \\ p^i q^{j+1} Z_N^* & \text{if } i \leq m, j = n + 1, k = 0 \\ p^{m+1} q^i Z_N^* & \text{if } i = m + 1, j \leq n, k = 0 \\ \{0\} & \text{if } i = m + 1, j = n + 1, k = 0. \end{cases} \]

So $D_0^{(i,n+1)} = p^i q^{n+1} Z_N^*$ for $i \leq m$, $D_0^{(m+1,j)} = p^{m+1} q^j Z_N^*$ for $j \leq n$, and $D_0^{(m+1,n+1)} = \{0\}$, and index sets for $0 \leq i \leq m + 1$ and $0 \leq j \leq n + 1$ are given as

\[I_i j \subset \begin{cases} \{0, 1, \ldots, e - 1\}, & \text{if } i \leq m, j \leq n \\ \{0\}, & \text{otherwise}. \end{cases} \]

Suppose that $\Omega = \bigcup_{i=0}^{m+1} \bigcup_{j=0}^{n+1} \bigcup_{k \in I_{i,j}} D_k^{(i,j)}$. We can define the generalized cyclotomic binary sequence $s$ of period $N$ as

\[s_i = \begin{cases} 1, & \text{if } i \pmod{N} \in \Omega, \text{ for all } i \geq 0. \end{cases} \]

(1.5)

Define

\[s(x) = s_0 + s_1 x + \cdots + s_{N-1} x^{N-1} = \sum_{i \in \Omega} p^i \]

(1.6)

as the characteristic polynomial of the sequence $s$. It is well known that the minimal polynomial of the binary sequence $s$ of period $N$ is given by

\[m(x) = \frac{x^N - 1}{\text{gcd}(x^N - 1, s(x))} \]

and that the linear complexity of $s$ is given by

\[L = N - \text{deg}(\text{gcd}(x^N - 1, s(x))). \]

In this paper, there are three main results. First, we show a method to compute the linear complexity of the aforementioned generalized cyclotomic sequences of period $p^{m+1} q^{n+1}$. Second, if $\gcd(p^m(p - 1), q^n(q - 1)) = e = 4$, we easily calculate the linear complexity of the aforementioned generalized cyclotomic sequences. Third, if $p \equiv q \equiv 5 \pmod{8}$, $\gcd(p - 1, q - 1) = 4$, and we fix a common primitive root $g$ of both $p$ and $q$, then $2 \in H_0 = \{g\}$, which is a subgroup of the multiplicative group $Z_{p^2}$, if and only if Whiteman's generalized cyclotomic numbers of order $4$ depend on the decomposition $pq = a^2 + 4b^2$ for $4b$.

II. GENERALIZED CYCLOTOMIC SEQUENCES OF PERIOD $p^{m+1} q^{n+1}$

In this section, we generalize the results from [9] and give a formula for the linear complexity of the generalized cyclotomic binary sequence $s$ of period $N = p^{m+1} q^{n+1}$ as defined in (1.5).

Lemma 2.1:

1) Let $e = \gcd(p - 1, q - 1)$ and $R = \frac{(p - 1)(q - 1)}{e}$; then

\[D_0 = \{y^k + hxq | k = 0, 1, \ldots, R - 1; h = 0, 1, \ldots, p^m q^n - 1\}. \]

If $i \leq m$ and $j \leq n$, then see the first equation shown at the bottom of the page.

2) If $i < m$ and $j = n + 1$, then see the second equation shown at the bottom of the page.

3) If $i = m + 1$ and $j \leq n$, then see the third equation shown at the bottom of the page.

Proof:

1) Since $g$ is a common primitive root of both $p^{m+1}$ and $q^{n+1}$, $g$ is also a common primitive root of both $p$ and $q$. Hence, in the multiplicative group $Z_{pq}$, ord$_{pq}(g) = (p - 1)(q - 1)/e = R$, so $g^k \neq g^{k'} \pmod{pq}$ for $0 \leq k \neq k' \leq R - 1$.

If $e \equiv 0 \pmod{pq}$ for $0 \leq k \leq R - 1$, then $p \nmid a$ and $q \nmid a$, so $a \notin Z_N^*$. Suppose that $a \in D_0$, where $r \in \{0, 1, \ldots, e - 1\}$; then $a = y^r g^{k_1} \pmod{p^{m+1} q^{n+1}}$. So $a \equiv y^r g^{k_1} \pmod{p^n}$ and $a \equiv g^{k_1} \pmod{q^n}$.

Hence, $p - 1 + k_1 - k$ and $q - 1 + k_1 - k$. Then by $\gcd(p - 1, q - 1) = e$, $r, r, s = 0$ and $a \in D_0$. Moreover, if $\langle k, h \rangle \neq \langle k', h' \rangle$ for $0 \leq k, k' \leq R - 1$ and $0 \leq h, h' \leq p^m q^n - 1$, then $q^{k} + h^2 q^{k'} \pmod{p^{m+1} q^{n+1}}$. By $|D_0| = p^m q^n (p - 1)(q - 1)/e$, this proves the first part of 1). Similarly, we can prove the second part of 1).
2) Since \( \eta : p^i q^{n+1} Z_q^0 \to p^i Z_{p^{m+1}}^0 \times \{0\} \) is a bijective map and by [9] \( p^i Z_{p^{m+1}}^0 = \{p^i(g^k + hp) | k = 0, 1, \ldots, p - 2, h = 0, 1, \ldots, p^m \} \), we prove 2). Similarly, we can prove 3). \( \square \)

Let \( \alpha \) be a primitive \( N \)th root of unity in an extension of \( GF(2) \). Then by Blahut’s theorem, the linear complexity of the sequence \( s \) defined as (1.5) is

\[
L = N - \{t | s(\alpha^t) = 0, t = 0, 1, \ldots, N - 1\} \quad (2.1)
\]

where \( s(x) = s_0 + s_1 x + \cdots + s_{N-1} x^{N-1} \) is the characteristic polynomial of the sequence \( s \). So the linear complexity of the sequence \( s \) reduces to counting the number of roots of \( s(x) \) in the set \( \{\alpha^t | t = 0, 1, \ldots, N - 1\} \).

