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Valuing contingent exotic options: a discounted density approach

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Based on a paper with Hans Gerber and Elias Shiu
Brownian motion (Wiener process)

- \( X(t) = \mu t + \sigma W(t) \)
- \( \{W(t)\} \): standard Wiener process
- notation: \( D = \frac{\sigma^2}{2} \)
- running minimum: \( m(t) = \min_{0 \leq s \leq t} X(s) \)
- running maximum: \( M(t) = \max_{0 \leq s \leq t} X(s) \)
2 Exponential stopping of Brownian motion

- $\tau$: exponential random variable independent of \( \{X(t)\} \)
  \[ f_\tau(t) = \lambda e^{-\lambda t}, \quad t > 0 \]

- We are interested in \( X(\tau), M(\tau), ... \)

- $\delta$: force of interest used for discounting
Three discounted density functions

\[ f_{X(\tau)}(x) = \int_0^\infty e^{-\delta t} f_X(t)(x) f_\tau(t) dt \]

\[ f_{M(\tau)}(m) = \int_0^\infty e^{-\delta t} f_M(t)(m) f_\tau(t) dt \]

\[ f_{X(\tau), M(\tau)}(x, m) = \int_0^\infty e^{-\delta t} f_{X(t), M(t)}(x, m) f_\tau(t) dt \]
\(\alpha < 0\) and \(\beta > 0\) solutions of the quadratic equation

\[D\xi^2 + \mu \xi - (\lambda + \delta) = 0\]

1. \(f^\delta_{X(\tau),M(\tau)}(x, m) = \frac{\lambda}{D} e^{-\alpha x - (\beta - \alpha)m} = \frac{\lambda}{D} e^{\alpha(m-x) - \beta m}, \quad -\infty < x \leq m, \quad m \geq 0\)

2. \(f^\delta_{M(\tau)}(m) = \frac{\lambda}{\lambda + \delta} \beta e^{-\beta m}, \quad m \geq 0\)

3. \(f^\delta_{X(\tau)}(x) = \begin{cases} 
\frac{\lambda}{D(\beta - \alpha)} e^{-\alpha x}, & \text{if } x < 0, \\
\frac{\lambda}{D(\beta - \alpha)} e^{-\beta x}, & \text{if } x > 0.
\end{cases}\)
\[ f_{X(\tau), M(\tau) - X(\tau)}(x, z) = \frac{\lambda}{D} e^{-\beta x - (\beta - \alpha) z}, \quad z \geq \max(-x, 0). \]

\[ f_{M(\tau), M(\tau) - X(\tau)}(y, z) = \frac{\lambda}{D} e^{-\beta y + \alpha z}, \quad y \geq 0, \quad z \geq 0. \]

\[ f_{M(\tau) - X(\tau)}(z) = \frac{\lambda}{\beta D} e^{\alpha z} = \frac{\lambda}{\lambda + \delta} (-\alpha) e^{\alpha z}, \quad z \geq 0. \]
\[ m(t) = - \max\{-X(s); \ 0 \leq s \leq t\} \]

If

\[ \mathbb{E}[e^{-\delta \tau} g(X(\tau), M(\tau))] = h(\alpha, \beta), \]

then we can translate it to

\[ \mathbb{E}[e^{-\delta \tau} g(-X(\tau), -m(\tau))] = h(-\beta, -\alpha). \]
\[ f_{\delta X(\tau),m(\tau)}(x, y) = \frac{\lambda}{D} e^{-\beta x + (\beta - \alpha) y}, \quad y \leq \min(x, 0), \]
\[ f_{\delta X(\tau),X(\tau)-m(\tau)}(x, z) = \frac{\lambda}{D} e^{-\alpha x - (\beta - \alpha) z}, \quad z \geq \max(x, 0), \]
\[ f_{\delta m(\tau),X(\tau)-m(\tau)}(y, z) = \frac{\lambda}{D} e^{-\alpha y - \beta z}, \quad y \leq 0, z \geq 0, \]
\[ f_{\delta m(\tau)}(y) = \frac{\lambda}{\beta D} e^{-\alpha y} = \frac{\lambda}{\lambda + \delta(-\alpha)e^{-\alpha y}}, \quad y \leq 0, \]
\[ f_{\delta X(\tau)-m(\tau)}(z) = \frac{\lambda}{-\alpha D} e^{-\beta z} = \frac{\lambda}{\lambda + \delta \beta e^{-\beta z}}, \quad z \geq 0. \]
Factorization formula

