<table>
<thead>
<tr>
<th><strong>Title</strong></th>
<th>Multi-objective optimization for a conventional suspension structure</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Author(s)</strong></td>
<td>Hu, Y; Chen, MZ</td>
</tr>
<tr>
<td><strong>Citation</strong></td>
<td>The 10th World Congress on Intelligent Control and Automation (WCICA 2012), Beijing, China, 6-8 July 2012. In Conference Proceedings, 2012, p. 1235-1240</td>
</tr>
<tr>
<td><strong>Issued Date</strong></td>
<td>2012</td>
</tr>
<tr>
<td><strong>URL</strong></td>
<td><a href="http://hdl.handle.net/10722/160289">http://hdl.handle.net/10722/160289</a></td>
</tr>
<tr>
<td><strong>Rights</strong></td>
<td>World Congress on Intelligent Control and Automation Proceedings. Copyright © IEEE.; ©2012 IEEE. Personal use of this material is permitted. However, permission to reprint/republish this material for advertising or promotional purposes or for creating new collective works for resale or redistribution to servers or lists, or to reuse any copyrighted component of this work in other works must be obtained from the IEEE.; This work is licensed under a Creative Commons Attribution-NonCommercial-NoDerivatives 4.0 International License.</td>
</tr>
</tbody>
</table>
Multi-objective optimization for a conventional suspension structure

Yinlong Hu and Michael Z. Q. Chen

Abstract—This paper investigates the multi-objective optimization of ride comfort, suspension deflection and tyre grip performance measures for a conventional suspension structure by deriving the analytical solutions for a quarter-car model. The optimization results are compared with two other configurations, one having the same complexity in construction but employing an inerter while the other being the simplest suspension network with one damper and one spring only. The motivation is to investigate the possibility and situations where the inerter can be replaced by some cheaper element such as the spring. The results show that for a low static stiffness and in the situations that ride comfort is less important than suspension deflection and tyre grip (such as race cars), the considered structure would be a reasonable alternative for the one employing an inerter.

Index Terms—passive vehicle suspension; quarter-car model; inerter

I. INTRODUCTION

Vehicle suspensions play a major role in a vehicle system and decide the overall vehicle performance like ride comfort, vehicle safety and handling. Generally, suspension systems can be divided into three categories, namely, passive suspension systems, semi-active suspension systems and active suspension systems. The advantage of the passive suspension system is its simplicity and low energy consumption.

Inerter is a recently proposed concept and device with the property that the applied force at the terminals is proportional to the relative acceleration between them [6, 12]. The inerter extends the class of mechanical realizations of complex impedances compared to the ones using only springs and dampers and has been applied to various mechanical systems, including vehicle suspensions [2, 13], motorcycle steering systems [7] and building vibration control [15]. It has also rekindled interest in passive network synthesis [1, 3–5].

The advantage of using the inerter for some performance requirements has been well demonstrated. However, since the performance optimization for a vehicle suspension is a compromise among a number of factors such as ride comfort, suspension deflection and tyre grip and for different kinds of vehicles, the requirements for suspension are different, for example, it is reasonable to improve tyre grip and suspension deflection performance at the cost of ride comfort for race cars. The problem that in which situations the inerter is not essential for suspension performance or the inerter can be replaced by other mechanical elements such as springs has not been considered, but such an issue is important since inerter is more expansive and complex to construct.

This paper investigates the multi-objective optimization of ride comfort, suspension deflection and tyre grip performance for a conventional suspension structure, which has been used in [8]. After deriving the analytical solutions for a quarter-car model and comparing it with the simplest suspension structure [10, 13] and one having a similar complexity but employing an inerter [10], the conditions where the considered structure would be an alternative choice compared the one with inerter are obtained and some general guidelines for practice are highlighted.

The rest of the paper is organized as follows: Section II introduces the relevant background on suspension structures and performance measures. Section III derives the optimal performance measures for ride comfort, suspension deflection and tyre grip individually for the considered conventional structure. Section IV investigates the multi-objective performance optimization and compares the three structures considered. Conclusions are drawn in Section V.

