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Realization of a Special Class of Admittances with One Damper and One Inerter

Michael Z. Q. Chen1,2,∗, Kai Wang2, Yun Zou2, and James Lam1

Abstract—In this paper, we investigate the realization problem of a special class of positive-real admittances in a form similar to biquadratic functions but with an extra pole at $s = 0$, which is common in vehicle suspension designs. The number of inerters and dampers is restricted to one in each case. This problem is analogous to restricting the number of resistors and capacitors in electrical circuit synthesis (see also [7]). In our realizations, we impose the condition that no transformer can be employed since large lever ratios can cause difficulties in practical implementation.

We present a necessary and sufficient condition for this special class of positive-real admittances to be realizable employing one damper and one inerter. In addition, an explicit construction is given comprising two different circuit arrangements, one employing four springs and the other two. This paper is organized as follows. In Section II, the problem of synthesizing a special class of passive mechanical networks with one damper and one inerter is formulated. Section III provides a necessary and sufficient condition for the realizability, and the networks to cover the realizability conditions. Conclusions are given in Section IV.

II. PROBLEM FORMULATION

The realization of the admittance $Y(s)$ in the form of

$$Y(s) = k \frac{a_0 s^2 + a_1 s + 1}{s(d_0 s^2 + d_1 s + 1)},$$

(1)

where $a_0, a_1, d_0, d_1 \geq 0$, and $k > 0$ was first discussed by Smith in [23]. It was pointed out that any positive-real $Y(s)$ in the form of (1) can in general be realized using one inerter, two dampers, and three springs. It is known that many mechanical admittances of suspension struts are in this form [24], [25], [27].

For mechanical systems, the spring is the easiest element to be realized practically [8]. Thus, the realization problem of admittance (1) when the number of inerters and dampers

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<th>Electrical</th>
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<td>$F \frac{F}{v} \frac{F}{v}$</td>
<td>$Y(s) = \frac{1}{s}$</td>
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<tr>
<td>$\frac{F}{v} = k(v_2 - v_1)$</td>
<td>spring</td>
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<td>$\frac{F}{v} = \frac{1}{s}(v_2 - v_1)$</td>
<td>inductor</td>
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<td>$F = \frac{1}{s}(v_2 - v_1)$</td>
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<tr>
<td>$F = \frac{1}{s}(v_2 - v_1)$</td>
<td>damper</td>
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Fig. 1. Electrical and mechanical circuit symbols and correspondences. In the force-current analogy forces substitute for currents and voltages substitute for voltages. The admittance $Y(s)$ maps velocity and voltage into force and current, respectively. (The symbol $s$ is the standard Laplace transform variable.)

Applications of the inerter to vehicle suspension [4], [15], [21], [25], motorcycle steering control [10], [11] and vibration absorption [23] have been identified which give performance advantages over more conventional passive solutions.

1 Department of Mechanical Engineering, The University of Hong Kong, Pokfulam Road, Hong Kong.
2 School of Automation, Nanjing University of Science and Technology, Nanjing, P.R. China.
This work was in part supported by GRF HKU 7140/11E, HKU CRCG 201008150001, NNSFC 61004093, and “973 Project” 2012CB720200. Correspondence: MZQ Chen, mzzchen@hku.hk.
is restricted to one in each case is meaningful. In [8], it is shown through a counter example that not all positive-real admittances (1) can be realized using one damper and one inerter. However, the realizability condition of this problem has not yet been given. The present paper addresses the following question: Given a positive-real admittance $Y(s)$ in the form of (1), what additional conditions for $Y(s)$ are needed to be realized with one inerter, one damper, and arbitrary of springs but no transformers?

III. MAIN RESULTS

It is obvious that the structure with one damper, one inerter and arbitrary number of springs can be shown in Fig. 2 where $b, c > 0$, and the network $X$ is realized with springs only. In this paper, we assume that $X$ has a well-defined impedance. Chen and Smith have derived a necessary and sufficient condition for any positive-real admittance to be realized as in the form of Fig. 2 [8].