To explore the roots of the polynomial \( s(x) \), we need the following auxiliary polynomials for \( i \leq m, j \leq n \)

\[
s^{i,j}(x) = \sum_{l \in D_i^{(j), (j)}} x^l = \sum_{k=0}^{p-1} x^{p^i} q^{p^j} \sum_{h=0}^{p^m-1} x^{p^{m+1} q^{p^m} h} \quad (2.2)
\]

\[
s_{i+1}(x) = \sum_{l \in D_i^{(j), (j+1)}} x^l = \sum_{k=0}^{p-2} x^{p^i} q^{q^{p^j}} \sum_{h=0}^{p^m-1} x^{p^{m+1} q^{p^m} h} \quad (2.3)
\]

\[
s^{i,j+1}(x) = \sum_{l \in D_i^{(j), (j+1)}} x^l = \sum_{k=0}^{q-2} x^{p^i} q^{q^{p^j} q^{p^m}} \sum_{h=0}^{p^m q^{p^m} h} \quad (2.4)
\]

Since \( D_i^{(j), (j+1)} = q^k D_0^{(i,j)} \) for \( i \leq m \) and \( j \leq n \), by the definition of \( s \) as (1.5)

\[
s(\alpha^l) = \sum_{i=0}^{m} \sum_{j=0}^{n} \sum_{k \in I_{i,j}} s_{i,j}(\alpha^{k^{n+m}}) + \sum_{i=0}^{m} \sum_{j=0}^{n} \sum_{k \in \hat{I}_{i,j}} s_{i+1,j}(\alpha^k)
\]

\[
+ \sum_{i=0}^{m} \sum_{j=0}^{n} \sum_{k \in \hat{I}_{i+1,j}} s_{i,j+1}(\alpha^k) \quad (2.3)
\]

where \( \sum_{k \in I_{i,j}} s_{i,j}(\alpha^{k^{n+m}}) = 0 \) if \( I_{i,j} = \emptyset \), \( i \leq m, j \leq n \), and for \( i \leq m \) and \( j \leq n + 1 \),

\[
\delta_{i,j} = \begin{cases} 1, & \text{if } I_{i,j} = \{0\}, \\ 0, & \text{if } I_{i,j} = \emptyset, \end{cases} \quad \delta_{i,j} = \begin{cases} 1, & \text{if } I_{i+1,j} = \{0\}, \\ 0, & \text{if } I_{i+1,j} = \emptyset. \end{cases}
\]

**Lemma 2.2:** For integers \( h \) and \( t \), we have equalities

\[
s_{i,j}(\alpha^{k^{n+m}}) = s_{i,j}(\alpha^k), i \leq m + 1, j \leq n + 1.
\]

**Proof:** Since \( g^h D_i^{(i,j)} = D_0^{(i,j)} \) for \( i \leq m + 1 \) and \( j \leq n + 1 \), we prove Lemma 2.2. \( \square \)

**Lemma 2.3:** Let \( p \nmid t \) and \( q \nmid t \). Suppose that \( i \leq m, j \leq n \), and \( i + j \leq m + n - 1 \). Then, \( s_{i,j}(\alpha^t) = 0 \).

**Proof:** Since \( \alpha \) is a \( p^{m+1} q^{n+1} \)th primitive root of unity, we have

\[
0 = \alpha^{p^{m+1} q^{n+1} - 1} = (\alpha^{p^iq^j} - 1)(1 + \alpha^{p^iq^j} + \alpha^{2p^iq^j} + \cdots + \alpha^{(p^m-1)q^{n+1} - 1} \alpha^{p^iq^j})
\]

for \( i + j \leq m + n + 1 \). Hence

\[
\sum_{h=0}^{p^{m+1} q^{n+1} - 1} \alpha^{hp^iq^j} = 0. \quad (2.5)
\]

Since \( p \nmid t \) and \( q \nmid t \), \( \alpha^t \) is also a \( p^{m+1} q^{n+1} \)th primitive root of unity. If \( i + j \leq m \), \( j \leq n \), and \( i + j \leq m + n + 1 \), then by (2.2) and (2.5), \( s_{i,j}(\alpha^t) = 0 \).

**Lemma 2.4:** Let \( p \nmid t \) and \( q \nmid t \). Suppose that \( u \leq m + 1 \) and \( v \leq n + 1 \).

1) Suppose that \( i \leq m, j \leq n \); then

\[
s_{i,j}(\alpha^{tp^q}) = \begin{cases} 1, & \text{if } u = m - i \text{ or } v = n - j, \\ 0, & \text{if } u \neq m - i. \end{cases}
\]

2) Suppose that \( i \leq m, j = n + 1 \); then

\[
s_{i,n+1}(\alpha^{tp^q}) = \begin{cases} 1, & \text{if } u = m - i, \text{ or } v = n - j, \\ 0, & \text{if } u = m - i \text{ or } v = n - j. \end{cases}
\]

3) Suppose that \( i - m + 1, j \leq n \), then

\[
s_{m+1,j}(\alpha^{tp^q}) = \begin{cases} 1, & \text{if } u = m - i \text{ or } v = n - j, \\ 0, & \text{if } u \neq m - i \text{ or } v \neq n - j. \end{cases}
\]

**Proof:**

1) Suppose that \( i \leq m, j \leq n \). Without loss of generality, we assume that \( u < n - j \); then for any \( b \in \{0, 1, \ldots, p^m - n - q^{n-m} - 1\} \), there exist \( p^m q^v \) elements \( h \in \{0, 1, \ldots, p^m - i q^m - j - 1\} \) such that \( b = h \mod p^m u^q v^{q^m} \). Hence by (2.2) and (2.5),

\[
s_{i,j}(\alpha^{tp^q}) = \begin{cases} 1, & \text{if } u = m - i \text{ or } v = n - j, \\ 0, & \text{if } u \neq m - i \text{ or } v \neq n - j. \end{cases}
\]

Similarly, if \( u < m - i \) and \( v \geq n - j \), then we get the same result.

If \( u = m - i \) and \( v = n - j \), then by (2.2) and \( \alpha^{p^m q^{n+1} q^{n} - 1} = s_{i,j}(\alpha^{tp^q}) = s_{m,n}(\alpha^t) \mod (2) \).