If $\tau$ is exponential with mean $1/\lambda$, then the following factorization formula holds,

$$E[e^{-\delta \tau} g_\tau(X)] = E[e^{-\delta \tau}] \times E[g_{\tau^*}(X)],$$

where $\tau^*$ is an exponential random variable with mean $1/(\lambda + \delta)$ and independent of $X$.

Remarks (i) $E[e^{-\delta \tau}] = \frac{\lambda}{\lambda + \delta}$.

(ii) The condition $\delta > 0$ can be replaced by the condition $\delta > -\lambda$. 

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Valuing contingent exotic options: a discounted density approach
3. Financial applications

- $S(t)$: stock price
- $S(t) = S(0)e^{X(t)} = S(0)e^{\mu t + \sigma W(t)}, \quad t \geq 0$
- a contingent option provides a payoff at time $\tau$
- Example: $\tau$: time of death
  GMDB (Guaranteed Minimum Death Benefits)
3. Lookback options

- Many equity-indexed annuities credit interest using a high water mark method or a low water mark method
Out-of-the-money fixed strike lookback call option

- Payoff:
  \[ [S(0)e^{M(\tau)} - K]_+ \]

- Time-0 value
  \[
  \int_k^\infty [S(0)e^y - K] f_{M(\tau)}(y) dy = \frac{\lambda}{\lambda + \delta} [S(0)\frac{\beta e^{-(\beta-1)k}}{\beta - 1} - Ke^{-\beta k}]
  \]
  \[
  = \frac{\lambda}{\lambda + \delta} \frac{K}{\beta - 1} \left[ \frac{S(0)}{K} \right]^\beta .
  \]

- Another expression for the option value
  \[
  \frac{\lambda}{D - \alpha \beta (\beta - 1)} \left[ \frac{S(0)}{K} \right]^\beta .
  \]
In-the-money fixed strike lookback call option

- Payoff

\[ \max(H, S(0)e^{M(\tau)}) - K. \]

- Rewriting as

\[ H - K + [S(0)e^{M(\tau)} - H]_+ \]

- Time-0 value

\[ \frac{\lambda}{\lambda + \delta} \left\{ H - K + \frac{H}{\beta - 1} \left[ \frac{S(0)}{H} \right]^\beta \right\}. \]
Floating strike lookback put option

- **Payoff**

\[
\text{max}(H, \max_{0 \leq t \leq \tau} S(t)) - S(\tau),
\]

where \( H \geq S(0) \).

- **Time-0 value**

\[
\frac{\lambda}{\lambda + \delta} \left\{ H + \frac{H}{\beta - 1} \left[ \frac{S(0)}{H} \right]^\beta \right\} - \mathbb{E}[e^{-\delta \tau} S(\tau)].
\]
Floating strike lookback put option

- Special case: $H = S(0)$, the time-0 value

\[
\frac{\lambda}{\lambda + \delta} \frac{\beta}{\beta - 1} S(0) - \mathbb{E}[e^{-\delta \tau} S(\tau)]
\]

\[
= 1 - \alpha \mathbb{E}[e^{-\delta \tau} S(\tau)] - \mathbb{E}[e^{-\delta \tau} S(\tau)]
\]

\[
= \frac{1}{-\alpha} \mathbb{E}[e^{-\delta \tau} S(\tau)].
\]

This result can be reformulated as

\[
\mathbb{E}[e^{-\delta \tau} \max_{0 \leq t \leq \tau} S(t)] = \left(\frac{1}{-\alpha} + 1\right) \mathbb{E}[e^{-\delta \tau} S(\tau)].
\]
Floating strike lookback put option