![Fig. 2. General one-port containing one damper and one inerter.](image)

The realizability condition of admittance $Y(s)$ is presented in terms of the elements of a third-order non-negative definite matrix $R$ as defined as

$$R := \begin{bmatrix} R_1 & R_4 & R_5 \\ R_4 & R_2 & R_6 \\ R_5 & R_6 & R_3 \end{bmatrix}.$$  \hspace{1cm} (2)

The following Lemmas 3.1–3.3 are reviewed, which will be used for later derivations.

**Lemma 3.1:** [8] Let $R$ be a non-negative definite matrix defined in (2). If any first- or second-order minor of $R$ is zero, then there exists an invertible diagonal matrix $D = \text{diag}\{1,x,y\}$ such that $DRD$ is a paramount matrix.

**Lemma 3.2:** [8] Let $R$ be a non-negative definite matrix defined in (2) and suppose that all first- and second-order minors are non-zero. Then there exists an invertible diagonal matrix $D = \text{diag}\{1,x,y\}$ such that $DRD$ is a paramount matrix if and only if one of the following conditions holds:

1. $R_4 R_5 R_6 < 0$;
2. $R_4 R_5 R_6 > 0$, $R_1 > (R_4 R_5 / R_6)$, $R_2 > (R_4 R_6 / R_5)$ and $R_3 > (R_5 R_6 / R_4)$;
3. $R_4 R_5 R_6 > 0$, $R_3 < (R_5 R_6 / R_4)$ and $R_1 R_2 R_3 + R_4 R_5 R_6 - R_1 R_5^2 - R_2 R_3^2 \geq 0$;
4. $R_4 R_5 R_6 > 0$, $R_3 < (R_4 R_5 / R_6)$ and $R_1 R_2 R_3 + R_4 R_5 R_6 - R_1 R_5^2 - R_3 R_2^2 \geq 0$;
5. $R_4 R_5 R_6 > 0$, $R_1 < (R_4 R_5 / R_6)$ and $R_1 R_2 R_3 + R_4 R_5 R_6 - R_1 R_5^2 - R_2 R_3^2 \geq 0$.

**Lemma 3.3:** [8] A positive-real function $Y(s)$ can be realized as the driving-point admittance of a network in the form of Fig. 2, where $X$ has a well-defined impedance and is realized with springs only and $b, c > 0$, if and only if $Y(s)$ can be written in the form of

$$Y(s) = \frac{(R_2 R_3 - R_6^2) \beta_2 + R_3 s^2 + R_2 s + 1}{s (\text{det} R_3^2 + (R_1 R_3 - R_2^2) s^2 + (R_1 R_2 - R_4^2) s + R_1)},$$  \hspace{1cm} (3)

where $R$ as defined in (2) is non-negative definite and there exists an invertible diagonal matrix $D = \text{diag}\{1,x,y\}$ such that $DRD$ is paramount, that is, the elements of $R$ satisfy the conditions of either Lemma 3.1 or Lemma 3.2.

It is noted that the above result is not directly based on the coefficients of the function. To make it easier to check the realizability condition for admittance $Y(s)$ in the form of (1), it seems natural to convert the admittance $Y(s)$ to the form

$$Y(s) = \frac{\alpha_3 s^3 + \alpha_2 s^2 + \alpha_1 s + 1}{\beta_4 s^3 + \beta_3 s^2 + \beta_2 s + \beta_1},$$  \hspace{1cm} (4)

and the conditions are in terms of the coefficients $\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3, \beta_4$ and the following equations are satisfied:

$$\alpha_3 = R_2 R_3 - R_6^2, \quad \alpha_2 = R_3, \quad \alpha_1 = R_2, \quad \beta_4 = \text{det}(R),$$

$$\beta_3 = R_1 R_3 - R_2^2, \quad \beta_2 = R_1 R_2 - R_4^2, \quad \beta_1 = R_1.$$  \hspace{1cm} (5)

Then we have the following lemma.