If \( u = m - i \) and \( v > n - j \), then \( \alpha^{p^m q^{n+1} q^{n} - 1} = 1 \) and \( \beta = \alpha^{p^{m+1} q^{n+1}} \) is a \( p \)th primitive root of unity. For any \( c \in \{1, \ldots, p - 1\} \), there are \( (q - 1)/r \) elements \( g^k \in \{g^k | k = 0, 1, \ldots, R - 1\} \) such that \( c \equiv g^k \mod p \).
Hence by (2.2) and (2.5), \(s_{i,j}(\alpha^p^j\beta^r^j) = p^{m-i}q^{n-j}(q-1)/e \sum_{r=1}^{q-1} \beta^r^{e(j)} \equiv (q-1)/e \pmod{2}\). Similarly, if \(u > m-i\) and \(v > n-j\), then \(s_{i,j}(\alpha^p^j\beta^r^j) \equiv (p-1)/e \pmod{2}\).

If \(u > m-i\) and \(v > n-j\), then by (2.2) and (2.5), \(p^{m-i}q^{n-j} = \beta^e \pmod{2}\).

2) Suppose that \(i \leq m\) and \(j \geq n+1\). If \(u \geq m-i\), then by (2.2) and (2.5), \(s_{i,n+1}(\alpha^p^j\beta^r^j) = p^{m-i} \sum_{k=0}^{p-2} \alpha^{p^k+q^{n+1}} \sum_{b=0}^{p-2} \alpha^{p^k+q^{n+1}} = 1\) \(\pmod{2}\).

If \(u < m-i\), then \(\alpha^{p^k+q^{n+1}} = 1\) and \(\alpha^{p^k+q^{n+1}}\) is a \(p^n\)-th primitive root of unity. Hence by (2.2) and (2.5), \(s_{i,n+1}(\alpha^p^j\beta^r^j) = p^{m-i} \sum_{k=0}^{p-2} \alpha^{p^k+q^{n+1}} \sum_{b=0}^{p-2} \alpha^{p^k+q^{n+1}} = 0\).

3) The proof is similar to that for (2).

By the previous lemmas, we know that the computation of the linear complexity of the sequence \(s\) turns into the computation of the values of \(s_{m,n}(x)\) for the generalized cyclotomic sequence.

We know that \(g\) is also a common primitive root of both \(p\) and \(q\). Let \(H_0 = \langle g \rangle\) be a subgroup of the multiplicative group \(Z*p^*_q\).

Let us introduce the polynomial \(T(x) = \sum_{i \in H_0} x^i\).

Let \(\beta = \alpha^p^j\beta^r^j\) be a \(p^n\)-th primitive root of unity in an extension field of \(GF(2)\). Then we have \(s_{m,n}(\alpha^i) = T(\beta^j)\).

For the computation of the linear complexity of the sequence \(s\) defined as (1.5), we need the following notations. For \(0 \leq u \leq m\) and \(0 \leq v \leq n\), set

\[
E_{u,v} = \left| \{ k \in H_0 \mid T(\beta^k) = 0, k = 0, 1, \ldots, e-1 \} \right|
\]

\[
F_{u,v} = \left| \{ k \in H_0 \mid T(\beta^k) = 1, k = 0, 1, \ldots, e-1 \} \right|
\]

Set for \(0 \leq u \leq m\) and \(0 \leq v \leq n\),

\[
\sigma_{u,v} = \frac{q-1}{e} \sum_{j=0}^{n-1} |I_{u,j}| + \frac{p-1}{e} \sum_{i=0}^{m} |I_{i,v}|
\]

\[
\sigma_{u,n+1} = \frac{q-1}{e} \sum_{j=0}^{n} |I_{u,j}| + \delta_{m,n+1} + \delta_{m+1,n+1}
\]

\[
\sigma_{m+1,v} = \frac{p-1}{e} \sum_{i=0}^{m} |I_{i,v}| + \delta_{m+1,v} + \delta_{m+1,n+1}
\]

where \(\delta_{m,n+1}\), \(\delta_{m,n+1}\), \(\delta_{m+1,n+1}\) are defined as (2.4) and

\[
\sum_{j=0}^{n} I_{u,j} = 0 \text{ if } v = 0 \text{ and } \sum_{i=0}^{m} I_{i,v} = 0 \text{ if } u = 0.
\]

\[
A_{u,v} = \begin{cases} E_{u,v}, & \text{if } \sigma_{u,v} \equiv 0 \pmod{2} \\ F_{u,v}, & \text{if } \sigma_{u,v} \equiv 1 \pmod{2} \end{cases}
\]

\[
A_{u,n+1} = \begin{cases} 1, & \text{if } \sigma_{u,n+1} \equiv 0 \pmod{2} \\ 0, & \text{if } \sigma_{u,n+1} \equiv 1 \pmod{2} \end{cases}
\]

\[
A_{m+1,v} = \begin{cases} 1, & \text{if } \sigma_{m+1,v} \equiv 0 \pmod{2} \\ 0, & \text{if } \sigma_{m+1,v} \equiv 1 \pmod{2} \end{cases}
\]

Now we get the most important theorem in this section.

**Theorem 2.5:** If the sequence \(s\) is defined as (1.5), then the linear complexity of the sequence \(s\) is

\[
L = p^{m+1}q^{n+1} - \sum_{u=0}^{m} \sum_{v=0}^{n} A_{u,v} p^u q^v R - \sum_{u=0}^{m} A_{u,n+1} p^u (p-1)
\]

\[
- \sum_{v=0}^{n} A_{m+1,v} q^v (q-1) - \delta
\]

where

\[
\delta = \begin{cases} 0, & \text{if } I_{m+1,n+1} \equiv \{ \} \\ 1, & \text{if } I_{m+1,n+1} \equiv \{ \} \pmod{2} \end{cases}
\]

**Proof:** If any \(t = p^u q^v g^h \in I_{k}^{(u,v)}\) for \(0 \leq u \leq m+1\), \(0 < v < n+1\) and \(0 < k < e-1\), then by Lemma 2.2 and (2.2)

\[
s(\alpha^u q^v g^h) = \sum_{i=0}^{m} \sum_{j=0}^{n} \sum_{l \in I_{i,j}} s_{i,j}(\alpha^u q^v g^h)
\]

\[
+ \sum_{i=0}^{m} \delta_{i+1,n} s_{i,n+1}(\alpha^u q^v g^h)
\]

\[
+ \sum_{j=0}^{n} \delta_{m+1,j} s_{m+1,j}(\alpha^u q^v g^h) + \delta_{m+1,n+1}
\]

If \(0 \leq u \leq m\) and \(0 \leq v \leq n\), then by Lemma 2.4

\[
s(\alpha^u) = \sum_{l \in I_{m,n}} s_{m,n}(\alpha^u) + \sigma_{m-u,n-1}
\]

\[
= \sum_{l \in I_{m,n-1}} T(\beta^{k+l+1}) + \sigma_{m-u,n-1}
\]

We conclude that \(s(\alpha^u) = 0\) if and only if \(I_{m,n} = I_{m,n-1} = 0\) \(\pmod{2}\). Hence, the order of set \(I_{m,n} = I_{m,n-1} = 0\) is \(A_{m-u,n-1} p^u q^{n+1} R\), so the order of set \(\{ t \in \cup_{u=0}^{m} \cup_{v=0}^{n} \ p^u q^v g^h \} s(\alpha^u) = 0\) is \(\sum_{u=0}^{m} 
\sum_{v=0}^{n} A_{u,v} p^u q^v R\).