Milevsky and Posner (2001) have evaluated (1) with a risk-neutral stock price process and \( H = S(0) \). They also assume that the stock pays dividends continuously at a rate proportional to its price. With \( I \) denoting the dividend yield rate, \( \delta = r \), and \( \mu = r - D - I \), the RHS of (2) is

\[
\frac{2D}{(r - D - I) + \sqrt{(r - D - I)^2 + 4D(\lambda + r)}} \times S(0) \frac{\lambda}{\lambda + I}.
\]

Although it seems rather different from formula (38) on page 117 of Milevsky and Posner (2001), they are the same.
Fractional floating strike lookback put option

Payoff

\[
[\gamma \max_{0 \leq t \leq \tau} S(t) - S(\tau)]_+ = S(0)[\gamma e^{M(\tau)} - e^{X(\tau)}]_+.
\]

Notice

\[
[\gamma e^{M(\tau)} - e^{X(\tau)}]_+ = e^{M(\tau)}[\gamma - e^{X(\tau) - M(\tau)}]_+.
\]
Fractional floating strike lookback put option

Hence

\[
E[e^{-\delta \tau} e^{M(\tau)} [\gamma - e^{X(\tau) - M(\tau)}]_+] = \int_0^\infty \int_0^\infty e^{y}[\gamma - e^{-z}] + f_M^{\delta}(M(\tau)) - X(\tau)(y, z) dy dz
\]

\[
= \frac{\lambda}{D} \left[ \int_0^\infty e^{y} e^{-\beta y} dy \right] \left[ \int_0^\infty [\gamma - e^{-z}]_+ e^{\alpha z} dz \right]
\]

\[
= \frac{\lambda}{D} \frac{1}{\beta - 1} \frac{\gamma^{1-\alpha}}{-\alpha(1 - \alpha)}
\]

\[
= \gamma^{1-\alpha} \frac{\lambda}{\lambda + \delta} \frac{\beta}{(1 - \alpha)(\beta - 1)}
\]

\[
= \gamma^{1-\alpha} \frac{1}{-\alpha} E[e^{-\delta \tau} e^{X(\tau)}].
\]
Fractional floating strike lookback put option

This can be rewritten as

$$E[e^{-\delta \tau} [\gamma e^{M(\tau)} - e^{X(\tau)}]_+] = \gamma^{1-\alpha} E[e^{-\delta \tau} (e^{M(\tau)} - e^{X(\tau)})].$$

Time-0 value

$$E[e^{-\delta \tau} [\gamma \max_{0 \leq t \leq \tau} S(t) - S(\tau)]_+] = \frac{\gamma^{1-\alpha}}{-\alpha} E[e^{-\delta \tau} S(\tau)],$$
Out-of-the-money fixed strike lookback put option

► Payoff

\[ [K - S(0)e^{m(\tau)}]_+ , \]

► Time-0 value

\[
\int_{-\infty}^{k} [K - S(0)e^{y}] f_{m(\tau)}(y)dy = \frac{\lambda}{\lambda + \delta} \frac{K}{1 - \alpha} \left[ \frac{K}{S(0)} \right]^{-\alpha}.
\]
In-the-money fixed strike lookback put option

- **Payoff**

\[ K - \min(H, S(0)e^{m(\tau)}) = K - H + [H - S(0)e^{m(\tau)}]_+ , \]

- **Time-0 value**

\[ \frac{\lambda}{\lambda + \delta} \left\{ K - H + \frac{H}{1 - \alpha} \left[ \frac{H}{S(0)} \right]^{-\alpha} \right\} . \]
Floating strike lookback call option

- **Payoff**

\[
S(\tau) - \min(H, \min_{0 \leq t \leq \tau} S(t)),
\]

where \(0 < H \leq S(0)\).