**Lemma 3.4:** The matrix $R$ as defined in (2) is non-negative definite if and only if all its principle minors are non-negative, which further indicates $\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3, \beta_4 \geq 0$ by (5).

**Proof:** The matrix $R$ is non-negative if and only if all its principle minors are non-negative, which further indicates $\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3, \beta_4 \geq 0$ by (5).

From the above lemma, it is obvious that any function in terms of (3) where $R$ is non-negative definite matrix as defined in (2) can be expressed as a function in form of (4) where $\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3, \beta_4 \geq 0$. Then the next lemma discusses the converse.

**Lemma 3.5:** Given any function $Y(s)$ in the form of (4) where $\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3, \beta_4 \geq 0$, then $Y(s)$ can also be expressed as (3) with non-negative definite $R$ defined in (2) and the entries of $R$ defined in (5) if and only if

$$\alpha_3 \leq \alpha_1 \alpha_2, \quad \beta_3 \leq \alpha_2 \beta_1, \quad \beta_2 \leq \alpha_1 \beta_1,$$

$$\beta_4 + 2\alpha_1 \alpha_2 \beta_1 - \beta_1 \alpha_3 - \alpha_3 \beta_2 - (\alpha_1 \beta_1 - \beta_1) \alpha_2 - (\alpha_3 - \alpha_2 \beta_1)^2 = 4(\alpha_1 \beta_1 - \beta_1)(\alpha_2 \beta_1 - \beta_1)(\alpha_3 - \alpha_2 \beta_1).$$

**Proof:** Sufficiency. Let $R_2 = \alpha_1$, $R_3 = \alpha_2$ and $R_1 = \beta_1$. Since $\alpha_3 \leq \alpha_1 \alpha_2$, $\beta_3 \leq \alpha_2 \beta_1$ and $\beta_2 \leq \alpha_1 \beta_1$, then by introducing the variables $R_4$, $R_5$, and $R_6$ we can make

$$\alpha_3 + R_6^2 = \alpha_1 \alpha_2 = R_2 R_3 \Rightarrow \alpha_3 = R_2 R_3 - R_6^2,$$

$$\beta_3 + R_5^2 = \alpha_2 \beta_1 = R_1 R_3 \Rightarrow \beta_3 = R_1 R_3 - R_5^2,$$

$$\beta_2 + R_4^2 = \alpha_1 \beta_1 = R_1 R_2 \Rightarrow \beta_2 = R_1 R_2 - R_4^2.$$

3846
From the above equations, we can also derive \( R_4 R_5 R_6 \) as
\[
R_4 R_5 R_6 = \pm \sqrt{(\alpha_1 \beta_1 - \beta_2) (\alpha_2 \beta_1 - \beta_3) (\alpha_3 \beta_1 - \beta_4)}. 
\]
Since \((\beta_4 + 2\alpha_1 \alpha_2 \beta_1 - \beta_1 \alpha_3 - \alpha_1 \beta_3 - \alpha_2 \beta_2)^2 = 4(\alpha_1 \beta_1 - \beta_2)(\alpha_2 \beta_1 - \beta_3)(\alpha_3 \beta_1 - \beta_4)\), then
\[
\beta_4 + 2\alpha_1 \alpha_2 \beta_1 - \beta_1 \alpha_3 - \alpha_1 \beta_3 - \alpha_2 \beta_2 = \pm 2\sqrt{(\alpha_1 \beta_1 - \beta_2)(\alpha_2 \beta_1 - \beta_3)(\alpha_3 \beta_1 - \beta_4)}. 
\]
The sign of \( R_4 R_5 R_6 \) will be the same as the term \((\beta_4 + 2\alpha_1 \alpha_2 \beta_1 - \beta_1 \alpha_3 - \alpha_1 \beta_3 - \alpha_2 \beta_2)\).

