Valuation of Variable Annuity Guarantees

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Hans U. Gerber and Elias Shiu
The term variable annuity is used to refer to a wide range of life insurance products, whose benefits can be protected against investment and mortality risks by selecting one or more guarantees out of a broad set of possible arrangements.

Variable annuities were introduced first in USA, in 1950s.

In 1990s, insurers included certain guarantees in such policies, guaranteed minimum death benefits (GMDB), guaranteed minimum living benefits (GMLB).
Background

- VAs were also successfully introduced in Asia market, e.g. in Japan, the volume of such contracts has grown to more than USD 100 bn.
- VAs become popular in Europe
- However, due to the complexity of such contracts (their valuation and hedging), some countries hesitate to offer VAs.
Introduction

- Option’s payoff, e.g. European call: $(S_T - K)_+$
- Contingent option’s payoff: $(S_\tau - K)_+$, where $\tau$ is a random variable, independent of $S_t$. 
Literature review


Distribution of $\tau$

- Any distribution on $(0, \infty)$ can be approximated by a linear combination of exponential distributions

$$ f_\tau(t) = \sum_{i=1}^{n} A_i \lambda_i e^{-\lambda_i t}, \quad t > 0, $$
Distribution of $\tau$

\[
E[e^{-\delta \tau} \Pi(S(\tau))]
= \int_0^\infty e^{-\delta t} E[\Pi(S(t))] f_\tau(t) dt
= \int_0^\infty e^{-\delta t} E[\Pi(S(t))] \left[ \sum_{i=1}^n A_i f_i(t) \right] dt
= \sum_{i=1}^n A_i \int_0^\infty e^{-\delta t} E[\Pi(S(t))] f_i(t) dt.
\]
Brownian motion (Wiener process)

- \( X(t) = \mu t + \sigma W(t) \)
- \( \{ W(t) \} \): standard Wiener process
- notation: \( D = \frac{\sigma^2}{2} \)
- running maximum: \( M(t) = \max_{0 \leq s \leq t} X(s) \)
Three probability density functions:

\( f_{X(t)}(x) \): pdf of \( X(t) \)

\( f_{M(t)}(m) \): pdf of \( M(t) \)

\( f_{X(t),M(t)}(x, m) \): joint pdf of \( X(t) \) and \( M(t) \)
Three probability density functions:

\[
f_X(t)(x) = \frac{1}{\sqrt{2\pi t}\sigma} e^{-\frac{(x-\mu t)^2}{2\sigma^2 t}}, \quad -\infty < x < \infty
\]

\[
f_M(t)(m) = \frac{1}{\sigma\sqrt{2\pi t}} e^{-\frac{(m-\mu t)^2}{2\sigma^2 t}} - \frac{2\mu}{\sigma^2} e^{\frac{2\mu m}{\sigma^2}} \Phi \left( \frac{-m - \mu t}{\sigma\sqrt{t}} \right) + \frac{1}{\sigma\sqrt{2\pi t}} e^{\frac{2\mu m}{\sigma^2}} \frac{(m+\mu t)^2}{2\sigma^2 t}, \quad m \geq 0
\]

\[
f_{X(t),M(t)}(x, m) = \frac{2(2m-x)}{\sigma^3\sqrt{2\pi t^3}} e^{\left(\mu x - \frac{1}{2} \mu^2 t - \frac{(2m-x)^2}{2t}\right)\sigma^{-2}},
\]

\[-\infty < x \leq m, m \geq 0\]
2 Exponential stopping of Brownian motion

- $\tau$: exponential random variable independent of $\{X(t)\}$
  $$f_\tau(t) = \lambda e^{-\lambda t}, \quad t > 0$$

- We are interested in $X(\tau), M(\tau), ...$

- $\delta$: force of interest used for discounting
Three discounted density functions

\[ f_{X(\tau)}(x) = \int_0^\infty e^{-\delta t} f_{X(t)}(x) f_{\tau}(t) \, dt \]

\[ f_{M(\tau)}(m) = \int_0^\infty e^{-\delta t} f_{M(t)}(m) f_{\tau}(t) \, dt \]

\[ f_{X(\tau), M(\tau)}(x, m) = \int_0^\infty e^{-\delta t} f_{X(t), M(t)}(x, m) f_{\tau}(t) \, dt \]
Theorem

\[ \alpha < 0 \text{ and } \beta > 0 \text{ solutions of the quadratic equation } D \xi^2 + \mu \xi - (\lambda + \delta) = 0 \]

1). \[ f^\delta_{X(\tau),M(\tau)}(x, m) = \frac{\lambda}{D} e^{-\alpha x - (\beta - \alpha)m} = \frac{\lambda}{D} e^{\alpha (m - x) - \beta m}, \]
\[ -\infty < x \leq m, \ m \geq 0 \]