- **Time-0 value**

\[
\mathbb{E}[e^{-\delta \tau} S(\tau)] + \frac{\lambda}{\lambda + \delta} \left\{ -H + \frac{H}{1 - \alpha} \left[ \frac{H}{S(0)} \right]^{-\alpha} \right\}.
\]

- In the special case where \(H = S(0)\), the time-0 value

\[
\mathbb{E}[e^{-\delta \tau} S(\tau)] - \frac{\lambda}{\lambda + \delta} \frac{-\alpha}{1 - \alpha} S(0)
\]

\[
= \mathbb{E}[e^{-\delta \tau} S(\tau)] - \frac{\beta - 1}{\beta} \mathbb{E}[e^{-\delta \tau} S(\tau)]
\]

\[
= \frac{1}{\beta} \mathbb{E}[e^{-\delta \tau} S(\tau)].
\]
Fractional floating strike lookback call option

Payoff

\[
[S(\tau) - \gamma \min_{0 \leq t \leq \tau} S(t)]_+ = S(0)[e^{X(\tau)} - \gamma e^{m(\tau)}]_+.
\]

\[
= S(0)e^{m(\tau)}[e^{X(\tau)} - m(\tau) - \gamma]_+
\]
Fractional floating strike lookback call option

- Its expected discounted value is $S(0)$ times the following expectation

$$
E[e^{-\delta\tau} e^{m(\tau)} [e^{X(\tau)-m(\tau)} - \gamma]^+] \\
= \frac{\lambda}{D} \left[ \int_{-\infty}^{0} e^{y} e^{-\alpha y} dy \right] \left[ \int_{0}^{\infty} [e^{z} - \gamma] e^{-\beta z} dz \right] \\
= \frac{\lambda}{D} \frac{1}{1 - \alpha \beta (\beta - 1)} \gamma^{1 - \beta} \\
= \frac{1}{\gamma^{\beta - 1}} \frac{\lambda}{\lambda + \delta (1 - \alpha)(\beta - 1)} - \alpha \\
= \frac{1}{\gamma^{\beta - 1}} \frac{1}{\beta} E[e^{-\delta\tau} e^{X(\tau)}].
$$
Fractional floating strike lookback call option

- This can be rewritten as

\[ E[e^{-\delta \tau}[e^{X(\tau)} - \gamma e^{m(\tau)}]^+] = \gamma^{-(\beta-1)} E[e^{-\delta \tau}(e^{X(\tau)} - e^{m(\tau)})]. \]

- We have

\[ E[e^{-\delta \tau}[S(\tau) - \gamma \min_{0 \leq t \leq \tau} S(t)]^+] = \frac{1}{\beta \gamma^{\beta-1}} E[e^{-\delta \tau} S(\tau)]. \]
High-low option

- **Payoff**

\[
\max(H, \max_{0 \leq t \leq \tau} S(t)) - \min(H, \min_{0 \leq t \leq \tau} S(t)),
\]

where \(0 < H \leq S(0) \leq \bar{H}\).

- **Time-0 value**

\[
\frac{\lambda}{\lambda + \delta} \left\{ \frac{\bar{H}}{H} + \frac{\bar{H}}{\beta - 1} \left[ \frac{S(0)}{\bar{H}} \right]^\beta - H + \frac{H}{1 - \alpha} \left[ \frac{H}{S(0)} \right]^{-\alpha} \right\}.
\]

- **In the special case** where \(H = S(0) = \bar{H}\), time-0 value

\[
S(0) \frac{\lambda}{\lambda + \delta} \frac{\beta - \alpha}{(\beta - 1)(1 - \alpha)} = \frac{\beta - \alpha}{-\alpha \beta} \mathbb{E}[e^{-\delta \tau} S(\tau)].
\]

- **This can be rewritten as**

\[
\left( \frac{1}{-\alpha} + \frac{1}{\beta} \right) \mathbb{E}[e^{-\delta \tau} S(\tau)],
\]
Barrier options

- A barrier option is an option whose payoff depends on whether or not the price of the underlying asset has reached a predetermined level or *barrier*.
- Knock-out options are those which go out of existence if the asset price reaches the barrier, and *knock-in options* are those which come into existence if the barrier is reached.
Parity relation

- Knock-out option + Knock-in option = Ordinary option.