Hence, if \( \beta_1 + 2\alpha_1 \alpha_2 \beta_1 - \beta_1 \alpha_3 - \alpha_1 \beta_3 - \alpha_2 \beta_2 = 2\sqrt{(\alpha_1 \beta_1 - \beta_2)(\alpha_2 \beta_1 - \beta_3)(\alpha_3 \beta_1 - \beta_4)} \), then let
\[
R_4 R_5 R_6 = \sqrt{(\alpha_1 \beta_1 - \beta_2)(\alpha_2 \beta_1 - \beta_3)(\alpha_3 \beta_1 - \beta_4)} \geq 0; 
\]
otherwise let
\[
R_4 R_5 R_6 = -\sqrt{(\alpha_1 \beta_1 - \beta_2)(\alpha_2 \beta_1 - \beta_3)(\alpha_3 \beta_1 - \beta_4)} \leq 0. 
\]

Then, the equation \( \beta_1 + 2\alpha_1 \alpha_2 \beta_1 - \beta_1 \alpha_3 - \alpha_1 \beta_3 - \alpha_2 \beta_2 = 2R_4 R_5 R_6 \) always holds, which gives \( \beta_4 = 2R_4 R_5 R_6 \).

By \( R_4 R_5 R_6 \), we can simplify the expressions, we define the following terms.

Given a non-negative definite matrix \( \det(\alpha_3 \beta_1 - \beta_4) \), then let \( \alpha_3 \beta_1 - \beta_4 = \alpha_4 \beta_5 + \alpha_5 \beta_1 - \beta_6 \).

Consequently, the following equations are obtained.
\[
\begin{align*}
R_1 - R_4 R_5 R_6 &= R_1 - \frac{2R_2 R_4 R_5 R_6}{R_6} = \beta_1 - \frac{W}{2W_1}, \\
R_2 - R_4 R_5 R_6 &= R_2 - \frac{2R_3 R_4 R_5 R_6}{R_5} = \alpha_1 - \frac{W}{2W_2}, \\
R_3 - R_4 R_5 R_6 &= R_3 - \frac{2R_1 R_4 R_5 R_6}{R_4} = \alpha_2 - \frac{W}{2W_3}. 
\end{align*}
\]

It is obvious that there exists at least one of the first- or second-order minors of \( R \) being zero if and only if one of the following 12 equations is satisfied: \( R_1 = 0, R_2 = 0, R_3 = 0, R_4 = 0, R_5 = 0, R_6 = 0, R_1 R_2 - R_3^2 = 0, R_1 R_3 - R_4^2 = 0, R_1 R_4 - R_5^2 = 0, R_1 R_5 - R_6^2 = 0 \).

Thus, the lemma is proved.
Similarly, we can also get $R_1 R_2 R_3 + R_4 R_5 R_6 - R_1 R_2 R_3 R_4 - R_1 R_2 R_4 R_5 = (\beta_4 + \alpha_2 \beta_2 + \alpha_3 \beta_1 - \alpha_1 \beta_3)/2$ and $R_1 R_5 R_3 - R_1 R_2 R_4 = (\beta_4 + \alpha_4 \beta_4 + \alpha_5 \beta_5 - \alpha_1 \beta_3)/2$. Since all the first- and second-order minors of $R$ are non-zero, $W_1$, $W_2$, $W_3$, and $W_4$ of the above equations never go to zero. Thus, the lemma is proved.

Therefore, the following theorem is obtained, which is equivalent to that in [8].