If \(u = m+1\) and \(v \leq n\), then by (2.2) and Lemma 2.4

\[
s(\alpha^{u+1}) = \frac{p-1}{e} \sum_{i=0}^{m} |I_{u,i} + \delta_{m+1,n+1} + \delta_{m+1,n+1} = \sigma_{m,u,n+1} + \sigma_{m-u,n-1}
\]

We conclude that \(s(\alpha^{u+1}) = 0\) if and only if \(I_{m+1,n} = I_{m+1,n-1} = 0\) \(\pmod{2}\). Hence, the order of set \(\{ t \in \cup_{u=0}^{m} \cup_{v=0}^{n} g^h \} s(\alpha^{u+1}) = 0\) is \(\sum_{u=0}^{m} A_{u,n+1} p^u (p-1)\).

Similarly, the order of set \(\{ t \in \cup_{u=0}^{m} \cup_{v=0}^{n} g^h \} s(\alpha^{u+1}) = 0\) is \(\sum_{u=0}^{m} A_{u,n+1} p^u (p-1)\).

If \(u = m+1\) and \(v = n+1\), then we conclude that

\[
s(\alpha^n) = s(1) = 0\] if and only if \(\delta_{m+1,n+1} = 0\) if and only if \(I_{m+1,n+1} = 0\).
TABLE I

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TABLE II

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Hence by the definition of $E_{a,n}$, $P_{a,n}$, $A_{a,n}$, $\delta$, we get the linear complexity of the sequence defined as (1.5).

III. GENERALIZED CYCLOTOMIC SETS OF ORDER 4

In this section, we will assume that $\gcd(p - 1, q - 1) = e = 4$ and $q$ is a primitive root of $p$ and $q$. We will generalize the results from [8] and give values of Gauss periods of Whiteman's generalized cyclotomy of order 4 over $GF(2)$. Moreover, we determine $h$ up to sign in Whiteman's generalized cyclomatic numbers of order 4 if $p \equiv q \equiv 5 \pmod{8}$.

Since $\gcd(p - 1, q - 1) = 4$,
\[ \text{ord}_{p^q}(g) = \text{lcm}(\text{ord}_p(g), \text{ord}_q(g)) = \text{lcm}(p - 1, q - 1) = \frac{(p - 1)(q - 1)}{4} = R. \]

Whiteman [15] defined generalized cyclotomic classes
\[ H_i = \{g^n y^i : s = 0, 1, \ldots, R - 1\}, i = 0, 1, 2, 3, \quad (3.1) \]
\[ y \equiv g \pmod{p}, y \equiv 1 \pmod{q}. \]

And we have $Z_{p^q}^* = H_0 \cup H_1 \cup H_2 \cup H_3$.

The corresponding generalized cyclotomic numbers of order 4 are defined by
\[ (i, j) = (H_i + 1) \cap H_j, \text{ for all } i, j = 0, 1, 2, 3. \]

By Gauss’s theorem, there are exactly two representations over $I$
\[ pq = a^2 + 4b^2, \quad pq = a'^2 + 4b'^2, \quad a \equiv a' \equiv 1 \pmod{4}. \quad (3.2) \]

**Lemma 3.1:** The 16 cyclotomic numbers $(i, j), i, j = 0, 1, 2, 3$, depend solely upon one of the two decompositions in (3.2).

If $(p - 1)(q - 1)/16$ is even, then in Table I $8a = -a + 2M + 3$, $8B = -a + 4b + 2M - 1$, $8C = 3a + 2M - 1$, $8D = -a + 4b + 2M - 1$, $8E = a + 2M + 1$, where $a, b$ is defined as (3.2) and $M = \frac{(p - 1)(q - 1)}{4}$.

If $(p - 1)(q - 1)/16$ is odd, then in Table II $8a = 3a + 2M + 5$, $8B = -a + 4b + 2M + 1$, $8C = -a + 2M + 1, 8D = -a + 4b + 2M + 1, 8E = a + 2M + 1$. In fact, $\frac{(p - 1)(q - 1)}{16}$ is even if and only if $p = q + 4 \pmod{8}$; $\frac{(p - 1)(q - 1)}{16}$ is odd if and only if $p \equiv q \equiv 5 \pmod{8}$.

**Lemma 3.2:** Let $m_1, m_2$ be two positive integers. The system of congruences
\[ y \equiv t_1 \pmod{m_1}, \quad y \equiv t_2 \pmod{m_2} \]
has solutions if and only if
\[ \gcd(m_1, m_2) \mid t_1 - t_2. \]

**Proof:** See [13, Theorem 2.9].

**Lemma 3.3:**
1) $-1 \in H_0$ if and only if $p \equiv q \equiv 5 \pmod{8}$, $-1 \in H_2$ if and only if $p \equiv q + 4 \pmod{8}$.
2) $2 \in H_0 \cup H_2$ if and only if $p \equiv q \equiv 5 \pmod{8}$, and $2 \in H_1 \cup H_3$ if and only if $p \equiv q + 4 \pmod{8}$.
3) Let $p \equiv q \equiv 5 \pmod{8}$ and
\[ 2 \equiv g^{t_1} \pmod{p}, 2 \equiv g^{t_2} \pmod{q}. \]

Then $2 \in H_0$ if and only if only if $4 t_1 - t_2$; in other words, $2 \in H_2$ if and only if only if $4 t_1 - t_2$.

