SIAM J. CONTROL AND OPTIMIZATION
Vol. 24, No. 4, July 1986

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STOCHASTIC ADAPTIVE CONTROL FOR EXPONENTIALLY CONVERGENT TIME-VARYING SYSTEMS*

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Abstract. This paper shows that the standard stochastic adaptive control algorithms for time-invariant systems have an inherent robustness property which renders them applicable, without modification, to time-varying systems whose parameters converge exponentially. One class of systems satisfying this requirement is those having non-steady-state Kalman filter or innovations representations. This allows the usual assumption of a stationary ARMAX representation to be replaced by a more general state space model.

Key words. stochastic control, adaptive control, martingale convergence, passivity, time-varying systems

1. Introduction. A stochastic adaptive controller is an algorithm which combines on-line parameter estimation with on-line control to generate a control law applicable to systems having unknown parameters and random disturbances [1]. Control laws based on this philosophy have been studied for at least three decades [2], but it is only recently that rigorous convergence analyses have appeared. To gain insight into the operation of these algorithms, several special cases have been studied in detail. For example, the authors of [3] have examined the convergence properties of a particular scheme which combines a simple stochastic gradient parameter estimator with a minimum variance control law.

A number of interesting properties of these simple stochastic adaptive control laws have been established. For example, the tracking error is known to converge to a minimum (in a specific sense) in a sample mean square sense [3]. In the case of regulation about a zero desired output, then it has been shown [5] that the parameter estimates converge to a fixed multiple of the true parameter values. However, if the desired output sequence is continuously disturbed and an identifiability condition holds, then the parameters can be shown to converge to their true values [4]. Various extensions of the above results have also been studied. For example convergence results have been established in [7], for least squares based adaptive control algorithms.

The above papers deal with systems having constant parameters. However, in practice one is often confronted with systems whose parameters vary with time in some fashion. This has motivated several authors to investigate special classes of time-varying systems in an effort to gain insights into the convergence properties relevant to this case. For example, Caines [8] has analyzed the performance of the stochastic gradient algorithm of [3] applied to systems with (converging) martingale parameters. Further results for systems having random parameters are described in [9].

The current paper also deals with systems whose parameters are time-varying. Indeed, the work has much in common with the results in [8], [9]. All three papers reduce to treatment of a near-super-martingale equation of a particular form—see equation (3.28) later. However, here the parameter time variations are deterministic and thus a different method of analysis is necessary from that used in [8], [9].

Our analysis has three key steps: a proof that a system which is convergent toward a minimum phase system has an input which grows no faster than the output; a proof that a system which is exponentially convergent toward a strictly passive system is eventually strictly passive in a certain sense (where the strict passivity concepts are

* Received by the editors May 29, 1984, and in revised form February 28, 1985.

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defined later); and a martingale convergence proof along the lines of [3], but using a modified martingale result as first proposed in [10], [11] in a different context.

One application of the results developed here is to systems described by a state space model corresponding to a non-steady-state innovations representation. Subject to the assumption that the system has no uncontrollable modes (in the filtering sense [12]) on the unit circle, then it is known [13] that the parameters in the innovations model are time-varying and converge exponentially fast toward those of the steady-state optimal filter. Thus the results of this paper allow global convergence to be established for the standard adaptive control algorithms when applied to these systems. This represents a relaxation of the usual modelling assumption employed elsewhere in the literature (e.g. [3] to [9]) that the system is described by an ARMAX model or equivalently a steady-state Kalman filter model. This particular robustness to modelling assumption is often implicitly assumed in the literature, and it is thus interesting for technical completeness to have a formal proof that the results go through in this case.

2. Preliminary result on passive systems. We verify a result which will be needed in the subsequent proof of convergence of an adaptive control algorithm. This concerns a passivity property of a system which is exponentially convergent toward an asymptotically stable and input strictly passive system. The definitions of passivity concepts used correspond to those presented in [15, Appendix C]. Consider the extended Hilbert space $l_{2_e}^n(\mathbb{Z}_+)$ of sequences $v: \mathbb{Z}_+ \to \mathbb{R}^n$ with truncated inner product $\langle u, v \rangle_T := \sum_{k=0}^{T-1} u^T(k)v(k) < \infty$. The main definition for present purposes is as follows.

DEFINITION 2.1. Consider a dynamical system represented by mapping $G: l_{2_e}^n(\mathbb{Z}_+) \to l_{2_e}^n(\mathbb{Z}_+)$. The system is input strictly passive (ISP) iff $\exists \delta > 0$ and β such that

$$\langle y, u \rangle_T \geqq \delta \|u\|_T^2 + \beta \quad \forall u \in l_{2_e}(\mathbb{Z}_+) \text{ and } T \geqq 0. \tag{2.1}$$

Consider the following special case of the time-varying linear system model (A.1), (A.2):

$$x(t+1) = A(t)x(t) + Bu(t), \tag{2.2}$$

$$y(t) = Cx(t). \tag{2.3}$$

Let $A(t) \to A$ exponentially fast and define $\{y^*(t)\}, \{x^*(t)\}$ by

$$x^*(t+1) = Ax^*(t) + Bu^*(t), \tag{2.4}$$

$$y^*(t) = Cx^*(t) \tag{2.5}$$

with (A, B) controllable and A asymptotically stable.

The following result establishes that if (2.4), (2.5) is also ISP then (2.2), (2.3) satisfies a property very close to ISP.

THEOREM 2.1. *Provided* (2.4), (2.5) *is input strictly passive then there exist* $\bar{n}_0$, β *and* $\delta > 0$ *such that*

$$\sum_{t=n_0}^{N} (y(t)u(t) - \delta u(t)^2) + \beta \geqq 0 \tag{2.6}$$

for all $N \geqq n_0 \geqq \bar{n}_0$ *and for all* $\{u(t)\} \in l_{2_e}(\mathbb{Z}_+)$. β, δ *depend on* $u(t)$ *for* $t < n_0$.