II. VEHICLE MODEL, SUSPENSION NETWORKS, AND PERFORMANCE MEASURES

The quarter-car model presented in Fig. 1 is the simplest model for suspension design. It consists of a sprung mass $m_s$, an unsprung mass $m_u$ and a tyre with spring stiffness $k_t$ [11]. Here, the suspension strut supplying an equal and opposite force on the sprung and unsprung masses is a passive mechanical admittance $Q(s)$. The equations of motion in the Laplace domain are:

$$m_s s^2 \ddot{z}_s = F_s - sQ(s)(\dot{z}_s - \dot{z}_u),$$

$$m_u s^2 \ddot{z}_u = sQ(s)(\dot{z}_s - \dot{z}_u) + k_t (\ddot{z}_r - \ddot{z}_u).$$

Fig. 2 is the suspension configuration under consideration, which is also a conventional suspension layout employed in [8]. The configuration contains a ‘center spring’ $k_1$ [13] and a ‘relaxation spring’ $k_2$ [9]. The static stiffness is

$$K = k + (k_1^{-1} + k_2^{-1})^{-1}. \quad (1)$$

The configurations $C_2$ and $C_3$ shown in Fig. 3 are used for comparison. $C_2$ is the simplest layout and has been discussed in [10, 13]. $C_3$ is the $S_2$ layout in [10], which is similar to the considered structure but employing an inerter.
The performance measures used in this paper are discussed in detail in [1, 14]. For ride comfort, we use the root-meansquare (rms) of body vertical acceleration in response to road disturbances, defined as $J_1$ as follows

$$J_1 = 2\pi (V \kappa)^{\frac{1}{2}} ||sT_{z_v-z_s}||_2,$$

where $V$ is the speed of the car, $\kappa$ is the road roughness parameter. $T_{z_v-z_s}$ denotes the transfer function from the road disturbance $\ddot{z}_s$ to the displacement of the sprung mass $\ddot{z}_v$ and $|| \cdot ||_2$ is the standard $H_2$ norm. The rms suspension deflection parameter $J_2$ is defined as

$$J_2 = 2\pi (V \kappa)^{\frac{1}{2}} \frac{1}{8} ||sT_{\ddot{z}_v-\ddot{z}_s}||_2^2.$$

The rms tyre grip parameter $J_3$ is defined as

$$J_3 = 2\pi (V \kappa)^{\frac{1}{2}} \frac{1}{8} ||sT_{\ddot{z}_v-k_s (\ddot{z}_s-\ddot{z}_v)}||_2^2.$$

The parameters for the quarter car model and performance measures in this paper are (unless otherwise stated): $m_s = 250$ kg, $k_t = 150$ kN m$^{-1}$, $\kappa = 5 \times 10^{-7}$ m$^3$cycle$^{-1}$, $V = 25$ ms$^{-1}$ and $m_u = 35$ kg or 20 kg. Throughout the paper, $F_s$ is equal to 0 since here we are only interested in the responses resulting from the road disturbances.

**III. OPTIMIZATION OF $J_1$, $J_2$ AND $J_3$ INDIVIDUALLY FOR THE CONSIDERED STRUCTURE**

In this section, we derive the analytical solutions of optimal $C_1$ for $J_1$, $J_2$ and $J_3$ respectively in the approach of [10].

An analytical expression of the $H_2$-norm of the (stable) transfer function $G(s)$ can be computed from a minimal state-space realization $G(s) = C(sI - A)^{-1}B$ as $||G||_2 = (CLCT)^{1/2}$, where the matrix $L$ is the unique solution of the Lyapunov equation $AL + LA^T + BB^T = 0$. The performance measures are given by $J_i = 2\pi (V \kappa H)^{1/2}$, where $H = CLCT^T$. We evaluate $H$ algebraically as follows.

**A. $J_1$ optimization results**

Let $m_s$, $m_u$, $k_t$ be fixed and positive. $K$ is the static stiffness shown in (1). Then

$$H_{C_1,J_1}(k_1, k_2, c, K) = c_1 c + c_2 c^{-1},$$

where $c_1 = d_1 k_2^{-2} + d_2 (k_1 + k_2)^{-1} + 3 k_2^2 (k_1 + k_2)$, $c_2 = d_4 (k_1 + k_2)^2 k_2^{-3}$, and

$$d_1 = \frac{k_1 K^2}{2 m_s^2}, \quad d_2 = \frac{K k_1}{m_s^2}, \quad d_3 = \frac{k_1}{2 m_s^2}, \quad d_4 = \frac{(m_u + m_s) K^2}{2 m_s^2}.$$

For any $K > 0$, the minimum of $H_{C_1,J_1}$ is achieved with

$$k_2^{-1} = 0, \quad \hat{c} = \frac{(m_u + m_s)}{k_t}^{1/2} K.$$

The result is consistent with the conclusion of [10] that the relaxation spring itself is not an advantage for ride comfort. When considering the ride comfort performance alone, the optima of $C_1$ and $C_2$ coincide, while for mixed performance measures discussed in Section IV, the relaxation spring should not be neglected.