**Proof:**
1) Since $q$ is a primitive root of $p$ and $q, -1 \equiv g^{t_1} \pmod{p}$ and $-1 \equiv g^{t_2} \pmod{q}$.

If $p \equiv q \equiv 5 \pmod{8}$, then $2 \parallel t_1$ and $2 \parallel t_2$ (see [10]), so $4 \parallel t_1 - t_2$. Hence, there is $k \in Z$ such that $k = t_1 \pmod{p - 1}$ and $k = t_2 \pmod{q - 1}$, so by Lemma 3.2 $-1 \equiv g^{k} \pmod{pq}$ and $-1 \in H_0$. If $p \equiv 1 \pmod{8}$ and $q \equiv 5 \pmod{8}$, then $4 \parallel t_1$ and $2 \parallel t_2$, so $4 t_1 - t_2$. Hence, there exists $k \in Z$ such that $k = t_1 - 2 \pmod{p - 1}$ and $k = t_2 \pmod{q - 1}$. Thus by Lemma 3.2 $-1 \equiv g^{k} \pmod{pq}$ and $-1 \in H_2$, where $y$ is defined as (3.1). The converse is straightforward.

2) Let $2 \equiv g^{t_1} \pmod{p}$ and $2 \equiv g^{t_2} \pmod{q}$. If $p \equiv q \equiv 5 \pmod{8}$, then $2 \parallel t_1$ and $2 \parallel t_2$, so $2 \parallel t_1 - t_2$. Similarly, we have $2 \in H_0 \cup H_2$. If $p \equiv 1 \pmod{8}$ and $q \equiv 5 \pmod{8}$, then $2 \parallel t_1$ and $2 \parallel t_2$, so $2 \parallel t_1 - t_2$. Similarly, we have $2 \in H_1 \cup H_3$. The converse is straightforward.

3) Since $p \equiv q \equiv 5 \pmod{8}$, $t_1$ and $t_2$ are odd in (3.3), so $2 \parallel t_1 - t_2$. By Lemma 3.2, we conclude that $4 \parallel t_1 - t_2$ if and only if there is $k \in Z$ such that $k \equiv t_1 \pmod{p - 1}$ and $k \equiv t_2 \pmod{q - 1}$ if and only if $2 \equiv g^{k} \pmod{pq}$ and $2 \in H_0$. Moreover, we have that $4 \parallel t_1 - t_2$ if and only if $4 \parallel t_1 - t_2$ if and only if $k \in Z$ such that $k \equiv t_1 - 2 \pmod{p - 1}$ and $k \equiv t_2 \pmod{q - 1}$ if and only if $2 \equiv g^{k} \pmod{pq}$ and $2 \in H_2$, where $y$ is defined as (3.1).

We define $P = \{p, 2p, \ldots, (q - 1)p\}$, $Q = \{q, 2q, \ldots, (p - 1)q\}$.

**Lemma 3.4:** For each $\omega \in P \cup Q$
\[ |H_1 \cap (H_1 + \omega)| = \begin{cases} |p - 1||q - 1|, & \text{if } i \neq j, \\ |p - 1||q - 1|, & \text{if } i = j, p | \omega, \\ |p - 5||q - 1|, & \text{if } i = j, q | \omega. \end{cases} \]
Proof: See [15, Lemmas 2 and 4].

**Lemma 3.5:** Let \( p \equiv q \equiv 5 \pmod{8} \). Then there are exactly two representations over \( \mathbb{Z} \)

\[
pq = a^2 + 4b^2 = a'^2 + 4b'^2, \ a \equiv a' \equiv 1 \pmod{4} \tag{3.4}
\]

where one of \( b \) and \( b' \) is divided by 4 and another is exactly divided by 2.

Proof: Let \( p = x_1^2 + 4y_1^2 \) and \( q = x_2^2 + 4y_2^2 \), \( x_j, y_j \in \mathbb{Z}, j = 1, 2 \); then\( 2 \nmid y_2, j = 1, 2 \) by \( p \equiv q \equiv 5 \pmod{8} \). Hence, \( pq = a^2 + 4b^2 = a'^2 + 4b'^2 \), \( b = x_1y_2 + x_2y_1 \), and \( b' = x_1y_2 - x_2y_1 \), where one of \( b \) and \( b' \) is divided by 4 and another is exactly divided by 2.

Let \( T(x) = \sum_{i \in H} x^i \) and \( \beta \) be a \( p \)th primitive root of unity in the extension over \( GF(2) \). Define

\[
T_k(\beta) = (T(\beta^n), T(\beta^{n^2}), T(\beta^{n^3})) \tag{3.5}
\]

where \( y \) is defined as (3.1) or (1.1).

**Lemma 3.6:** If \( p \equiv q + 4 \pmod{8} \), then \( T_k(\beta) = (\gamma, \gamma^2, \gamma^3, \gamma^4) \) or \( T_k(\beta') = (\gamma, \gamma^2, \gamma^3, \gamma^4) \), where \( \gamma^2 + \gamma + 1 = 0 \) or \( \gamma^3 + \gamma^2 + 1 = 0 \).

Proof: If \( p = q + 4 \pmod{8} \), then by Lemma 3.3, \( 2 \in H_1 \cup H_2 \). Suppose that \( 2 \in H_1 \); then \( T(\beta)^2 = \sum_{i \in H_0} \beta^{2i} = \sum_{i \in H_1} \beta^i = T(\beta) \). Similarly, \( T(\beta)^4 = T(\beta^{n^2}), T(\beta)^8 = T(\beta^{n^3}) \). Let \( \gamma := T(\beta) \); then \( T_k(\beta) = (\gamma, \gamma^2, \gamma^3, \gamma^4) \) satisfies \( \gamma^2 + \gamma + 1 = 0 \), \( \gamma^3 + \gamma^2 + 1 = 0 \), and \( \gamma^3 + \gamma^2 + \gamma = -1 \). Suppose that \( 2 \in H_3 \); then \( T_k(\beta) = (\gamma, \gamma^2, \gamma^3, \gamma^4) \).

The following is a well-known result.

**Lemma 3.7:** Let \( p \equiv q \equiv 5 \pmod{8} \) be distinct primes with \( \gcd(p - 1, q - 1) = 1 \). Fix \( q \) a common primitive root of \( p \) and \( q \). Then \( 2 \in H_0 \cup H_2 \). Suppose that \( 2 \in H_0 \) if and only if \( T(\beta^{n^i}) \in GF(2), i = 0, 1, 2 \); \( 2 \in H_2 \) if and only if either \( T(\beta), T(\beta^{n^i}) \in GF(2) \) or \( T(\beta), T(\beta^{n^i}) \in GF(2) \).