Proof. Since (2.4), (2.5) is ISP, there exists $\delta^* < 0$ and $\beta^*(x^*(n_0))$ such that

$$\sum_{t=n_0}^{N} (y^*(t)u^*(t) - \delta^* u^*(t)^2) + \beta^*(x^*(n_0)) \geqq 0 \tag{2.7}$$

for all $N \geqq n_0$ and for all $u(t) \in l_{2_e}(\mathbb{Z}_+)$. From (2.7), we have

$$\sum_{t=n_0}^{N} (y^*(t)-y(t))u(t)+\sum_{t=n_0}^{N} y(t)u(t) \geqq \delta^* \sum_{t=n_0}^{N} u(t)^2-\beta^*(x^*(n_0)).$$

Let

$$\alpha(n_0, N) := \sum_{t=n_0}^{N} (y^*(t)-y(t))u(t).$$

Then

$$\sum_{t=n_0}^{N} y(t)u(t) \geqq \delta^* \sum_{t=n_0}^{N} u(t)^2-\alpha(n_0, N)-\beta^*(x^*(n_0)). \tag{2.8}$$

The remainder of the proof involves establishing a bound for $\alpha(n_0, N)$. Now

$$y^*(t) = C\Phi^*(t, n_0)x^*(n_0)+\sum_{i=n_0}^{t-1} C\Phi^T(t, i+1)Bu^*(i),$$

$$y(t) = C\Phi(t, n_0)x(n_0)+\sum_{i=n_0}^{t-1} C\Phi(t, i+1)Bu(i)$$

where $\Phi^*(t, n_0)=A^{t-n_0}$ and $\Phi(t, n_0)$ is given by (A.3). Then

$$\begin{aligned}\alpha(n_0, N) &= \sum_{t=n_0}^{N} u(t)C\left(A^{t-n_0}x^*(n_0)-\prod_{i=n_0}^{t-1} A(i)\right)x(n_0)\\ &\quad+\sum_{t=n_0}^{N} u(t)C\left(\sum_{i=n_0}^{t-1}\left(A^{t-i-1}-\prod_{j=i+1}^{t-1} A(j)\right)Bu(i)\right)\\ &:= \alpha_1(n_0, N)+\alpha_2(n_0, N).\end{aligned}$$

Now without loss of generality choose $u^*(t)$ so that

$$x^*(n_0)=x(n_0) \quad \text{and} \quad y^*(n_0)=y(n_0).$$

This is possible because (2.4), (2.5) is controllable. We can consider $u(t)=u^*(t)$ for $t \geqq n_0$. We have

$$|\alpha(n_0, N)| \leqq |\alpha_1(n_0, N)|+|\alpha_2(n_0, N)|.$$

The remainder of the proof involves bounding α_1 and α_2 in terms of $\|u(t)\|_{n_0}^N$.

We will use $|\cdot|$ for the Euclidean norm.

$$\begin{aligned}|\alpha_1(n_0, N)| &= \left|\sum_{t=n_0}^{N} u(t)C\left(A^{t-n_0}-\prod_{i=n_0}^{t-1} A(i)\right)x(n_0)\right|\\ &\leqq \|u(t)\|_{n_0}^N \left\|C\left(A^{t-n_0}-\prod_{i=n_0}^{t-1} A(i)\right)x(n_0)\right\|_{n_0}^N\\ &\qquad\text{using the Schwarz inequality}\\ &\leqq \|u(t)\|_{n_0}^N \sum_{t=n_0}^{N}\left|C\left(A^{t-n_0}-\prod_{i=n_0}^{t-1} A(i)\right)x(n_0)\right|\\ &\qquad\text{since } \|\cdot\|_2 \leqq \|\cdot\|_1\\ &\leqq \|u(t)\|_{n_0}^N |C||x(n_0)| \sum_{t=n_0}^{N}\left|A^{t-n_0}-\prod_{i=n_0}^{t-1} A(i)\right|\\ &\leqq \|u(t)\|_{n_0}^N |C||x(n_0)|\eta^{n_0-1}\sum_{t=n_0}^{N} \chi\eta^{t-n_0}\\ &\qquad\text{choosing } n_0 \geqq \bar{n} \text{ and using Lemma A.2}\\ &\leqq 2\varepsilon_1 \|u(t)\|_{n_0}^N |x(n_0)|\end{aligned}$$

where

$$\varepsilon_1=\frac{|C|\chi\eta^{n_0-1}}{2(1-\eta)}\leqq \varepsilon_1(\|u(t)\|_{n_0}^N)^2+\varepsilon_1|x(n_0)|^2.$$

So $|\alpha_1(n_0, N)|\leqq \varepsilon_1(\|u(t)\|_{n_0}^N)^2+\varepsilon_1|x(n_0)|^2$.