**Theorem 3.1:** A positive-real function $Y(s)$ can be realized as the driving-point admittance of a network in the form of Fig. 2, where $X$ has a well-defined impedance and is realizable with springs only and $b, c > 0$, if and only if $Y(s)$ can be written in the form of (4), namely

$$Y(s) = \frac{\alpha_3 s^3 + \alpha_2 s^2 + \alpha_1 s + 1}{\beta_4 s^4 + \beta_3 s^3 + \beta_2 s^2 + \beta_1 s},$$

where the coefficients $\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3, \beta_4 \geq 0$, and further satisfy the conditions of Lemma 3.5, and the conditions of either Lemma 3.6 or Lemma 3.7.

**Proof:** Sufficiency. Since $Y(s)$ can be written in the form of (4) with $\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3, \beta_4 \geq 0$ and satisfying conditions of Lemma 3.5, it follows immediately by that lemma that $Y(s)$ can also be expressed as (3), where $R$ is defined in (2) and non-negative definite. Expressing (4) as (3), we see that the non-negative coefficients $\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3, \beta_4 \geq 0$ and satisfy (5). Furthermore, if the conditions of Lemma 3.6 hold, then there must exist at least one minor of $R$ being zero, which by Lemma 3.1 implies that there must exist an invertible $D = \text{diag}\{1, x, y\}$ such that $DRD$ is a paramount matrix; if the conditions of Lemma 3.7 hold, then there also exists such an invertible matrix. Thus, if the conditions of either Lemma 3.6 or Lemma 3.7 hold, by Lemma 3.3 it can be seen that $Y(s)$ can be realized as the driving-point admittance of a network in the form of Fig. 2, where $X$ has a well-defined impedance and is realizable with springs only and $b, c \neq 0$.

**Necessity.** By Lemma 3.3, $Y(s)$ can be written in the form of (3) where $R$ as defined in (2) is non-negative definite. Then it is obvious to see that $Y(s)$ can also be expressed as (4) with the coefficients defined in (5), which indicates that $\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3, \beta_4 \geq 0$, and satisfy the conditions of Lemma 3.5. Since the conditions of either Lemma 3.1 or Lemma 3.2 hold, we can conclude that the conditions of either Lemma 3.6 or Lemma 3.7 are satisfied.

Now we are focusing on the realization of admittance (1). First, it is necessary to show the positive-realness of admittance (1).

**Lemma 3.8:** [9], [23] The $Y(s)$ in the form of (1) with $a_0, a_1, a_2, d_0, d_1 \geq 0$ and $k > 0$ is positive-real if and only if $a_0d_1 - a_1d_0 \geq 0$, and $a_0 - d_0 \geq 0$, and $a_1 - d_1 \geq 0$.

Defining the resultant of $p(s) := a_0 s^2 + a_1 s + 1$ and $q(s) := d_0 s^2 + d_1 s + 1$ in $s$, we have [12]

$$R_k := \det R(p, q) = \begin{vmatrix} a_0 & a_1 & 1 \\ 0 & a_0 & a_1 \\ d_0 & d_1 & 1 \\ 0 & d_0 & d_1 \end{vmatrix} = (a_0 - a_0)^2 - (a_0d_1 - a_1d_0)(a_1 - d_1).$$

It is widely known that $R_k = 0$ if and only if a positive-real $Y(s)$ in the form of (1) can be written in the form:

$$Y(s) = k \frac{a s + 1}{s(ds + 1)},$$

where $a \geq 0$, $d \geq 0$, and $a - d \geq 0$ [12]. Therefore we have the following theorem.

**Theorem 3.2:** Given a positive-real function (1) where $a_0$, $a_1$, $d_0$, $d_1 \geq 0$ and $k > 0$. If $R_k = 0$, then it can be realized with at most one damper and two springs.

**Proof:** From the discussion above, it is known that if $R_k = 0$, then $Y(s)$ can be written as (11) where $a \geq 0$, $d \geq 0$, and $a - d \geq 0$. Furthermore, it is obvious that $Y(s) = k/s + k(a - d)/(ds + 1)$, which is a realization of at most one damper and two springs, which proves this theorem.