2). \[ f^\delta_{M(\tau)}(m) = \frac{\lambda}{\lambda + \delta} \beta e^{-\beta m}, \quad m \geq 0 \]

Kyprianou (2006, Equ.8.2)

3). \[ f^\delta_{X(\tau)}(x) = \begin{cases} \frac{\lambda}{D(\beta - \alpha)} e^{-\alpha x}, & \text{if } x < 0, \\ \frac{\lambda}{D(\beta - \alpha)} e^{-\beta x}, & \text{if } x > 0. \end{cases} \]

Albrecher, Cheung, Thonhauser (2010, Ex.4.1)
Comparison of discounted density functions with probability density functions

\[ f_{X(\tau), M(\tau)}^\delta(x, m) = \frac{\lambda}{D} e^{\alpha(m-x) - \beta m} \]

versus

\[ f_{X(t), M(t)}(x, m) = \frac{2(2m-x)}{\sigma^3 \sqrt{2\pi} t^3} e^{(\mu x - \frac{1}{2} \mu^2 t - \frac{(2m-x)^2}{2t}) \sigma^{-2}} \]
Note that 2) and 3) follow from 1):

\[
f_{\delta M(\tau)}(m) = \int_{-\infty}^{m} f_{X(\tau), M(\tau)}(x, m) \, dx \\
= \int_{-\infty}^{m} \frac{\lambda}{D} e^{\alpha(m-x)-\beta m} \, dx = \frac{\lambda}{\lambda + \delta} \beta e^{-\beta m}
\]

\[
f_{X(\tau)}(x) = \int_{\max(x,0)}^{\infty} f_{X(\tau), M(\tau)}(x, m) \, dm \\
= \kappa e^{-\alpha x - (\beta - \alpha) \max(x,0)} \\
= \begin{cases} 
\kappa e^{-\alpha x}, & \text{if } x < 0, \\
\kappa e^{-\beta x}, & \text{if } x > 0.
\end{cases}
\]

with \( \kappa = \frac{\lambda}{D(\beta - \alpha)} \)
Proof of 1):

- Does not use $f_{X(t), M(t)}(x, m)$
- Does not use the reflection principle
- Idea: For an arbitrary bounded function $\pi(u, m)$, consider

$$V(u, m) = E[e^{-\delta \tau} \pi(u + X(\tau), \max(u + M(\tau), m))]$$

In particular, determine $V(0, 0)$
\[ D \frac{\partial^2 V}{\partial u^2} + \mu \frac{\partial V}{\partial u} - (\lambda + \delta) V + \lambda \pi = 0 \]

Note that \( \alpha < 0 \) and \( \beta > 0 \) are the roots of the characteristic equation.

\[ \frac{\partial V}{\partial m}(u, m)|_{u=m} = 0, \quad \text{etc.} \]

We find that

\[ V(0, 0) = \int_{0}^{\infty} \int_{-\infty}^{m} \frac{\lambda}{D} e^{-\alpha x - (\beta - \alpha)m \pi(x, m)} dx dm \]

Because this is for arbitrary \( \pi \),

we conclude that ...
Another proof of 1):

\[ f_{X(\tau)}(x) = \kappa e^{-\alpha x}, \quad \text{if } x < 0. \]

Let \( f_{X(\tau), \tau}(x, t) \) denote the joint probability density function of \( X(\tau) \) and \( \tau \). Thus

\[ f_{X(\tau)}(x) = \int_0^\infty e^{-\delta t} f_{X(\tau), \tau}(x, t) dt. \]

Let \( \widehat{f_{X(\tau)}}(z) \) denote the two-sided Laplace transform of \( f_{X(\tau)}(x) \), we have

\[
\begin{align*}
\widehat{f_{X(\tau)}}(z) &= \int_{-\infty}^{\infty} e^{-zx} f_{X(\tau)}(x) dx \\
&= \int_{-\infty}^{\infty} \int_0^\infty e^{-zx-\delta t} f_{X(\tau), \tau}(x, t) dt dx = E[e^{-zX(\tau)-\delta \tau}].
\end{align*}
\]
Another proof of 1):