Now turning to α_2, we have

$$|\alpha_2(n_0, N)|=\left|\sum_{t=n_0}^{N} u(t)C\left(\sum_{i=n_0}^{t-1}\left(A^{t-i-1}-\prod_{j=i+1}^{t-1}A(j)\right)Bu(i)\right)\right|$$

$$\leqq \|u(t)\|_{n_0}^N\left\|C\sum_{i=n_0}^{t-1}\left(A^{t-i-1}-\prod_{j=i+1}^{t-1}A(j)\right)Bu(i)\right\|_{n_0}^N.$$

Observe that

$$\left\|C\sum_{i=n_0}^{t-1}\left(A^{t-i-1}-\prod_{j=i+1}^{t-1}A(j)\right)Bu(i)\right\|_{n_0}^N$$

$$=\left\{\sum_{t=n_0}^{N}\left(\sum_{i=n_0}^{t-1}C\left(A^{t-i-1}-\prod_{j=i+1}^{t-1}A(j)\right)Bu(i)\right)^2\right\}^{1/2}$$

$$\leqq\left\{\sum_{t=n_0}^{N}\left(\sum_{i=n_0}^{t-1}\left(C\left(A^{t-i-1}-\prod_{j=i+1}^{t-1}A(j)\right)B\right)^2\right)\left(\sum_{i=n_0}^{t-1}u(i)^2\right)\right\}^{1/2}$$

using the Schwarz inequality

$$\leqq\left\{\sum_{t=n_0}^{N}\sum_{i=n_0}^{t-1}\left(C\left(A^{t-i-1}-\prod_{j=i+1}^{t-1}A(j)\right)B\right)^2\right\}^{1/2}\|u(t)\|_{n_0}^N$$

$$\leqq|C||B|\,\|u(t)\|_{n_0}^N\left\{\sum_{t=n_0}^{N}\sum_{i=n_0}^{t-1}\chi^2\eta^{2(t-1)}\right\}^{1/2}\qquad\text{using Lemma A.2}$$

$$=|C||B|\chi\|u(t)\|_{n_0}^N\left\{\sum_{t=n_0}^{N}\sum_{i=n_0}^{t-1}\eta^{2(t-1)}\right\}^{1/2}.$$

Now

$$\sum_{t=n_0}^{N}\sum_{i=n_0}^{t-1}\eta^{2(t-1)}=\sum_{p=0}^{N-n_0}\sum_{q=0}^{p-1}\eta^{2(p+n_0-1)}$$

$$\leqq\eta^{2(n_0-1)}\sum_{p=0}^{N-n_0}p\eta^{2p}$$

$$\leqq\eta^{2(n_0-1)}\frac{\eta^2}{(1-\eta^2)^2}\qquad\text{since }|\eta|<1$$

$$=\frac{\eta^{2n_0}}{(1-\eta^2)^2}.$$

So we have

$$|\alpha_2(n_0, N)|\leqq(\|u(t)\|_{n_0}^N)^2|C||B|\chi\frac{\eta^{2n_0}}{(1-\eta^2)^2}.$$

Let

$$\varepsilon_2=|C||B|\chi\frac{\eta^{2n_0}}{(1-\eta^2)^2}.$$

So

$$|\alpha_2(n_0, N)| \leqq \varepsilon_2(\|u(t)\|_{n_0}^N)^2.$$

We then have

$$|\alpha(n_0, N)| \leqq (\varepsilon_1+\varepsilon_2)(\|u(t)\|_{n_0}^N)^2 + \varepsilon_1|x(n_0)|^2.$$

Note that ε_1, ε_2 can be made arbitrarily small by taking n_0 large enough.

Let

$$\varepsilon := \varepsilon_1 + \varepsilon_2.$$

Substituting into (2.8)

$$\sum_{t=n_0}^{N} y(t)u(t) \geqq (\delta^* - \varepsilon)(\|u(t)\|_{n_0}^N)^2 - \beta^*(x(n_0)) - \varepsilon_1|x(n_0)|^2.$$

Let

$$\delta := \delta^* - \varepsilon, \qquad \beta := \beta^*(x(n_0)) + \varepsilon_1|x(n_0)|^2.$$

By taking n_0 large enough, it can be guaranteed that $\delta > 0$. □

Remarks. 1. Since system (2.4), (2.5) is both input strictly passive and asymptotically stable, it is in fact very strictly passive (see [15, Appendix C]).

2. The system does not become input strictly passive as usually defined because δ is dependent on $x(n_0)$.

3. The adaptive control algorithm. We are concerned here with the adaptive control of a linear time-varying finite dimensional system admitting an autoregressive moving average representation of the form:

$$\begin{aligned} &y(t) + a_1(t)y(t-1) + \cdots + a_n(t)y(t-n) \\ &\quad = b_0(t)u(t-d) + \cdots + b_m(t)u(t-d-m) \\ &\quad\quad + \omega(t) + c_1(t)\omega(t-1) + \cdots + c_l(t)\omega(t-l). \end{aligned} \tag{3.1}$$

We shall express (3.1) in compact notation as

$$A(t, q^{-1})y(t) = B(t, q^{-1})q^{-d}u(t) + C(t, q^{-1})\omega(t) \tag{3.2}$$

where q^{-1} represents the delay operator and $A(t, q^{-1}) = 1 + a_1(t)q^{-1} + \cdots + a_n(t)q^{-n}$; $B(t, q^{-1}) = b_0(t) + b_1(t)q^{-1} + \cdots + b_m(t)q^{-m}$; $C(t, q^{-1}) = 1 + c_1(t)q^{-1} + \cdots + c_l(t)q^{-l}$. The corresponding initial condition is $x_0 := \{y(0) \cdots y(1-k);\ u(1-d), \cdots, u(1-k);\ \omega(0), \cdots, \omega(1-k)\}$ where $k = \max\{n, m+d, l\}$.

The process $\{x_0, \omega(1), \omega(2), \cdots\}$ is defined on the underlying probability space $(\Omega, \mathcal{F}, P)$ and we define $\mathcal{F}_0$ to be the σ-algebra generated by x_0. Further, for all $t \geqq 1$ $\mathcal{F}_t$ shall denote the σ-algebra generated by the observations up to time t. The distributions of the random variables $x_0, \omega(1), \omega(2), \cdots$ are assumed mutually absolutely continuous with respect to Lebesgue measure.