From the above theorem, we see that in order to investigate the realizability conditions of (1), it is only necessary to consider the case when $R_k \neq 0$. Then the next theorem presents the realizability condition for admittance (1) to be realized in the form of Fig. 2.

**Theorem 3.3:** Given a positive-real function

$$Y(s) = k \frac{a_0 s^2 + a_1 s + 1}{s(ds^2 + d_1 s + 1)},$$

where $a_0, a_1, a_2, d_0, d_1 \geq 0, k > 0$ and $R_k \neq 0$. It can be realized as the driving-point admittance of a network in the form of Fig. 2, where $X$ has a well-defined impedance and is realizable with springs only and $b, c > 0$, if and only if

$$\frac{d_0^2}{a_0 d_1 - a_1 d_0}(a_1 - d_1) \geq 1$$

or

$$a_0 d_1 - a_1 d_0 = 0.$$  

**Proof:** Necessity. By Theorem 3.1, it is known that $Y(s)$ in the form of (1) with $R_k \neq 0$ can be expressed as (4) with $\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3, \beta_4 \geq 0$. Thus there are two possible cases to reduce (4) to (1).

For the first case, the coefficients in (4) can be regarded as follows:

$$\alpha_3 = 0, \alpha_2 = a_0, \alpha_1 = a_1, \beta_4 = 0, \beta_3 = \frac{d_0}{k}, \beta_2 = \frac{d_1}{k}, \beta_1 = \frac{1}{k},$$

which are all non-negative. Furthermore, the coefficients satisfy the conditions of Lemma 3.5, and either the conditions of Lemma 3.6 or Lemma 3.7. It is obvious that the conditions of Lemma 3.6 must hold. By Lemma 3.8, the positive-realness of $Y(s)$ guarantees the first three conditions of Lemma 3.5, which are shown as follows

$$\alpha_3 - \alpha_1 \alpha_2 \leq 0,$$  

$$\beta_3 - \alpha_2 \beta_1 \leq 0,$$  

$$\beta_2 - \alpha_1 \beta_1 \leq 0.$$
For the fourth equation, we can see that
\[
(\beta_4 + 2\alpha_1 \alpha_2 \beta_1 - \beta_1 \alpha_3 - \alpha_1 \beta_3 - \alpha_2 \beta_2)^2 = 4(\alpha_1 \beta_1 - \beta_2)(\alpha_2 \beta_1 - \beta_3)(\alpha_1 \alpha_2 - \alpha_3),
\]
which indicates (13).

For the second case, the coefficients in (4) can be regarded as follows:
\[
Y(s) = \frac{a_0 T^3 + (a_0 + a_1 T) s^2 + (a_1 + T) s + 1}{s^2(a_0 + a_1 T)^2 + a_1 + T s + 1},
\]
where
\[
\alpha_3 = a_0 T, \quad \alpha_2 = a_0 + a_1 T, \quad \alpha_1 = a_1 + T, \quad \beta_4 = d_0 T/k, \quad \beta_3 = (d_0 + d_1 T)/k, \quad \beta_2 = (d_1 + T)/k, \quad \beta_1 = 1/k, \quad T > 0.
\]

It is obvious that the coefficients are all non-negative, and we know the conditions of Lemma 3.5, and the conditions of either Lemma 3.6 or Lemma 3.7 must hold. The first three equations of Lemma 3.5 hold obviously for the positive-realness of \(Y(s)\), which are shown as follows:
\[
\begin{align*}
\alpha_3 - \alpha_1 \alpha_2 &= -a_0 a_1 - a_1^2 T - a_1 T^2 \leq 0, \\
\beta_3 - \alpha_2 \beta_1 &= \frac{(d_0 - a_0) + (d_1 - a_1) T}{k} \leq 0, \\
\beta_2 - \alpha_1 \beta_1 &= \frac{d_1 - a_1}{k} \leq 0.
\end{align*}
\]