- Notation: $L$ denotes the barrier and $\ell = \ln[L/S(0)]$
Up-and-out and up-and-in options ($L > S(0)$ ($\ell > 0$))

▶ Payoffs

\[ I_{\left[ \max_{0 \leq t \leq \tau} S(t) \right] < L} b(S(\tau)) = I_{\left( M(\tau) < \ell \right)} b(S(0)e^{X(\tau)}) \]

▶

\[ I_{\left[ \max_{0 \leq t \leq \tau} S(t) \right] \geq L} b(S(\tau)) = I_{\left( M(\tau) \geq \ell \right)} b(S(0)e^{X(\tau)}) \]
The expected discounted values

- **Up-and-out**

\[
\int_0^\infty \left[ \int_{-\infty}^y I_{(y<\ell)} b(S(0)e^x)f_{X(\tau),M(\tau)}(x, y)dx \right]dy \\
= \frac{\lambda}{D} \int_0^\ell \left[ \int_{-\infty}^y b(S(0)e^x)e^{-\alpha x}dx \right] e^{-(\beta-\alpha)y}dy
\]

- **Up-and-in**

\[
\frac{\lambda}{D} \int_\ell^\infty \left[ \int_{-\infty}^y b(S(0)e^x)e^{-\alpha x}dx \right] e^{-(\beta-\alpha)y}dy;
\]
Down-and-out and down-and-in options  
$(0 < L < S(0) \ (\ell < 0))$

- Payoffs

\[
I([\min_{0 \leq t \leq \tau} S(t)] > L) b(S(\tau)) = I(m(\tau) > \ell) b(S(0) e^{X(\tau)})
\]

- Payoffs

\[
I([\min_{0 \leq t \leq \tau} S(t)] \leq L) b(S(\tau)) = I(m(\tau) \leq \ell) b(S(0) e^{X(\tau)})
\]
The expected discounted values

\[ \frac{\lambda}{D} \int_{\ell}^{0} \left[ \int_{0}^{\infty} b(S(0)e^{x})e^{-\beta x} \, dx \right] e^{(\beta-\alpha)y} \, dy \]

\[ \frac{\lambda}{D} \int^{\ell}_{-\infty} \left[ \int_{y}^{\infty} b(S(0)e^{x})e^{-\beta x} \, dx \right] e^{(\beta-\alpha)y} \, dy, \]
Notation

\[ A_1(n) = \frac{\lambda S(0)^n}{D (n - \alpha)(\beta - n)}, \]
\[ A_2(n) = \frac{\lambda L^n}{D (n - \alpha)(\beta - n)} \left[ \frac{S(0)}{L} \right]^\beta, \]
\[ A_3(n) = \frac{\lambda L^n}{D (n - \alpha)(\beta - n)} \left[ \frac{L}{S(0)} \right]^{-\alpha}, \]
\[ A_4 = \frac{\lambda K^n}{D (n - \alpha)(\beta - \alpha)} \left[ \frac{K}{S(0)} \right]^{-\alpha} = \frac{\kappa K^n}{n - \alpha} \left[ \frac{K}{S(0)} \right]^{-\alpha}, \]
Notation

\[ A_5 = \frac{\lambda K^{n-\alpha} L^{\alpha}}{D (n-\alpha)(\beta-\alpha)} \left[ \frac{S(0)}{L} \right]^\beta = \frac{\kappa K^{n-\alpha} L^{\alpha}}{n-\alpha} \left[ \frac{S(0)}{L} \right]^\beta, \]

\[ A_6 = \frac{\lambda K^n}{D (\beta-n)(\beta-\alpha)} \left[ \frac{S(0)}{K} \right]^\beta = \frac{\kappa K^n}{\beta-n} \left[ \frac{S(0)}{K} \right]^\beta, \]

\[ A_7 = \frac{\lambda K^{-(\beta-n)} L^\beta}{D (\beta-n)(\beta-\alpha)} \left[ \frac{L}{S(0)} \right]^{-\alpha} = \frac{\kappa K^{-(\beta-n)} L^\beta}{\beta-n} \left[ \frac{L}{S(0)} \right]^{-\alpha}, \]

\[ A_8 = \frac{\lambda K}{D -\alpha(1-\alpha)(\beta-\alpha)} \left[ \frac{K}{S(0)} \right]^{-\alpha} = \frac{\kappa K}{-\alpha(1-\alpha)} \left[ \frac{K}{S(0)} \right]^{-\alpha}, \]
Notation