Now we give the values of \( T_k(\beta) \).

**Theorem 3.8:** Let \( \beta \) be a \( p \)th primitive root of unity. Suppose that the cyclotomic numbers of Lemma 3.1 depend upon the decomposition \( pq = a^2 + 4b^2, a \equiv 1 \pmod{4} \). Let \( T_k(\beta) = (T(\beta^n), T(\beta^{n^2}), T(\beta^{n^3})) \). Then by a choice of \( \beta \) (i.e., a \( p \)th primitive root of unity), we have:

1) \( T_k(\beta) = (0, 0, 1, 0) \) or \( (1, 0, 0, 0) \) if \( a \equiv 1 \pmod{8} \) and \( 4 \mid b 
\)
2) \( T_k(\beta) = (0, 1, 1, 1) \) or \( (1, 1, 0, 1) \) if \( a \equiv 5 \pmod{8} \) and \( 4 \mid b 
\)
3) \( T_k(\beta) = (\mu, 1, \mu + 1, 1) \) or \( (\mu + 1, 1, \mu), 1 \) if \( a \equiv 1 \pmod{8} \) and \( 2 \nmid b 
\)
4) \( T_k(\beta) = (\mu, 0, \mu + 1, 0) \) or \( (\mu + 0, 1, \mu, 0) \) if \( a \equiv 5 \pmod{8} \) and \( 2 \nmid b 
\)
5) \( T_k(\beta) = (\gamma, \gamma^2, \gamma^3, \gamma^4) \) if \( b = 1 \pmod{2} \); where \( \mu \) satisfies \( \mu^2 + \mu + 1 = 0 \), \( \gamma \) satisfies either \( \gamma^2 + \gamma^3 + \gamma + 1 = 0 \) or \( \gamma^4 + \gamma^3 + 1 = 0 \).

Proof: Set

\[
\Psi_i := T(\beta^n), i = 0, 1, 2, 3.
\]

If \( p \equiv q \equiv 5 \pmod{8} \), then \( -1 \in H_0 \), and then by Lemmas 3.1, 3.3, and 3.4

\[
\Psi_0 \Psi_2 = \sum_{i \in H_0} \beta^i \sum_{m \in H_2} \beta^m = \sum_{i \in H_0} \sum_{m \in H_2} \beta^{i-m} = (2, 0)\Psi_0 + (1, 3)\Psi_1 + (0, 2)\Psi_2 + (3, 1)\Psi_3 - \frac{(p-1)(q-1)}{8}.
\]

By Lemma 3.3, we have \( 2 \in H_0 \cup H_2 \) and by Lemma 3.7 we have \( \Psi_0 + \Psi_2, \Psi_1 + \Psi_3 \in GF(2) \). Since \( \Psi_0 + \Psi_2 + \Psi_3 = 1 \), without loss of generality (i.e., by a choice of \( \beta \)) we may assume that

\[
\Psi_0 + \Psi_2 = 1, \Psi_1 + \Psi_3 = 0.
\]

Then by (3.6) and (3.7), we have

\[
\Psi_0 \Psi_2 = s - t^2 - s^2 \equiv t[mod \ 2], \Psi_1 \Psi_3 = s^2 - t^2 - s^2 \equiv s + t[mod \ 2].
\]

Solving systems (3.8) and (3.9), we obtain:

1) \( T_k(\beta) = (0, 0, 1, 0) \) or \( (1, 0, 0, 0) \) if \( s \equiv 0 \pmod{2} \) and \( t \equiv 0 \pmod{2} \);
2) \( T_k(\beta) = (0, 1, 1, 1) \) or \( (1, 1, 0, 1) \) if \( s \equiv 1 \pmod{2} \) and \( t \equiv 0 \pmod{2} \);
3) \( T_k(\beta) = (\mu, 1, \mu + 1, 1) \) or \( (\mu + 1, 1, \mu, 1) \) if \( s \equiv 0 \pmod{2} \) and \( t \equiv 1 \pmod{2} \);
4) \( T_k(\beta) = (\mu, 0, \mu + 1, 0) \) or \( (\mu + 0, 1, \mu, 0) \) if \( s \equiv 1 \pmod{2} \) and \( t \equiv 1 \pmod{2} \);

where \( \mu \) is a root of the equation \( x^2 + x + 1 = 0 \).

If \( p \equiv q \equiv 5 \pmod{8} \), then \( b \) is odd and (5) is clear from Lemma 6.3.

**Corollary 3.9:** Let \( p \equiv q \equiv 5 \pmod{8} \). Fix a common primitive root \( g \) of \( p \) and \( q \). Then \( 2 \in H_0 \) if and only if the generalized cyclotomic numbers of Lemma 3.1 depend on the decomposition \( N = a^2 + 4b^2 \) with \( 4 \nmid b \).

Proof: It is clear from Lemma 3.7 and Theorem 3.8.

\[\blacksquare\]
By Corollary 3.9 and Lemma 3.3, we can determine $b$ up to
sign in Whiteman’s generalized cyclotomic numbers of order 4
in the case $p \equiv q \equiv 5 \pmod{8}$ if fixing a common primitive
root $g$ of $p$ and $q$.

IV. APPLICATIONS

A. Sequence of Period $pq$

We can use the method in Sections II and III to compute the
linear complexity of the generalized cyclotomic $pq$-periodic bi-
ary sequence of order 4 in [1]. But we cannot use the method in
[1] to calculate the linear complexity of the following sequence.

The generalized cyclotomic $pq$-periodic binary sequence $s$ of
order 4 with respect to the primes $p$ and $q$ is defined as

$$s_i = \begin{cases} 1, & \text{if } i \equiv j \pmod{N} \in \Omega \\ 0, & \text{otherwise} \end{cases} \quad (4.1)$$

where $P = \{p, 2p, \ldots, (q - 1)p\}$ and $\Omega = \mathbb{P} \cup H_0$.

Now we compute the linear complexity $L$ and the minimal
polynomial $m(x)$ of Whiteman’s generalized cyclotomic se-
quence of order 4. Let $\beta$ be a $pq$th primitive root of unity in
an extension over $GF(2)$. Set

$$d_i(x) = \prod_{\nu \in H_i} (x - \beta^\nu), i = 0, 1, 2, 3.$$ 

By Theorem 2.5 and 3.8, we can get the following result.