After calculation, the fourth equation \((\beta_4 + 2\alpha_1 \alpha_2 \beta_1 - \beta_1 \alpha_3 - \alpha_1 \beta_3 - \alpha_2 \beta_2)^2 = 4(\alpha_1 \beta_1 - \beta_2)(\alpha_2 \beta_1 - \beta_3)(\alpha_1 \alpha_2 - \alpha_3)\) is equivalent to
\[
(a_1 - d_1)T^2 - (a_0 d_1 - a_1 d_0) = 0.
\]

If \(a_1 = d_1\), then we have \(a_0 d_1 - a_1 d_0\), which is the condition derived in the first case. If \(a_1 \neq d_1\), then \(a_0 d_1 - a_1 d_0\) must hold, which make the positive-realness conditions reduce to \(a_0 d_1 - a_1 d_0 > 0, a_0 - d_0 \geq 0, \) and \(a_1 - d_1 > 0\), which indicates that \(a_0, a_1, d_1 > 0\) and \(d_0 \geq 0\). Hence, \(\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3 > 0,\) and \(\beta_4 \geq 0\). Satisfying the conditions of Lemma 3.5 gives
\[
T = \sqrt{\frac{a_0 d_1 - a_1 d_0}{a_1 - d_1}}.
\]

And we have \(W_1 = a_1 T^2 + a_1^2 T + a_0 a_1 > 0, W_2 = ((a_1 - d_1) T + (a_0 - d_0))/k, W_3 = (a_1 - d_1)/k,\) and
\[
\begin{align*}
\beta_1 - \frac{W}{2 W_1} &= (a_1 - d_1) T^2 + 2a_1 d_1 T + (a_0 d_1 - a_1 d_0) > 0, \\
\alpha_1 - \frac{W}{2 W_2} &= (a_1 - d_1) T^2 + 2(a_0 - d_0) T + (a_0 d_1 - a_1 d_0) > 0, \\
\alpha_2 - \frac{W}{2 W_3} &= (a_1 - d_1) T^2 + (a_0 d_1 - a_1 d_0) > 0.
\end{align*}
\]

Thus, the conditions of Lemma 3.6 cannot be satisfied, which indicates that the conditions of Lemma 3.7 must hold. We see that
\[
W = 2a_1 \sqrt{\frac{(a_0 d_1 - a_1 d_0)(a_1 - d_1)}{k}} + (a_0 - d_0) > 0.
\]

Combining the above equation with (25), we conclude that only condition (3) of Lemma 3.7 holds. Thus, it follows that \(\beta_4 + \alpha_1 \beta_3 + \alpha_3 \beta_1 - \alpha_2 \beta_2 \geq 0\). Since the equation
\[
\beta_4 + \alpha_1 \beta_3 + \alpha_3 \beta_1 - \alpha_2 \beta_2 = \frac{2\sqrt{a_0 d_1 - a_1 d_0} (d_0 - \sqrt{(a_0 d_1 - a_1 d_0)(a_1 - d_1)})}{k a_1 - d_1}
\]
holds, we can obtain (12).

Sufficiency. When \(a_0 d_1 - a_1 d_0 = 0\), express \(Y(s)\) in the form of (4) with the coefficients \(\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3, \beta_4\) satisfying (14), which indicates that they are all non-negative. The positive-realness of \(Y(s)\) guarantees (15), (16), and (17). Since \(a_0 d_1 - a_1 d_0 = 0\), by (18) we see the equation
\[
(\beta_4 + 2\alpha_1 \alpha_2 \beta_1 - \beta_1 \alpha_3 - \alpha_1 \beta_3 - \alpha_2 \beta_2)^2 = 4(\alpha_1 \beta_1 - \beta_2)(\alpha_2 \beta_1 - \beta_3)(\alpha_1 \alpha_2 - \alpha_3)^2
\]
holds. Therefore, the conditions of Lemma 3.5 are satisfied. It is obvious that the conditions of Lemma 3.6 must be satisfied. Hence, by Theorem 3.1, \(Y(s)\) can be realized as the driving-point admittance of a network in the form of Fig. 2, where \(X\) has a well-defined impedance and is realizable with springs only and \(b, c > 0\).