\[ A_9 = \frac{\lambda}{D} \frac{K^{1-\alpha} L^\alpha}{-\alpha(1-\alpha)(\beta-\alpha)} \left[ \frac{S(0)}{L} \right]^\beta \]

\[ = \frac{\kappa K^{1-\alpha} L^\alpha}{-\alpha(1-\alpha)} \left[ \frac{S(0)}{L} \right]^\beta, \]

\[ A_{10} = \frac{\lambda}{D} \frac{K}{\beta(\beta-1)(\beta-\alpha)} \left[ \frac{S(0)}{K} \right]^\beta = \frac{\kappa K}{\beta(\beta-1)} \left[ \frac{S(0)}{K} \right]^\beta, \]

\[ A_{11} = \frac{\lambda}{D} \frac{K^{-(\beta-1)} L^\beta}{\beta(\beta-1)(\beta-\alpha)} \left[ \frac{L}{S(0)} \right]^{-\alpha} \]

\[ = \frac{\kappa K^{-(\beta-1)} L^\beta}{\beta(\beta-1)} \left[ \frac{L}{S(0)} \right]^{-\alpha}. \]
Up-and-out all-or-nothing call option

The option value is

\[
\begin{cases}
0, & \text{if } L < K, \\
\frac{\lambda}{D} \int_0^\ell \int_k^y S(0)^n e^{nx} e^{-\alpha x} dx [e^{-(\beta-\alpha)y} dy], & \text{if } L \geq K \text{ and } S(0) > K, \\
\frac{\lambda}{D} \int_k^\ell \int_k^y S(0)^n e^{nx} e^{-\alpha x} dx [e^{-(\beta-\alpha)y} dy], & \text{if } L \geq K \text{ and } S(0) \leq K
\end{cases}
\]

\[
= \begin{cases}
0, & \text{if } L < K, \\
A_1(n) - A_2(n) - A_4 + A_5, & \text{if } L \geq K \text{ and } S(0) > K, \\
A_6 - A_2(n) + A_5, & \text{if } L \geq K \text{ and } S(0) \leq K.
\end{cases}
\]
Up-and-out all-or-nothing put option

The option value is

\[
\begin{cases}
\frac{\lambda}{D} \int_0^\ell \left[ \int_{-\infty}^{y} S(0)^n e^{nx} e^{-\alpha x} dx \right] e^{-(\beta-\alpha)y} dy, & \text{if } L < K, \\
\frac{\lambda}{D} \int_0^\ell \left[ \int_{-\infty}^{k} S(0)^n e^{nx} e^{-\alpha x} dx \right] e^{-(\beta-\alpha)y} dy, & \text{if } L \geq K \& S(0) > K, \\
\frac{\lambda}{D} \left\{ \int_0^k \left[ \int_{-\infty}^{y} S(0)^n e^{nx} e^{-\alpha x} dx \right] e^{-(\beta-\alpha)y} dy \\
+ \int_k^\ell \left[ \int_{-\infty}^{k} S(0)^n e^{nx} e^{-\alpha x} dx \right] e^{-(\beta-\alpha)y} dy \right\}, & \text{if } L \geq K \& S(0) \leq K.
\end{cases}
\]

\[
\begin{cases}
A_1(n) - A_2(n), & \text{if } L < K, \\
A_4 - A_5, & \text{if } L \geq K \text{ and } S(0) > K, \\
A_1(n) - A_5 - A_6, & \text{if } L \geq K \text{ and } S(0) \leq K.
\end{cases}
\]
up-and-out option with payoff $S(\tau)^n$

\[
\frac{\lambda}{D} \int_0^\ell \left[ \int_{-\infty}^y S(0)^n e^{nx} e^{-\alpha x} \, dx \right] e^{-(\beta-\alpha)y} \, dy = A_1(n) - A_2(n).
\]

This is the sum of the value of the up-and-out all-or-nothing put option and the value of the up-and-out all-or-nothing call option.
Up-and-out call option