**B. $J_2$ optimization results**

Let $m_s$, $m_u$, $k_t$ be fixed and positive. $K$ is the static stiffness shown in (1). Then

$$H_{C_1,J_2}(k_1, k_2, c, K) = e_1 k_2^{-2} c + e_2 (k_1 + k_2)^{-1}$$

where $e_1 = k_t/2$ and $e_2 = (m_u + m_s)/2$. For $K > 0$, $\min H_{C_1,J_2} = 0$ with $k_2^{-1} = 0$ and $c^{-1} = 0$.

**C. $J_3$ optimization results**

Let $m_s$, $m_u$, $k_t$ be fixed and positive. $K$ is the static stiffness shown in (1). Then

$$H_{C_1,J_3}(k_1, k_2, c, K) = c_3 c + c_4 c^{-1},$$

where $c_3 = a_1 k_2^{-2} + a_2 (k_1 + k_2)^{-1} + a_3 k_2^2 (k_1 + k_2)^2$, $c_4 = a_4 (k_1 + k_2)^2 k_2^{-3}$, and

$$a_1 = \frac{m_u^2 K^2 k_t}{2 m_s^2} + \frac{m_u k_t (2 K^2 - 2 K k_t)}{2 m_s^2},$$

$$a_2 = \frac{K k_t m_u^2}{2 m_s^2} + \frac{2 K k_t m_u - k_t^2 m_u}{m_s} + K k_t - \frac{1}{2} k_t^2,$$

$$a_3 = \frac{(m_u + m_s) k_t}{2 m_s^2},$$

$$a_4 = \frac{(m_u^3 + m_s^3) K^2}{2 m_s^4} + \frac{m_u^2 (3 K - 2 k_t) K}{2 m_s} + \frac{m_u (3 K^2 - 2 K k_t + k_t^2)}{2}.$$

Denote $K_0 = \frac{(2 m_u + m_s) k_t}{2 (m_u + m_s)^2}$. For $K \geq K_0$, $\min H_{C_1,J_3} = \frac{(m_u + m_s) K^2}{2 m_s^2}$. For $K < K_0$, $\min H_{C_1,J_3}$ is achieved with

$$k_2^{-1} = 0, \quad \hat{c} = \frac{(m_u + m_s)}{k_t}^{1/2} K.$$
2(a3a1)1/2 for \( k_1 = 0, k_2^2 = 0 \) and \( c = (a4/a3)^{1/2} \). The network is effectively reduced to \( C2 \). For \( K < K_0 \), the unique minimum of \( H_{C1,j} \) is \( \left( \frac{a4(a3a1-a2)}{a1} \right)^{1/2} \) and the minimum is achieved with

\[
k_2 > \frac{2a_1}{a_2}, \quad k_1 = -\left( \frac{a_2}{2a_1} k_2^2 + k_2 \right), \quad c = \left( \frac{c_2}{c_1} \right)^{1/2}.
\]

(6)

From the analytical solution of \( J_3, C_1 \) performs better than \( C_2 \) for \( K < K_0 \). The result is shown in Fig. 4.

IV. MULTI-OBJECTIVE PERFORMANCE OPTIMIZATION

We have obtained the analytical expressions of \( C_1 \) for optimal \( J_1, J_2 \) and \( J_3 \), respectively. The analytical expressions of \( C_2 \) and \( C_3 \) for optimal \( J_1 \) and \( J_3 \) are obtained in [10]. \( J_2 \) performance measure for \( C_2 \) and \( C_3 \) can be derived similarly to the \( C_1 \) case. The analytical expressions of \( C_2 \) and \( C_3 \) are shown below:

\[
H_{C2,j_1} = d_3 c + d_4 c^{-1},
\]