Let $f_{\tau}(t)$ denote the probability density function of $\tau$ and $\hat{f}(z)$ its Laplace transform. Then

$$\hat{f}_{X(\tau)}(z) = \mathbb{E}[\mathbb{E}[e^{-zX(\tau)}|\tau]] = \mathbb{E}[e^{Dz^2\tau - \mu z \tau - \delta \tau}]$$

$$= \hat{f}(-Dz^2 + \mu z + \delta).$$

Note that $\hat{f}_{X(\tau)}(z)$ is well defined for $z$ such that $Dz^2 - \mu z - \delta < 0$; this is an open interval containing 0.
Another proof of 1):

\[
\hat{f}(z) = \frac{\lambda}{z + \lambda}
\]

and therefore,

\[
\hat{f}_{\chi(\tau)}(z) = \frac{\lambda}{-Dz^2 + \mu z + \delta + \lambda}.
\]

We note that \(-\beta\) and \(-\alpha\) are the zeros of the denominator.

\[
f_{\chi(\tau)}(x)\] can be obtained by inverting the Laplace transform.
Another proof of 1)

- For $m \geq \max(x, 0)$,

$$Pr(X(t) \leq x, M(t) > m) = e^{Rm}Pr(X(t) \leq x - 2m),$$

where $R = \mu / D$ (the adjustment coefficient).

- Since this identity is true for each $t > 0$, we can replace $t$ by $T$.

- For $x \leq 0$,

$$F_{X(\tau)}(x) = \frac{\kappa}{-\alpha} e^{-\alpha x} = \frac{\lambda}{-\alpha (\beta - \alpha)D} e^{-\alpha x}.$$
Another proof of 1)

For $m \geq \max(x, 0)$,

$$
Pr(X(\tau) \leq x, M(\tau) > m) = e^{Rm} Pr(X(\tau) \leq x - 2m)
$$

$$
= e^{Rm} F_X(x - 2m) = \frac{\lambda}{-\alpha(\beta - \alpha)D} e^{Rm - \alpha(x-2m)}
$$

$$
= \frac{\lambda}{-\alpha(\beta - \alpha)D} e^{-(\beta-\alpha)m-\alpha x}.
$$

$$
f_{X(\tau), M(\tau)}(x, m) = -\frac{\partial^2}{\partial x \partial m} Pr(X(\tau) \leq x, M(\tau) > m)
$$

$$
= \frac{\lambda}{D} e^{-(\beta-\alpha)m-\alpha x}, \quad m \geq \max(x, 0).
$$
other ways to proof 1)

- The density $f_{X(t),M(t)}(x, m)$ is known
- $f_{X(\tau),M(\tau)}(x, m) = \int_0^\infty e^{-\delta t} f_{X(t),M(t)}(x, m) f_{\tau}(t) dt$
- $f_{X(\tau),M(\tau)}(x, m) = \int_0^\infty f_{X(t),M(t)}(x, m) f_{\tau}(t) dt$
Lemma 1: If $\tau$ is exponential with mean $1/\lambda$, then the following factorization formula holds,

$$E[e^{-\delta \tau} g_\tau(X)] = E[e^{-\delta \tau}] \times E[g_{\tau^*}(X)],$$

where $\tau^*$ is an exponential random variable with mean $1/(\lambda + \delta)$ and independent of $X$.

Remarks (i) $E[e^{-\delta \tau}] = \frac{\lambda}{\lambda + \delta}$.
(ii) The condition $\delta > 0$ can be replaced by the condition $\delta > -\lambda$. 
Proof of the factorization formula

\[ E[e^{-\delta \tau} g_{\tau}(X)] = \int_0^\infty e^{-\delta t} E[g_t(X)] \lambda e^{-\lambda t} \, dt \]

\[ = \frac{\lambda}{\lambda + \delta} \int_0^\infty (\lambda + \delta) e^{-(\lambda + \delta) t} E[g_t(X)] \, dt \]

\[ = E[e^{-\delta \tau}] \times E[g_{\tau^*}(X)]. \]
3. Financial applications

- $S(t)$: stock price
- $S(t) = S(0)e^{X(t)} = S(0)e^{\mu t + \sigma W(t)}, \quad t \geq 0$
- a contingent option provides a payoff at time $\tau$
- Example: $\tau$: time of death
  - GMDB (Guaranteed Minimum Death Benefits)
A contingent option is exercised at time $\tau$

Payoff:

- $[K - S(\tau)]_+$: contingent put option
- $[S(\tau) - K]_+$: contingent call option
- Exotic expressions in terms of $S(\tau)$ and $\max_{0 \leq t \leq \tau} S(t)$
- $[K - S(\tau)]_+ I S_0 e^{M(\tau)} \geq H \} \quad \text{contingent up-and-in put option for } S(0) < H$ where $H$ is the barrier level
The cost of the contingent put option

\[ p = \mathbb{E}[e^{-\delta \tau} [K - S(\tau)]_+] = \mathbb{E}[e^{-\delta \tau} [K - S(0)e^{X(\tau)}]]_+ \]

\[ = \int_{-\infty}^{\ln(K/S(0))} [K - S(0)e^x] f^{X(\tau)}(x) dx, \]

\[ = \begin{cases} \frac{\kappa}{\alpha(\alpha-1)} K \left( \frac{K}{S(0)} \right)^{-\alpha} & \text{if } K \leq S(0), \\ \frac{\kappa}{\beta(\beta-1)} K \left( \frac{K}{S(0)} \right)^{-\beta} + K \frac{\lambda}{\lambda+\delta} - S(0) \frac{\lambda}{\lambda+\delta-\mu-D} & \text{if } K > S(0). \end{cases} \]
Other options