When \(d_0^2/[(a_0 d_1 - a_1 d_0)(a_1 - d_1)] \geq 1\), combining with the positive-realness of \(Y(s)\), we conclude that \(a_0, a_1, d_1, d_0 > 0\). Multiply the numerator and the denominator of (1) with the factor \((T s + 1)\) simultaneously, where \(T\) is defined as (24), then \(Y(s)\) can be expressed as (4) with the coefficients satisfying (19), which are all positive. The positive-realness of \(Y(s)\) guarantees (20), (21), and (22). It is known that defining \(T\) as (24) gives (23), which is equivalent to (26).

Thus the conditions of Lemma 3.5 hold. We can see from the sufficiency part that the third condition of Lemma 3.7 must be satisfied with the defined \(T\). Thus, by Theorem 3.1 \(Y(s)\) can be realized by the required network of this theorem.

We now provide explicit network constructions that will satisfy the realizability conditions. We treat the two conditions (12) and (13) of Theorem 3.3 in Theorems 3.4 and 3.5, respectively.

**Theorem 3.4:** Given a positive-real function \(Y(s)\) in the form of (1) where \(a_0, a_1, d_0, d_1 \geq 0, k > 0,\) and \(R_k \neq 0\). If condition (12) holds, then \(Y(s)\) can be realized as in Fig. 3 with
\[
\begin{align*}
k_1 &= \frac{ka_0 d_1 (a_0 d_1 - a_1 d_0) [(a_1 - d_1) T + (a_0 - d_0)]}{d_0 (a_1 - d_1) (d_1 T + d_0) [(a_0 - d_0) T + (a_0 d_1 - a_1 d_0)]}, \\
k_2 &= \frac{ka_1 d_1 T [(a_1 - d_1) T + (a_0 - d_0)]^2}{[(a_1 - d_1) T + (a_0 - d_0)]^2}, \\
k_3 &= \frac{ka_0 T [(a_1 - d_1) T + (a_0 - d_0)]}{[(d_1 T + d_0) [(a_0 - d_0) T + (a_0 d_1 - a_1 d_0)]}, \\
k_4 &= \frac{ka_0 d_1 (a_0 d_1 - a_1 d_0) [(a_1 - d_1) T + (a_0 - d_0)]}{(a_1 - d_1) [(a_0 - d_0) T + (a_0 d_1 - a_1 d_0)]}, \\
b &= \frac{ka_0^2 d_1^2 [(a_0 d_1 - a_1 d_0) [(a_1 - d_1) T + (a_0 - d_0)]^2}{(a_1 - d_1)^2 [(a_0 - d_0) T + (a_0 d_1 - a_1 d_0)]}.
\end{align*}
\]
where $T$ is defined in (24).

\[ \begin{align*} k_1 + k_2 + c &= 0, \\
& \quad \text{with one damper and one inerter,} \\
& \text{one damper, and two springs.} \\
& \text{and two springs.} \\
& \text{two springs,} \\
& \text{one damper,} \\
& \text{and at most} \\
& \text{each case and number of the springs} \\
& \text{is arbitrary. To solve the problem, we first converted} \\
& \text{previous result by Chen and Smith [8] to a more direct form.} \\
& \text{Then a necessary and sufficient condition for realizability} \\
& \text{was derived. Furthermore, explicit circuit arrangements} \\
& \text{were provided to cover the realizability conditions. We broke down} \\
& \text{the realizable admittances into two groups, where the first} \\
& \text{group can be realized with one inerter, one damper, and four} \\
& \text{springs while the second group with at most one inerter, one} \\
& \text{damper, and two springs.} \\
& \end{align*} \]

**REFERENCES**