(7)

\[
H_{C2,j_1} = (d_3 + d_3b^{-1} + d_6b^{-2})c + ((d_7 + d_8b^{-1}) + d_6b^{-2})k^2 - (d_9 + 2d_9b^{-1})k + (d_4c)^{-1},
\]

(8)

\[
H_{C2,j_2} = e_2c^{-1},
\]

(9)

\[
H_{C2,j_2} = e_2b^{-3}c + ((e_4b^{-3} + e_5b^{-1})k)^2 - 2e_3c^{-1} k + e_2c^{-1},
\]

(10)

\[
H_{C2,j_3} = a_3c + a_4c^{-1},
\]

(11)

\[
H_{C2,j_3} = a_3c + a_4c^{-1},
\]

(12)

where

\[
d_5 = \frac{-(m_u + m_s)K}{m_s^2}, \quad d_6 = \frac{(m_u + m_s)^2K^2 + k_t m_s^2K}{2m_s^2 k_t},
\]

\[
d_7 = \frac{m_u + m_s}{2m_s^2}, \quad d_8 = \frac{-(m_u + m_s)^2K + m_s^2 k_t}{2m_s^2 k_t},
\]

\[
d_9 = \frac{(m_u + m_s)^3K^2 + 2m_s[m_u + m_s]k_t K + m_s^2 k_t^2}{2(m_s k_t)^2},
\]

\[
e_3 = \frac{K(m_u + m_s) + k_t m_s^2}{2K k_t}, \quad e_5 = \frac{m_s^2}{2K^2},
\]

\[
e_4 = \frac{(m_u + m_s)^3K^2 + 2(m_u + m_s)k_t m_s^2 K + k_t^2 m_s^4}{2K^2 k_t^2},
\]

\[
a_5 = \frac{-(m_u + m_s)^3K + m_u m_s (m_u + m_s) k_t}{m_s^2 k_t},
\]

\[
a_6 = \frac{(m_u + m_s)^4K^2 + (m_u + m_s)^2(m_s - 2m_u)m_s k_t K}{2m_s^2 k_t} + \frac{m_s^2 k_t}{2}, \quad \alpha_7 = \frac{(m_u + m_s)^3}{2m_s^2},
\]

\[
a_8 = \frac{2(m_u + m_s)^3K + m_u(m_u + m_s)^2(2m_u - m_s)k_t}{2m_s^2 k_t},
\]

\[
a_9 = \frac{m_u + m_s}{2m_s k_t^2}((m_u + m_s)^4K^2 + 2m_s(m_u + m_s)^2(m_u - m_s)k_t m_s k_t - m_u) k_t K + (m_s^2 - m_u m_u + m_s^2)(m_s k_t)^2).
\]

A. Mixed performance of \( J_1 \) and \( J_2 \)

We now derive the global optimum for a combined measure \( H_{C1,1,2} = (1 - \alpha)H_{C1,j_1} + \alpha m_s^2 H_{C1,j_2} \), where \( \alpha \in [0,1] \) is a weighting between \( J_1 \) and \( J_2 \). The scaling factor \( m_s^2 \) is inserted to approximately normalize the measures.

1) Mixed performance of \( J_1 \) and \( J_2 \) for \( C_1 \): Let \( m_u, m_s, k_t \) be fixed and positive. \( K \) is the static stiffness in (1).

Consider the mixed performance

\[
H_{C1,1,2}(k_1, k_2, c, K) = \left( (1 - \alpha)H_{C1,j_1} + \alpha m_s^2 H_{C1,j_2} \right),
\]

(13)

where \( H_{C1,j_1} \) and \( H_{C1,j_2} \) are given by (2) and (4). For any fixed \( K \) and \( \alpha, H_{C1,1,2} \) has a unique minimum with

\[
k_2^{-1} = 0, \quad c = \left( \frac{(1 - \alpha)d_4 + \alpha m_s^2 e_2}{(1 - \alpha)d_3} \right)^{1/2}.
\]

(14)

The optimal mixed performance of \( J_1 \) and \( J_2 \) for \( C_1 \) requires \( k_2^{-1} = 0 \), then \( C_1 \) reduces to \( C_2 \) and gives no improvement compared with \( C_2 \).

2) Mixed performance of \( J_1 \) and \( J_2 \) for \( C_2 \): Let \( m_u, m_s, k_t \) be fixed and positive. \( K \) is the static stiffness in (1).