The value is

\[
\begin{cases}
0, & \text{if } L < K, \\
A_1(1) - A_2(1) - A_1(0)K + A_2(0)K + A_8 - A_9, & \text{if } L \geq K \text{ and } S(0) > K, \\
A_2(0)K + A_{10} - A_2(1) - A_9, & \text{if } L \geq K \text{ and } S(0) \leq K.
\end{cases}
\]
Up-and-out put option

The value is

\[
\begin{cases}
A_1(0)K - A_2(0)K - A_1(1) + A_2(1), & \text{if } L < K, \\
A_8 - A_9, & \text{if } L \geq K \text{ and } S(0) > K, \\
A_1(0)K - A_1(1) + A_{10} - A_9, & \text{if } L \geq K \text{ and } S(0) \leq K.
\end{cases}
\]
Double barrier option

Payoff:

\[ \pi(S(\tau)) I\{a < m(\tau), M(\tau) < b\} \]
Double barrier option

Using a formula from B & S (2002), we have

\[
E[b(S(0)e^{X(\tau)}) I\{a < m(\tau), M(\tau) < b\}] = \frac{\kappa}{\Xi - \Xi^{-1}} \left[ \Xi \int_{a}^{0} b(S(0)e^{z})e^{-\alpha z} dz + \Xi^{-1} \int_{a}^{0} b(S(0)e^{z})e^{-\beta z} dz 
+ \Xi \int_{0}^{b} b(S(0)e^{z})e^{-\beta z} dz + \Xi^{-1} \int_{0}^{b} b(S(0)e^{z})e^{-\alpha z} dz
- \gamma^{-1} \int_{a}^{b} b(S(0)e^{z})e^{-\alpha z} dz - \gamma \int_{a}^{b} b(S(0)e^{z})e^{-\beta z} dz \right],
\]

where \( \Xi = e^{\frac{1}{2}(b-a)(\beta-\alpha)} \) and \( \gamma = e^{\frac{1}{2}(a+b)(\beta-\alpha)}. \)
Double barrier option

When $\delta > 0$, we have

$$E[e^{-\delta \tau} b(S(0)e^{X(\tau)})I\{a < m(\tau), M(\tau) < b\}]$$

$$= \lambda \int_0^\infty e^{-(\lambda + \delta)t} b(S(0)e^{X(t)})I\{a < m(t), M(t) < b\} \, dt$$

$$= \frac{\kappa}{\Xi - \Xi^{-1}} \left[ \Xi \int_0^a b(S(0)e^z)e^{-\alpha z} \, dz + \Xi^{-1} \int_a^0 b(S(0)e^z)e^{-\beta z} \, dz \\
+ \Xi \int_0^b b(S(0)e^z)e^{-\beta z} \, dz + \Xi^{-1} \int_a^b b(S(0)e^z)e^{-\alpha z} \, dz \\
- \gamma^{-1} \int_a^b b(S(0)e^z)e^{-\alpha z} \, dz - \gamma \int_a^b b(S(0)e^z)e^{-\beta z} \, dz \right].$$
Several stocks

\[ X(t) = (X_1(t), X_2(t), \cdots, X_n(t))' \]

an \( n \)-dimensional Brownian motion.

- **\( \mu \)** the mean vector
- **\( C \)** the covariance matrix of \( X(1) \)

\[ g_t(X) \]

a real-valued functional of the process up to time \( t \).

- **\( h \)** an \( n \)-dimensional vector of real numbers
\[ E[e^{-\delta \tau} e^{h'X(\tau)} g_{\tau}(X)] = E[e^{-\delta(h)\tau} g_{\tau}(X); h], \quad (3) \]

where

\[ \delta(h) = \delta - \ln[M_{X(1)}(h)] \]

\[ = \delta - h'\mu - \frac{1}{2}h'Ch. \]
Proof of (3)

Conditioning on \( \tau = t \), the LHS (3) is

\[
\int_0^\infty e^{-\delta t} E[e^{h'X(t)} g_t(X)] f_\tau(t) dt.
\]

By the factorization formula in the method of Esscher transforms, the expectation inside the integrand can be written as the product of two expectations,

\[
E[e^{h'X(t)}] \times E[g_t(X); h] = [M_{X(1)}(h)]^t \times E[g_t(X); h].
\]