We make the following assumptions on the process $\{\omega(t)\}$:

(3.3) N.1 $\quad E\{\omega(t)|\mathcal{F}_{t-1}\} = 0 \quad$ a.s. $t \geqq 1$.

(3.4) N.2 $\quad E\{\omega(t)^2|\mathcal{F}_{t-1}\} = \sigma_t^2 \leqq \sigma^2 < \infty, \qquad t \geqq 1$.

(3.5) N.3 $\quad \limsup_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} \|\omega(t)\|^2 < \infty \quad$ a.s.

We wish to design an adaptive control law to cause $\{y(t)\}$ to track (in some sense) a given desired output sequence $\{y^*(t)\}$ and to ensure that $\{y(t)\}$, $\{u(t)\}$ remain bounded (in some sense). Reference [3] presents further background to this problem as well as giving a convergence analysis for a particular adaptive control algorithm for the case when $A(t, q^{-1})$, $B(t, q^{-1})$, $C(t, q^{-1})$ do not depend on t.

In order to specify the algorithm, we assume:

S.1. d is known.

S.2. Upper bounds for n, m, and l are known.

In addition, the following properties of system (3.2) will be assumed in the analysis of the algorithm:

S.3
$$a_i(t) \to a_i, \qquad i = 1, \cdots, n,$$
$$b_i(t) \to b_i, \qquad i = 0, \cdots, m,$$
$$c_i(t) \to c_i, \qquad i = 1, \cdots, l,$$

exponentially fast.

S.4. $B(z)$ and $C(z)$ have all zeros outside the closed unit circle, where

$$B(z) = b_0 + b_1 z + \cdots + b_m z^m,$$
$$C(z) = 1 + c_1 z + \cdots + c_l z^l.$$

S.5. The system $C(q^{-1})z(t) = b(t)$ is input strictly passive.

For simplicity we shall treat the single input single output unit delay ($d = 1$) case. However, natural extensions exist for the multi-input multi-output nonunit delay as explored for non-time-varying systems in [3], [15], etc.

The model (3.2) can be rearranged into the following predictor form:

$$C(t, q^{-1})[y(t) - \omega(t)] = \alpha(t, q^{-1})y(t-1) + \beta(t, q^{-1})u(t-1) \tag{3.6}$$

where

$$\alpha(t, q^{-1}) := [C(t, q^{-1}) - A(t, q^{-1})]q, \tag{3.7}$$

$$\beta(t, q^{-1}) := B(t, q^{-1})q. \tag{3.8}$$

The adaptive control algorithm which we propose to analyze is the following stochastic gradient minimum variance algorithm:

(3.9) A.1
$$\hat{\theta}(t) = \hat{\theta}(t-1) + \frac{\phi(t-1)}{r(t-2) + \phi(t-1)^T \phi(t-1)} e(t)$$

where $\hat{\theta}(t)$ is an estimate of $\theta(t)$ and $\theta^T(t) = (\alpha_1(t), \alpha_2(t), \cdots, \alpha_n(t), b_0(t), \cdots, b_m(t), c_1(t), \cdots, c_l(t))$. $\hat{\theta}(0)$ is given such that $\hat{\theta}_{n+1}(0) \neq 0$.

(3.10) A.2 $\quad r(t-1) = r(t-2) + \phi(t-1)^T \phi(t-1), \qquad r(0) > 0$ given.

A.3 $\quad y^*(t) = \phi(t-1)^T \hat{\theta}(t-1)$.

(3.11) A.4
$$\phi(t-1)^T = (y(t-1), \cdots, y(t-\bar{n}), u(t-1), \cdots, u(t-\bar{n}), -\bar{y}(t-1), \cdots, -y(t-\bar{n}))$$

where $\bar{n}$ = upper bound on max $(n, m+1, l)$.

(3.12)
A.5 $\quad \bar{y}(t) = \phi(t-1)^T \hat{\theta}(t)$.

A.6 $\quad e(t) = y(t) - y^*(t)$.

This algorithm differs slightly from the time-invariant version in [3] by using a posteriori predictions. A discussion of the significance of this can be seen in [15].

The theoretical possibility of division by zero while evaluating $u(t)$ can be avoided since it can be argued inductively [3] that division by zero is a zero probability event. Since all the results in this paper are almost sure results, then division by zero can only affect the convergence on a set of measure zero. This argument depends on the above assumption of absolute continuity of the distribution functions. Hence, the algorithm is well-posed in the sense that all variables remain bounded in finite time (a.s.).

We then have the following global convergence result:

THEOREM 3.1. *Let Assumptions* N.1–N.3 *and* S.1–S.5 *hold for the system* (3.1) *and the algorithm* A.1–A.6. *Then with probability one, for any initial parameter estimate* $\hat{\theta}(0)$

$$\limsup_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} y(t)^2 < \infty, \tag{3.13}$$

$$\limsup_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} u(t)^2 < \infty, \tag{3.14}$$

$$\lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} [E\{(y(t)-y^*(t))^2 | \mathcal{F}_{t-1}\} - \sigma_t^2] = 0 \tag{3.15}$$

where σ_t^2 is the minimum mean square control error achievable at time t by $\mathcal{F}_{t-1}$ measurable controls.

Proof. We shall present an outline proof only, highlighting the key departures from the usual proofs for time-invariant systems as in [3], [15].