Consider the mixed performance

\[
H_{C2,1,2}(c, K) = (1 - \alpha)H_{C2,j_1} + \alpha m_s^2 H_{C2,j_2},
\]

(15)

where \( H_{C2,j_1} \) and \( H_{C2,j_2} \) are given by (7) and (9). For any fixed \( K \) and \( \alpha, H_{C2,1,2} \) has a unique minimum with

\[
c = \left( \frac{(1 - \alpha)d_4 + \alpha m_s^2 e_2}{(1 - \alpha)d_3} \right)^{1/2}.
\]

(16)

3) Mixed performance of \( J_1 \) and \( J_2 \) for \( C_3 \): Let \( m_u, m_s, k_t \) be fixed and positive. Consider

\[
H_{C3,1,2} = (1 - \alpha)H_{C3,j_1} + \alpha m_s^2 H_{C3,j_2} = f_1 c + f_2 c^{-1},
\]

(17)

where \( H_{C3,j_1} \) and \( H_{C3,j_2} \) are given by (8) and (10),

\[
f_1 = (1 - \alpha)(d_3 + d_5b^{-1} + d_6b^{-2}) + \alpha m_s^2 e_3b^{-2},
\]

\[
f_2 = t_2 k^2 + t_1 k + t_0,
\]

where

\[
t_2 = (1 - \alpha)(d_3 + d_5b^{-1} + d_6b^{-2}) + \alpha m_s^2 e_3b^{-2}, \quad t_1 = -(1 - \alpha)(d_5 + 2d_9b^{-1}) - 2\alpha m_s^2 e_3b^{-1}, \quad t_0 = (1 - \alpha)d_4 + \alpha m_s^2 e_2.
\]
For any fixed $K$ and $\alpha$, $H_{C_5:1.2}$ has a unique minimum by $b=b$ and

$$k = \frac{t_1}{2t_2} \quad \text{and} \quad c = \left( \frac{f_2}{f_1} \right)^{1/2}.$$  \hspace{1cm} (18)

Let $Q$ be the set of real positive solutions of function obtained by substituting (18) into (17) and differentiating with respect to $b^{-1}$. With $b_0 = \frac{2(d_1(1-\alpha) + \alpha a_c)}{(\alpha - 1)d_2^2}$, $b$ is given by $b_0$ or $Q \cap (0, b_0)$.

4) Numerical example: We simulate the cases with static stiffness $K = 15, 35, 55$ kN$m^{-1}$, respectively. Fig. 6 and Fig. 7 are given by $m_u = 35$ kg.

Fig. 5 shows that the performance difference is decreasing with increasing static stiffness. The use of inerter ($C_5$) can improve ride comfort performance greatly compared with $C_1$ and $C_2$, which is shown in Fig. 6. However, as Fig. 7 shows, $C_1$ and $C_2$ requiring less suspension deflection than $C_3$ does. In some situations that the suspension travel distance is limited or suspension deflection is more essential than ride comfort performance, $C_1$ and $C_2$ are better.

B. Mixed performance of $J_1$ and $J_2$

In this section, we derive the global optima for a combined measure $H_{C_5:1.3} = (1-\alpha)m_2^2H_{C_1:J_1} + \alpha H_{C_2:J_2}$, where $\alpha \in [0, 1]$ is a weighting between $J_1$ and $J_2$. The scaling factor $m_2^2$ is inserted to approximately normalize the measures.

1) Mixed performance of $J_1$ and $J_3$ for $C_1$: Let $m_s, m_u, k_t$ be fixed and positive. $K$ is the static stiffness in (1). Consider the mixed performance

$$H_{C_1:1.3} = (1-\alpha)m_2^2H_{C_1:J_1} + \alpha H_{C_2:J_3} = J_3 c + f_4 c^{-1},$$

where

$$f_3 = (1-\alpha)m_2^2d_1 + \alpha a_2)(k_1 + k_2)^{-1} + (1-\alpha)m_2^2d_2 + \alpha a_2)(k_1 + k_2)^{-1} + (1-\alpha)m_2^2d_3 + \alpha a_3)k_1(k_1 + k_2)^{-2},$$

$$f_4 = (1-\alpha)m_2^2d_4 + \alpha a_4)(k_1 + k_2)^2k_2^{-2}.$$

Denote

$$K_1 = \frac{\alpha k_1 m_s (2m_u + m_s)}{2(m_s^2 + 2m_u m_s + \alpha m_2^2)^1} ,$$

$$K_2 = \frac{2(m_2^2d_1(1-\alpha) + \alpha a_1)}{m_2^2d_2(1-\alpha) + \alpha a_2}.$$