- barrier and double barrier options
- all or nothing options
- Margrabe option
- look back options
- policies has roll-up and/or dividends
- ...

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Valuation of Variable Annuity Guarantees
Consider options that will expire at a fixed time $T$, $T > 0$. Thus, the time-$\tau$ payoff is

$$[K - S(\tau)]_+ I_{(\tau \leq T)},$$

and

$$[K - S(\tau)]_+ - [K - S(\tau)]_+ I_{(\tau > T)}.$$
\( T \)-year \( K \)-strike contingent put option

\[
\begin{align*}
E[e^{-\delta \tau} [K - S(\tau)]_+ I(\tau > T)] &= Pr(\tau > T) E[e^{-\delta \tau} [K - S(\tau)]_+ | \tau > T] \\
&= e^{-(\lambda + \delta) T} E[e^{-\delta \tau} [K - S(T)e^{\mu \tau + \sigma W(\tau)}]_+]
\end{align*}
\]

by the memoryless property.
\( \tau \) is an uniform distribution

- For \( \tau \) exponential, define

\[
V(\delta, \lambda, T) = E\left[ e^{-\delta \tau} \pi(S(\tau), \tau) I_{\{\tau < T\}} \right] \\
= \int_0^T \lambda e^{-(\lambda-\delta)t} E[\pi(S(t), t)] dt.
\]

- For \( \tau \sim U(0, T) \), the cost of option is

\[
\int_0^T \frac{1}{T} e^{-\delta t} E[\pi(S(t), t)] dt = \frac{1}{T} \int_0^T e^{-\delta t} E[\pi(S(t), t)] dt \\
= \frac{1}{T} V(\delta, 0, T).
\]
Erlang distribution

- $\tau$ has an Erlang distribution

$$f_\tau(t) = \frac{\lambda^n t^{n-1}}{(n-1)!} e^{-\lambda t}, \quad t > 0,$$

- $$\hat{f}(z) = \left(\frac{\lambda}{z + \lambda}\right)^n.$$
Erlang distribution

$$
\hat{f}_{X(\tau)}(z) = \left( \frac{\lambda}{-Dz^2 + \mu z + \delta + \lambda} \right)^n \\
= \left( \frac{\lambda}{-D(z + \beta)(z + \alpha)} \right)^n \\
= \kappa^n \left( \frac{1}{z + \beta} - \frac{1}{z + \alpha} \right)^n,
$$

$$
f_{X(\tau)}(x) = \begin{cases} \\
\kappa^n e^{-\alpha x} \sum_{j=1}^{n} \frac{(2n-j-1)}{(n-j)} \frac{(\beta-\alpha)^{n-j}}{(j-1)!(\beta-\alpha)^n}(x)^{-1}, & \text{if } x < 0 \\
\kappa^n e^{-\beta x} \sum_{j=1}^{n} \frac{(2n-j-1)}{(n-j)} \frac{(\beta-\alpha)^{n-j}}{(j-1)!(\beta-\alpha)^n}(x)^{j-1}, & \text{if } x > 0
\end{cases}
$$
A numerical example

- $n \to \infty$, this family of Erlang distributions converges to the degenerate distribution at $T$.
- The price of a European $T$-year $K$-strike put option can be approximated by

$$p = \int_{-\infty}^{\ell} [K - S(0)e^x]f_{X(\tau)}(x)dx.$$
A numerical example

Let $S(0) = 42$, $K = 40$, $\delta = 0.1$, $\sigma = 0.2$ and $T = 0.5$. Using the Black-Scholes formula, the put option price is 0.809.

Table: The prices of put option for various $n$

<table>
<thead>
<tr>
<th>$n$</th>
<th>1</th>
<th>10</th>
<th>20</th>
<th>30</th>
<th>50</th>
<th>100</th>
<th>250</th>
</tr>
</thead>
<tbody>
<tr>
<td>Price</td>
<td>0.624</td>
<td>0.786</td>
<td>0.797</td>
<td>0.801</td>
<td>0.804</td>
<td>0.806</td>
<td>0.808</td>
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