Set

$$\eta(t) = y(t) - \bar{y}(t). \tag{3.16}$$

We then have the following preliminary properties of the algorithm:

$$\text{P.1} \quad \eta(t) = \frac{r(t-2)}{r(t-1)} e(t). \tag{3.17}$$

$$\text{P.2} \quad \lim_{N\to\infty} \sum_{t=1}^{N} \frac{\phi(t-1)^T \phi(t-1)}{r(t-1) r(t-2)} < \infty. \tag{3.18}$$

$$\text{P.3} \quad C(t, q^{-1}) z(t) = b(t) \tag{3.19}$$

where

$$z(t) = \eta(t) - \omega(t), \tag{3.20}$$

$$b(t) = -\phi(t-1)^T \tilde{\theta}(t), \tag{3.21}$$

$$\tilde{\theta}(t) = \hat{\theta}(t) - \theta(t). \tag{3.22}$$

$$\text{P.4} \quad E\{b(t)\omega(t) | \mathcal{F}_{t-1}\} = -\frac{\phi(t-1)^T \phi(t-1)}{r(t-1)} \sigma_t^2. \tag{3.23}$$

The above properties are as in [3], [13].

Now subtracting $\theta(t)$ from both sides of (3.9) and using (3.10), (3.17), we have

$$\tilde{\theta}(t) = \tilde{\theta}(t-1) + \theta_e(t) + \frac{\phi(t-1)}{r(t-2)} \eta(t)$$

where

$$\theta_e(t)=\theta(t)-\theta(t-1),$$

$$\tilde{\theta}(t)-\frac{\phi(t-1)\eta(t)}{r(t-2)}=\tilde{\theta}(t-1)+\theta_e(t). \tag{3.24}$$

Define

$$V(t)=\tilde{\theta}(t)^T\tilde{\theta}(t).$$

Then squaring both sides of (3.24) and using (3.21) gives

$$V(t)+\frac{2b(t)\eta(t)}{r(t-2)}+\frac{\phi(t-1)^T\phi(t-1)}{r(t-2)^2}\eta^2(t)=V(t-1)+2\tilde{\theta}(t-1)^T\theta_e(t)+\|\theta_e(t)\|^2.$$

Hence using (3.20), (3.23) we have

$$\begin{aligned} E\{V(t)|\mathscr{F}_{t-1}\}&=V(t-1)-\frac{2}{r(t-2)}E\{b(t)z(t)|\mathscr{F}_{t-1}\}\\ &\quad+\frac{2\phi(t-1)^T\phi(t-1)}{r(t-2)r(t-1)}\sigma_t^2-E\left\{\frac{\phi(t-1)^T\phi(t-1)}{r(t-2)^2}\eta(t)^2|\mathscr{F}_{t-1}\right\}\\ &\quad+\|\theta_e(t)\|^2+2\tilde{\theta}(t-1)^T\theta_e(t). \end{aligned} \tag{3.25}$$

From Assumption S.3, there exists a G, λ with $0<G<\infty$, $|\lambda|<1$ such that

$$\|\theta_e(t)\|^2\leqq G|\lambda|^t. \tag{3.26}$$

Also, for $0<a(t)<\infty$, we have

$$2[\tilde{\theta}(t-1)^T\theta_e(t)]\leqq a(t)^2\|\tilde{\theta}(t-1)\|^2+\frac{1}{a(t)^2}\|\theta_e(t)\|^2.$$

Thus selecting $a(t)^2=|\lambda|^{t/2}$ and using (3.26), we have

$$\begin{aligned} 2[\tilde{\theta}(t-1)^T\theta_e(t)]&\leqq|\lambda|^{t/2}\|\tilde{\theta}(t-1)\|^2+G|\lambda|^{t/2}\\ &=|\lambda|^{t/2}V(t-1)+G|\lambda|^{t/2}. \end{aligned} \tag{3.27}$$

Substituting (3.27) into (3.25) gives

$$\begin{aligned} E\{V(t)|\mathscr{F}_{t-1}\}&\leqq V(t-1)[1+|\lambda|^{t/2}]-\frac{2}{r(t-2)}E\{b(t)z(t)|\mathscr{F}_{t-1}\}\\ &\quad+\frac{2\phi(t-1)^T\phi(t-1)}{r(t-1)r(t-2)}\sigma_t^2-E\left\{\frac{\phi(t-1)^T\phi(t-1)}{r(t-2)^2}\eta(t)^2\,\Big|\,\mathscr{F}_{t-1}\right\}+2G|\lambda|^{t/2}. \end{aligned} \tag{3.28}$$

Define

$$S(t):=2\sum_{j=n_0}^{t}[b(j)z(j)-\delta z(j)^2]+K, \tag{3.29}$$

$$X(t):=V(t)+\frac{S(t)}{r(t-2)}+2\delta\sum_{j=n_0}^{t}\frac{z(j)^2}{r(j-2)}+\sum_{j=n_0}^{t}\frac{\phi(j-1)^T\phi(j-1)}{r(j-2)^2}\eta(j)^2. \tag{3.30}$$

From Assumptions S.5, S.3, Property P.3 and Theorem 2.1, we know that there exist a n_0, $\delta>0$ and a K (depending on the conditions at n_0) such that $S(t)\geqq 0$ for all $t\geqq n_0$. Under these conditions $X(t)\geqq 0$.