Theorem 4.1:

(I) If $p \equiv 1 \pmod{8}$ and $q \equiv 5 \pmod{8}$, then

$$L = pq - 1, \quad m(x) = \frac{x^{pq} - 1}{x - 1}.$$ 

(II) If $p \equiv 5 \pmod{8}$ and $q \equiv 1 \pmod{8}$, then

$$L = pq - p - q + 1, \quad m(x) = \frac{(x^{pq} - 1)(x - 1)}{(x - p)(x - q)}.$$ 

(III) Let $2 \in H_0$ and $pq \equiv 1 \pmod{16}$. Then

$$L = \frac{(p - 1)(3q + 1)}{4}, \quad m(x) = \frac{x^{pq} - 1}{d_0(x)(x^q - 1)}.$$ 

(IV) Let $2 \in H_0$ and $pq \equiv 9 \pmod{16}$. Then

$$L = \frac{(p - 1)(q + 3)}{4}, \quad m(x) = \frac{x^{pq} - 1}{d_0(x)x^q - 1}.$$ 

(V) Let $2 \in H_2$ and $pq \equiv 1 \pmod{16}$. Then

$$L = \frac{(p - 1)(q + 1)}{2}, \quad m(x) = \frac{x^{pq} - 1}{d_1(x)d_2(x)(x^q - 1)}.$$ 

(VI) Let $2 \in H_2$ and $pq \equiv 9 \pmod{16}$. Then

$$L = pq - q, \quad m(x) = \frac{x^{pq} - 1}{x^q - 1}.$$ 

Proof: By Theorems 2.5 and 3.8, we compute the linear
complexity of the sequence $s$ defined as (4.1). About Theorem

2.5, we know that $n - m = 0, D_{b}^{(0,0)} - H_0, D_{b}^{(1,0)} - PZ_{p}^{*} - 
\delta_1 = 1, \delta_{0,1} = 0, \delta_{1,1} = 0, \sigma_{0,0} = 1, \sigma_{1,0} \equiv \frac{p - 1}{4} + 
1 \pmod{2}, \sigma_{0,1} \equiv \frac{q - 1}{4} \pmod{2}, \delta = 1.$

(I) If $p \equiv 1 \pmod{8}$ and $q \equiv 5 \pmod{8}$, then $\sigma_{1,0} \equiv 1 \pmod{2}, \sigma_{0,1} \equiv 1 \pmod{2}, \text{and } E_{0,0} = F_{0,0} = 0.$

Hence, $A_{0,0} = 0, A_{1,0} = A_{1,1} = 1,$ so

$$L = pq - (p - 1) - (q - 1) - 1 = pq - p - q + 1.$$ 

(II) If $p \equiv 5 \pmod{8}$ and $q \equiv 1 \pmod{8}$, then $\sigma_{1,0} \equiv 0 \pmod{2}, \sigma_{0,1} \equiv 0 \pmod{2}, \text{and } E_{0,0} = F_{0,0} = 0.$

Hence, $A_{0,0} = 0, A_{1,0} = A_{1,1} = 1,$ so

$$L = pq - (p - 1)(q - 1) - (q - 1) - 1 = \frac{(p - 1)(q + 3)}{4}.$$ 

Choosing $\beta$ with $T_4(\beta) = (1, 0, 0, 0)$ in Theorem 3.8
(1), we have

$$m(x) = \frac{x^{pq} - 1}{d_0(x)(x^q - 1)}.$$ 

(III) If $2 \in H_0$ and $pq \equiv 1 \pmod{16}$, then $\sigma_{1,0} \equiv 0 \pmod{2}, \sigma_{0,1} \equiv 1 \pmod{2}, F_{0,0} = 3.$

Hence, $A_{0,0} = 3, A_{1,0} = 1, \text{and } A_{0,1} = 0,$ so

$$L = pq - 3\frac{(p - 1)(q - 1)}{4} - (q - 1) - 1 = \frac{(p - 1)(q + 3)}{4}.$$ 

Choosing $\beta$ with $T_4(\beta) = (0, 1, 1, 1)$ in Theorem 3.8
(2), we have

$$m(x) = \frac{x^{pq} - 1}{d_0(x)d_2(x)(x^q - 1)} - \frac{x^{p - 1}d_{0}(x)}{x - 1}.$$ 

(IV) If $2 \in H_0$ and $pq \equiv 9 \pmod{16}$, then $\sigma_{1,0} \equiv 0 \pmod{2}, \sigma_{0,1} \equiv 1 \pmod{2}, F_{0,0} = 3.$

Hence, $A_{0,0} = 3, A_{1,0} = 1, \text{and } A_{0,1} = 0,$ so

$$L = pq - 3\frac{(p - 1)(q - 1)}{4} - (q - 1) - 1 = \frac{(p - 1)(q + 3)}{4}.$$ 

Choosing $\beta$ with $T_4(\beta) = (\mu, \mu, 0, 0, 0)$ in Theorem
3.8 (3), we have

$$m(x) = \frac{x^{pq} - 1}{d_1(x)d_2(x)(x^q - 1)}.$$ 

(V) If $2 \in H_2$ and $pq \equiv 1 \pmod{16}$, then $\sigma_{1,0} \equiv 0 \pmod{2}, \sigma_{0,1} \equiv 1 \pmod{2}, F_{0,0} = 2.$

Hence, $A_{0,0} = 2, A_{1,0} = 1, \text{and } A_{0,1} = 0,$ so

$$L = pq - 2\frac{(p - 1)(q - 1)}{4} - (q - 1) - 1 = \frac{(p - 1)(q + 1)}{2}.$$ 

Choosing $\beta$ with $T_4(\beta) = (\mu, \mu, 0, 0, 0)$ in Theorem
3.8 (3), we have

$$m(x) = \frac{x^{pq} - 1}{d_1(x)d_2(x)(x^q - 1)}.$$ 

(VI) If $2 \in H_2$ and $pq \equiv 9 \pmod{16}$, then $\sigma_{1,0} \equiv 0 \pmod{2}, \sigma_{0,1} \equiv 1 \pmod{2}, F_{0,0} = 2.$

Hence, $A_{0,0} = 2, A_{1,0} = 1, A_{0,1} = 0,$ so

$$L = pq - (q - 1) - 1 = pq - q, m(x) = \frac{x^{pq} - 1}{x^q - 1}.$$
B. Sequence of Period $N = p^{m+1}q^n+1$

Suppose $\Omega = \bigcup_{i=0}^{m} \bigcup_{j=0}^{n} p^i q^j D_0^{(i,j)}$ ($m > 0$, $n > 0$) and

$$s_i = \begin{cases} 1, & \text{if } i \equiv (\text{mod } N) \in \Omega \\ 0, & \text{otherwise} \end{cases} \quad (4.2)$$

Then by Theorem 2.5, we get the linear complexity of the sequence in (4.2).