Hence

\[
\int_0^\infty e^{-\delta t} E[e^{h'X(t)} g_t(X)] f_\tau(t) dt = \int_0^\infty e^{-\delta(h)t} E[g_t(X); h] f_\tau(t) dt.
\]
Application of (3)

- \( \mathbf{k} \) \( n \)-dimensional vector of real numbers
- \( q_t(k'X) \) real-valued functional of the process up to time \( t \)

\[
E[e^{-\delta \tau} e^{h'X(\tau)} q_\tau(k'X)] = E[e^{-\delta(h)\tau} q_\tau(k'X); h].
\]

- The quadratic equation becomes

\[
\frac{1}{2} \text{Var}[k'X(1); h] \xi^2 + E[k'X(1); h] \xi - [\lambda + \delta(h)] \\
= \frac{1}{2} k'Ck \xi^2 + k' (\mu + Ch) \xi - (\lambda + \delta - h'\mu - \frac{1}{2} h'Ch)
\]
Special case: $n = 2$

- $S_1(t) = S_1(0)e^{X_1(t)}$ and $S_2(t) = S_2(0)e^{X_2(t)}$
- $\mu = (\mu_1, \mu_2)'$
- $C = \begin{pmatrix} \sigma_1^2 & \rho \sigma_1 \sigma_2 \\ \rho \sigma_1 \sigma_2 & \sigma_2^2 \end{pmatrix}$
Margrabe option

Payoff:

\[ [S_1(\tau) - S_2(\tau)]_+ . \]  \hspace{1cm} (4)

If we rewrite (4) as

\[ e^{X_2(\tau)} [S_1(0) e^{X_1(\tau) - X_2(\tau)} - S_2(0)]_+ , \]

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Valuing contingent exotic options: a discounted density approach
$$E[e^{-\delta\tau}[S_1(\tau) - S_2(\tau)]_+|S_1(0) < S_2(0)] = \frac{\kappa^* S_2(0)}{\beta^* (\beta^* - 1)} \left[ \frac{S_1(0)}{S_2(0)} \right]^\beta^*.$$ 

Here,

$$\kappa^* = \frac{\lambda}{D^*(\beta^* - \alpha^*)},$$

$$D^* = \frac{1}{2} \text{Var}[X_1(1) - X_2(1)] = \frac{1}{2} (\sigma_1^2 + \sigma_2^2 - 2 \rho \sigma_1 \sigma_2),$$

and $\alpha^* < 0$ and $\beta^* > 0$ are the zeros of

$$D^* \xi^2 + (\mu_1 - \mu_2 + \rho \sigma_1 \sigma_2 - \sigma_2^2) \xi - (\lambda + \delta - \mu_2 - \frac{1}{2} \sigma_2^2) = \ln[M_{X(1)}((\xi, 1 - \xi)')] - (\lambda + \delta).$$
If we write (4) as
\[ e^{X_1(\tau)}[S_1(0) - S_2(0)e^{X_2(\tau)-X_1(\tau)}]_{+}, \]

\[ E[e^{-\delta \tau}[S_1(\tau) - S_2(\tau)]_{+}|S_1(0) < S_2(0)] = \kappa^{**}S_1(0) + \alpha^{**}(1 - \alpha^{**}) \left[ \frac{S_1(0)}{S_2(0)} \right]^{-\alpha^{**}}. \]

Here,
\[ \kappa^{**} = \frac{\lambda}{D^{**}(\beta^{**} - \alpha^{**})}, \]
\[ D^{**} = \frac{1}{2} \text{Var}[X_2(1) - X_1(1)] = D^*, \]
and \( \alpha^{**} < 0 \) and \( \beta^{**} > 0 \) are the zeros of
\[ \ln[M_{X(1)}((1 - \xi, \xi)')] - (\lambda + \delta). \]
Hence

\[ \alpha^* = 1 - \beta^{**} \]

and

\[ \beta^* = 1 - \alpha^{**}. \]

Thus, \( \kappa^* = \kappa^{**} \)