It is readily seen using (3.28), (3.30) that

$$E\{X(t)|\mathcal{F}_{t-1}\} \leqq X(t-1)[1+|\lambda|^{t/2}] + \frac{2\phi(t-1)^T\phi(t-1)}{r(t-2)r(t-1)}\sigma_t^2 + 2G|\lambda|^{t/2} \quad \text{for } t \geqq n_0.$$

From Property P.2, and Assumption N.2, we have that

$$\sum_{t=0}^{\infty}\left[\frac{\phi(t-1)^T\phi(t-1)}{r(t-2)r(t-1)}\sigma_t^2 + G|\lambda|^{t/2}\right] < \infty.$$

Thus we can apply the martingale convergence theorem (Appendix B) to conclude

$$X(t) \to X < \infty \quad \text{a.s.} \tag{3.31}$$

Using (3.30), we see that

$$\lim_{N\to\infty} \sum_{t=1}^{N} \frac{z(t)}{r(t-2)} < \infty \quad \text{a.s.,} \tag{3.32}$$

$$\lim_{N\to\infty} \sum_{t=1}^{N} \frac{\phi(t-1)^T\phi(t-1)}{r(t-2)^2}\eta(t)^2 < \infty \quad \text{a.s.} \tag{3.33}$$

The lower summation limits in (3.32), (3.33) can be extended from n_0 to 1 because the algorithm ensures all variables remain bounded in finite time (a.s.).

A simple argument by contradiction can now be used to conclude (3.13) to (3.15) using (3.32) together with Assumptions S.3, S.4 and Theorem A.1. The steps are exactly as in [3] and as explained in general in [15]. □

4. Adaptive control with general state space model. Consider a linear finite dimensional system described by the following time-invariant state space model:

$$x(t+1) = Fx(t) + Gu(t) + v_1(t), \tag{4.1}$$

$$y(t) = Hx(t) + v_2(t) \tag{4.2}$$

where $\{v_1(t)\}$, $\{v_2(t)\}$ are zero mean Gaussian white noise sequences satisfying:

$$E\{v_1(t)v_1(t)^T\} = Q = DD^T \geqq 0, \tag{4.3}$$

$$E\{v_2(t)v_2(t)^T\} = R \geqq 0. \tag{4.4}$$

The initial state $x(0)$ is also assumed to have a Gaussian distribution with mean $\bar{x}_0$ and covariance P_0. We make the following assumptions:

S.S.1: (H, F) is observable.

S.S.2(a): (F, D) has no uncontrollable modes on the unit circle and $P_0 > 0$

or

(b): (F, D) is stabilizable and $P_0 \geqq 0$.

Using standard Kalman filtering ideas [12], the innovations model for (4.1), (4.2) is

$$\hat{x}(t+1) = F\hat{x}(t) + Gu(t) + K(t)\omega(t), \qquad \hat{x}(0) = \bar{x}_0, \tag{4.5}$$

$$y(t) = H\hat{x}(t) + \omega(t). \tag{4.6}$$

Here $K(t)$ is obtained from the solution of the following matrix Riccati equation:

$$\Sigma(t+1) = F\Sigma(t)F^T - F\Sigma(t)H^T(H\Sigma(t)H^T+R)^{-1}H\Sigma(t)F^T + Q, \tag{4.7}$$

$$\Sigma(0) = P_0, \tag{4.8}$$

$$K(t) = F\Sigma(t)H^T(H\Sigma(t)H^T+R)^{-1}. \tag{4.9}$$

In view of Assumption S.S.1, we can transform the system state such that (H, F) are in observer canonical form, i.e. (4.5), (4.6) can be written as

$$\hat{x}(t+1)=\begin{bmatrix} -a_1 & 1 & & 0 \\ \vdots & & \ddots & \\ \vdots & & & 1 \\ -a_n & 0 & \cdots & 0 \end{bmatrix}\hat{x}(t)+\begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}u(t)+\begin{bmatrix} k_1(t) \\ \vdots \\ k_n(t) \end{bmatrix}\omega(t), \tag{4.10}$$

$$y(t)=[1 \quad 0 \quad \cdots \quad 0]\hat{x}(t)+\omega(t). \tag{4.11}$$

Using (4.11) in (4.10) gives the following *time varying* ARMAX model:

$$A(q^{-1})y(t)=B(q^{-1})u(t)+C(t, q^{-1})\omega(t) \tag{4.12}$$

where

$$\begin{aligned} A(q^{-1}) &= 1+a_1q^{-1}+\cdots+a_nq^{-n}, \\ B(q^{-1}) &= b_1q^{-1}+\cdots+b_nq^{-n}, \\ C(t, q^{-1}) &= 1+(k_1(t-1)+a_1)q^{-1}+\cdots+(k_n(t-n)+a_n)q^{-n}. \end{aligned} \tag{4.13}$$

It is known [13] that Assumptions S.S.1, S.S.2 are sufficient to ensure:

$$K(t)\to\bar{K} \quad \text{exponentially fast} \tag{4.14}$$

and

$$C(q^{-1})=1+(\bar{k}+a_1)q^{-1}+\cdots+(\bar{k}_n+a_n)q^{-n} \text{ is asymptotically stable.} \tag{4.15}$$

We make the following additional assumptions:

S.S.3: $b_1\neq 0$ (corresponding to $d=1$ in § 3).

S.S.4: An upper bound for n is known. (As in S.2, this is an assumption on the data supplied to the algorithm.)

S.S.5: The system

$$C(q^{-1})z(t)=b(t)$$

is input strictly passive.

S.S.6: $B(z)$ has all zeros outside the unit circle and $b_1\neq 0$ (the latter for simplicity only).

We then have the following elementary corollary to Theorem 3.1:

COROLLARY 4.1. *Let Assumptions* S.S.1-S.S.6 *hold for the system* (4.1), (4.2) *and the algorithm* A.1-A.6. *Then with probability one, for any initial parameter estimate* $\hat{\theta}(0)$

$$\limsup_{N\to\infty}\frac{1}{N}\sum_{t=1}^{N}y(t)^2<\infty, \tag{4.16}$$

$$\limsup_{N\to\infty}\frac{1}{N}\sum_{t=1}^{N}u(t)^2<\infty, \tag{4.17}$$

$$\lim_{N\to\infty}\frac{1}{N}\sum_{t=1}^{N}E\{(y(t)-y^*(t))^2|\mathcal{F}_{t-1}\}=\sigma^2 \tag{4.18}$$

where

$$\sigma^2=H\bar{\Sigma}H^T+R \tag{4.19}$$

and $\bar{\Sigma}$ *is the steady state solution of* (4.7).