**Theorem 4.2:** Let $m_2$ and $n_2$ be the largest even integers such that $m_2 \leq m$ and $n_2 \leq n$, respectively. Let $m_1$ and $n_1$ be the largest odd integers such that $m_1 \leq m$ and $n_1 \leq n$, respectively.

1) Suppose that $p \equiv 1 \pmod{8}$ and $q \equiv 5 \pmod{8}$; then

$$L = (p^{m+1} - 1)(q^n+1 - \delta_n)$$

where

$$\delta_n = \begin{cases} 0, & \text{if } n \text{ is even} \\ 1, & \text{if } n \text{ is odd} \end{cases}$$

2) Suppose that $p \equiv q \equiv 5 \pmod{8}$.

   (I) If $2 \in H_2$ and $pq \equiv 1 \pmod{16}$, then see the first equation shown at the bottom of the page.

   (II) If $2 \in H_0$ and $pq = 9 \pmod{16}$, then see the second equation shown at the bottom of the page.

   (III) If $2 \in H_2$ and $pq \equiv 1 \pmod{16}$, then see the third equation shown at the bottom of the page.

   (IV) If $2 \in H_2$ and $pq \equiv 9 \pmod{16}$, then see the fourth equation shown at the bottom of the page.

**Proof:** By Theorems 2.5 and 3.8, we compute the linear complexity of the sequence. About Theorem 2.5, we know that $\delta_0, n+1 = 0$, $i = 0, 1, \ldots, m$, $\delta_{m+1}, j = 0, j = 0, 1, \ldots, n+1$, $\sigma_{m,v} = \frac{q+1}{4}(n-v) + \frac{q-1}{4}(m-u)$, $\sigma_{n+1} = \frac{q+1}{4}(n+1)$, $\sigma_{m+1,v} = \frac{q+1}{4}(m+1)$ for $0 \leq u \leq m$, $0 \leq v \leq n$, and $\delta = 1$.

1) Since $p \equiv 1 \pmod{8}$ and $q \equiv 5 \pmod{8}$, by Lemma 3.6 we know $E_{n,v} = 0 = F_{n,v}$, so $A_{n,v} = 0$ for $0 \leq u \leq m$ and $0 \leq v \leq n$. By $\sigma_{m+1,v} = \frac{q+1}{4}\sum_{j=0}^{m} I_{u,j} \equiv 0 \pmod{2}$, $\sigma_{n+1} = \frac{q+1}{4}\sum_{i=0}^{n} I_{u,j} \equiv n+1 \pmod{2}$. Hence, by Theorem 2.5 we have

$$L = N - \sum_{i=0}^{n} p^n(q-1) - \delta_n \sum_{u=0}^{m} p^u(p-1) - 1$$

$$- (p^{m+1} - 1)(q^n+1 - \delta_n)$$

where $\delta_n = 1$ if $n$ is odd and $\delta_n = 0$ if $n$ is even.

2) If $p \equiv q \equiv 5 \pmod{8}$.

   (I) If $2 \in H_0$ and $pq \equiv 1 \pmod{16}$, then for $0 \leq u \leq m$ and $0 \leq v \leq n$, by Theorem 3.8 $E_{n,v} = 3$ and $F_{n,v} = 1$, $A_{n,v} = \frac{q+1}{4}\sum_{j=0}^{m} I_{u,j} + \frac{q+1}{4}\sum_{i=0}^{n} I_{u,j} \equiv m - u + n - v \pmod{2}$, $\sigma_{n+1} \equiv n+1 \pmod{2}$, and $\sigma_{m+1,v} \equiv m+1 \pmod{2}$. Hence, we have

$$L = N - \sum_{u=0}^{m} \sum_{v=0}^{n} p^n q^n R$$

$$- 2 \sum_{m+n-u-v \text{ even}} p^n q^n H - \delta_n(q^n+1 - 1) - \delta_n(p^{m+1} - 1) - 1.$$
Moreover, we have the first equation shown at the bottom of the page. Hence, we prove (I).

(II) If $2 \in H_0$ and $pq \equiv 9 \pmod{16}$, then $E_{u,v} = 1$ and $F_{u,v} = 3$ for $0 \leq u \leq m$ and $0 \leq v \leq n$. Similarly, we have

$$L = N - \sum_{u=0}^{m} \sum_{v=0}^{n} p^u q^v R$$

$$- 2 \sum_{m+n-u-v \text{ odd}} p^u q^v R - \delta_m (q^{n+1} - 1) - \delta_n (p^{m+1} - 1) - 1.$$

Moreover, we have the second equation shown at the bottom of the page. Hence, we prove (II).

(III) If $2 \notin H_2$ and $pq \equiv 1 \pmod{16}$, then by Theorem 3.8 $E_{u,v} = 0$ and $F_{u,v} = 2$ for $0 \leq u \leq m$ and $0 \leq v \leq n$. Similarly, we have

$$L = N - 2 \sum_{m+n-u-v \text{ odd}} p^u q^v R - \delta_m (q^{n+1} - 1) - \delta_n (p^{m+1} - 1) - 1.$$

So we prove (III).

(IV) If $2 \in H_2$ and $pq \equiv 9 \pmod{16}$, then by Theorem 3.8 $E_{u,v} = 2$ and $F_{u,v} = 0$ for $0 \leq u \leq m$ and $0 \leq v \leq n$. Similarly, we have

$$L = N - 2 \sum_{m+n-u-v \text{ even}} p^u q^v R - \delta_m (q^{n+1} - 1) - \delta_n (p^{m+1} - 1) - 1.$$

So we prove (IV).

\[\square\]

V. OPEN PROBLEM

If $p = q + 4 \pmod{8}$, how do Whiteman’s generalized cyclotomic numbers of order 4 depend on the two decompositions $pq = a^2 + 4b^2 = a'^2 + 4b'^2$, $a \equiv a' \equiv 1 \pmod{4}$?

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