Proof. The result is immediate from Theorem 3.1 on noting that $\{\omega(t)\}$ is a Gaussian innovations sequence and therefore satisfies N.1–N.3. Also, $\sigma_t^2 \to \sigma^2$ exponentially fast [13]. Hence

$$\lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} \sigma_t^2 = \sigma^2. \tag{4.20}$$ □

Remarks. 1. It should be pointed out that this result can be obtained directly from the corresponding result for the steady-state ARMAX model [3], [15]. Firstly, note that the covariance P_0 for the Kalman filter (with $\bar{x}_0$) defines a Gaussian distribution for x_0. So use of the steady-state gain $\bar{K}$ corresponds to a particular choice of the initial condition distribution. Since the martingale convergence theorem leads to a sample path convergence result, modification of the initial state distribution does not affect this result. The proof for this comes by noting that once a.s. convergence has been established with respect to one distribution, then it is also true for any other distribution which is absolutely continuous with respect to the original one [18]. Thus, having proved global convergence for the distribution corresponding to a steady-state Kalman filter, it also holds for other distributions corresponding to the non-steady-state case. However, it should be realized that the original martingale properties will no longer apply.

2. If a general initial condition distribution is used and one replaces (4.12) by the corresponding steady state ARMAX model, then the prediction error so defined will not satisfy N.1.

3. The result in Corollary 4.1 also applies to degenerate distributions, i.e. when the initial state is exactly known. In this case the argument in Remark 1 above cannot be used and the more complicated machinery of § 3 is necessary to deal with this case.

5. Conclusions. This paper has analyzed a robustness property of the discrete time stochastic adaptive control algorithm based on gradient estimation and minimum variance control. The algorithm is shown to be globally convergent when the system parameters are exponentially convergent to values satisfying the conditions for a globally convergent time-invariant system. This result is applied to the special case where the time variation is derived from a non-steady-state Kalman filter.

Appendix A—Properties of convergent linear systems. We present some properties of time-varying linear systems which are convergent toward an asymptotically stable system.

We consider the following time-varying system:

$$x(t+1) = A(t)x(t) + B(t)u(t), \tag{A.1}$$

$$y(t) = C(t)x(t) + D(t)u(t) \tag{A.2}$$

and we introduce the notation:

$$\Phi(n, n_0) = \prod_{k=n_0}^{n-1} A(k) \tag{A.3}$$

for the state transition matrix.

We will use $|\cdot|$ for the Euclidean norm and $\|\cdot\|$ for the l_2 norm (similarly for the induced norms).

LEMMA A.1. *Let A be asymptotically stable. Suppose $A(k) \to A$. Then there exist $\bar{n}$, $v > 0$, and $0 < \beta < 1$ such that*

$$|\Phi(n, n_0)| \leqq v\beta^{n-n_0} \tag{A.4}$$

for all $n > n_0 \geqq \bar{n}$.

Proof. Since A is asymptotically stable, $\exists P>0$ such that

$$P - A^T P A = Q \quad \text{with } Q>0. \tag{A.5}$$

Now consider the autonomous time-varying system

$$x(k+1) = A(k)x(k) \tag{A.6}$$

and try the Lyapunov function

$$V(x) = x^T P x. \tag{A.7}$$

Then

$$\begin{aligned} V(x(k+1)) - V(x(k)) &= x(k)^T [A^T(k) P A(k) - P] x(k) \\ &:= -x(k)^T Q(k) x(k). \end{aligned} \tag{A.8}$$

Now since $Q(k) \to Q > 0$, then $\exists \bar{n}$ such that for $k \geqq \bar{n}$, we have some μ such that

$$V(x(k+1)) - V(x(k)) \leqq -\mu |x(k)|^2. \tag{A.9}$$

Introducing $|x|_P = (x^T P x)^{1/2}$, we have

$$M|x|_P \leqq |x| \quad \text{where } M = (1/\lambda_{\max} P)^{1/2}. \tag{A.10}$$

Hence from (A.9)

$$\begin{aligned} |x(k+1)|_P^2 &\leqq x(k)^T P x(k) - \mu |x(k)|^2 \\ &\leqq (1-\mu M^2)|x(k)|_P^2 \quad \text{for all } x(k). \end{aligned}$$

Thus

$$|A(k)|_P \leqq (1-\mu M^2) := \beta < 1.$$

Hence for $n > n_0 \geqq \bar{n}$

$$|\Phi(n, n_0)|_P = \left| \prod_{k=n_0}^{n-1} A(k) \right|_P \leqq \prod_{k=n_0}^{n-1} |A(k)|_P \leqq \beta^{n-n_0}.$$

So

$$|\Phi(n, n_0)| \leqq v\beta^{n-n_0} \quad \text{for all } n > n_0 \geqq \bar{n}$$

where

$$v = (1/\lambda_{\min} P)^{1/2}.$$

□

The authors suspect that Lemma 2.1 has been established elsewhere though they have no specific reference. Fuchs [14] states a related (but differing in detail) result. We can now immediately establish:

THEOREM A.1. *Consider the system* (2.1), (2.2). *Then provided* $A(k)\to A$ *with* A *asymptotically stable and* $B(k)$, $C(k)$, $D(k)$ *are bounded*:

(a) *There exist* $\bar{n}$, $0<K_1<\infty$, $0\leqq K_2<\infty$ *independent of* N *such that*

$$\sum_{t=n_0}^{N}|y(t)|^2\leqq K_1\sum_{t=n_0}^{N}|u(t)|^2+K_2\quad \text{for all } N\geqq n_0\geqq \bar{n}. \tag{A.11}$$

(b) *There exist* $0\leqq m_3\leqq\infty$, $0\leqq m_4<\infty$ *which are independent of* t *such that*

$$|y_i(t)|\leqq m_3+m_4\max_{n_0\leqq\tau\leqq N}|u(\tau)|\quad \text{for all } N\geqq t\geqq n_0. \tag{A.12}$$

Proof. The proof mimics the corresponding proof for the time-invariant case given elsewhere (see for example [15, Appendix B]). □

In the sequel, we use the notation $\|x(t)\|_{n_0}^{N}:=(\sum_{t=n_0}^{N}|x(t)|^2)^{1/2}$. If the sum is finite for all $N\geqq n_0$, we say $x(t)\in l_{2_e}(\mathbb{Z}_+)$ where $\mathbb{Z}_+$ is the set of integers $n_0, n_0+1,\cdots$.

LEMMA A.2. *Consider a matrix* A *and sequence* $A(k)$ *as in Lemma* A.1. *Suppose* $A(k)\to A$ *exponentially. Then there exist* $\bar{n}$, $\chi>0$, *and* $0<\eta<1$ *such that*

$$\left|A^{n-n_0}-\prod_{k=n_0}^{n-1}A(k)\right|\leqq\chi\eta^{n-1} \tag{A.13}$$

for all $n>n_0\geqq\bar{n}$.

Proof. Let $K(k):=A-A(k)$. We find the following observation useful

$$\begin{aligned}A^{n-n_0}-\prod_{k=n_0}^{n-1}A(k)&=AA^{n-n_0-1}-(A-K(n-1))\prod_{k=n_0}^{n-2}A(k)\\&=A\left(A^{n-n_0-1}-\prod_{k=n_0}^{n-2}A(k)\right)+K(n-1)\prod_{k=n_0}^{n-2}A(k).\end{aligned}$$

Repeated application of this gives

$$\begin{aligned}A^{n-n_0}-\prod_{k=n_0}^{n-1}A(k)&=A^{n-n_0-1}K(n_0)+A^{n-n_0-2}K(n_0+1)A(n_0)\\&\quad+A^{n-n_0-3}K(n_0+2)\prod_{k=n_0}^{n_0+1}A(k)\\&\quad+\cdots+AK(n-2)\prod_{k=n_0}^{n-3}A(k)+K(n-1)\prod_{k=n_0}^{n-2}A(k).\end{aligned}$$

Now since A is stable, $\exists\gamma>0, 0<\xi<1$ such that

$$|A^k|\leqq\gamma\xi^k.$$

Lemma A.1 gives that for $n>n_0\geqq\bar{n}\exists v>0$, $0<\beta<1$ such that

$$\left|\prod_{k=n_0}^{n-1}A(k)\right|\leqq v\beta^{n-n_0}.$$

Further, exponential convergence of $A(k)$ to A gives that $\exists\Omega>0$, $0<\lambda<1$ such that

$$|K(k)|\leqq\Omega\lambda^k.$$

Let

$$\psi:=\max(\gamma,v,\Omega),\qquad \eta:=\max(\xi,\beta,\lambda).$$

Then

$$\left|A^{n-n_0}-\prod_{k=n_0}^{n-1} A(k)\right| \leqq |K(n_0)||A^{n-n_0-1}|+|A(n_0)||K(n_0+1)||A^{n-n_0-2}|$$
$$+\left|\prod_{k=n_0}^{n_0+1} A(k)\right| |K(n_0+2)||A^{n-n_0-3}|$$
$$+\cdots+\left|\prod_{k=n_0}^{n-3} A(k)\right| |K(n-2)||A|$$
$$+\left|\prod_{k=n_0}^{n-2} A(k)\right| |K(n-1)|$$
$$\leqq \psi^2\eta^{n-1}(1+\psi\eta+\psi\eta^2+\cdots+\psi\eta^{n-n_0-2}+\eta^{n-n_0-1})$$

using the above exponential bounds

$$=\psi^2\eta^{n-1}\left(1+\eta^{n-n_0-1}+\psi\eta\sum_{k=0}^{n-n_0-3}\eta^k\right)$$

assuming that $n-n_0 \geqq 3$

$$\leqq \psi^2\eta^{n-1}\left(2+\frac{\psi\eta}{1-\eta}\right).$$

Let

$$\chi := \phi^2\left(2+\frac{\psi\eta}{1-\eta}\right).$$

Then we have the required inequality. (It is easy to see that the result holds for $n > n_0$.) □

Appendix B—Modified martingale convergence theorem. Let $\{X(t)\}$ be a sequence of nonnegative random variables adapted to an increasing sequence of sub σ-algebras $\{\mathcal{F}_t\}$. If

$$E\{X(t+1)|\mathcal{F}_t\} \leqq (1+\gamma(t))X(t)-\alpha(t)+\beta(t) \quad \text{a.s.}$$

where $\alpha(t) \geqq 0$, $\beta(t) \geqq 0$ and $E\{X(0)\} < \infty$, $\sum_{j=1}^{\infty} |\gamma(t)| < \infty$, $\sum_{j=0}^{\infty} \beta(j) < \infty$ a.s. then $X(t)$ converges almost surely to a finite random variable and

$$\lim_{N\to\infty} \sum_{t=0}^{N} \alpha(t) < \infty \quad \text{a.s.}$$

Proof. See Neveu [16]. □

The above theorem was first employed in a stochastic adaptive control proof in [10], [11]. Other applications are given in [15].

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