For $K \geq K_1$, $H_{C_1:1.3}$ has a unique minimum given by

$$k_2^{-1} = 0, \quad c = \left( \frac{1-\alpha}{1-\alpha} \right)^{1/2}.$$

For $K < K_1$, $H_{C_1:1.3}$ has a unique minimum given by

$$k_2 > K_2, \quad c = \left( \frac{f_1}{f_3} \right)^{1/2},$$

$$k_1 = \frac{(m_2^2d_2(1-\alpha) + \alpha a_2)k_2^2}{2(m_2^2d_1(1-\alpha) + \alpha a_1) - (2m_2^2d_1(1-\alpha) + 2\alpha a_1)k_2 - 2(m_2^2d_1(1-\alpha) + \alpha a_1)}.$$

Define $\hat{k} = \frac{k_{num}}{k_{den}}$ and $\hat{c} = \left( \frac{k_{num}}{d_{num}} \right)^{1/2}$, $\gamma = m_2^2$ and

$$k_{num} = 2((\gamma a_4 - d_4)\alpha + d_4)\hat{k}^{-1} + ((\gamma a_5 - d_5)\alpha + d_5),$$

$$k_{den} = 2((\gamma a_9 - d_9)\alpha + d_9)\hat{k}^{-2} + ((\gamma a_8 - d_8)\alpha + d_8)\hat{k}^{-1} + ((\gamma a_7 - d_7)\alpha + d_7),$$

$$c_{num} = ((\gamma a_9 - d_9)\hat{k}^2\hat{k}^{-2} + ((\gamma a_8 - d_8)\hat{k}^{-1} + (\gamma a_7 - d_7)\hat{k},$$

$$c_{den} = ((\gamma a_9 - d_9)\gamma + d_9)\hat{k}^{-2} + ((\gamma a_8 - d_8)\alpha + d_8)\hat{k}^{-1} + (\gamma a_5 - d_5)\hat{k} + (a_4^2 - d_4)\alpha + d_4, \gamma = m_2^2$$

Let $Q$ be the set of real, positive solutions of the equation obtained after substituting (8) and (12) into (22) and differentiating with respect to $b^{-1}$. For any $k \geq 0$, the minimum of $H_{C_1:1.3}$ is achieved with $\hat{k}$, $\hat{c}$ above and $b = -2((\gamma a_6 - d_6)\alpha + d_6) := b_1$ or $b \in Q \cap (0, b_1)$.

4) Numerical results: We simulate cases with static stiffness $K = 20, 55, 75$ kN$m^{-1}$, respectively. Fig. 9 and Fig. 10 are given by $m_u = 35$ kg.

At low static stiffness, $C_1$ performs better than $C_2$ as shown in Fig. 8, but tends to coincide with $C_2$ with increasing
static stiffness, which can be explained by the expression of optimal $H_{C3:1, 3}$. There exists a critical point of static stiffness $K_1$ for the global optimization of $H_{C3:1, 3}$ and for $K > K_1$ $C_1$ reduces to $C_2$. From Fig. 9, $C_1$ has no advantage for ride comfort for both low and high static stiffness compared with $C_3$. However, for tyre grip performance, $C_1$ does better than $C_2$ and $C_3$ do at low static stiffness, as shown in Fig. 10. In summary, the use of inerter ($C_3$) has a considerable advantage both in ride comfort and tyre grip performance in general, especially for a high static stiffness. However, if one is more concerned about tyre grip performance (like race cars) for a low static stiffness, $C_1$ would be a good alternative, since spring is cheaper and easier to construct than inerter.

C. Mixed performance of $J_2$ and $J_3$

We now derive the global optima for a combined measure $H_{C1:2, 3} = (1 - \alpha)m_s^2H_{C1, 2} + \alpha H_{C1, 3}$ where $\alpha \in [0, 1]$ is a weighting between $J_2$ and $J_3$.

1) Mixed performance of $J_2$ and $J_3$ for $C_1$: Let $m_s, m_u, k_t$ be fixed and positive. $K$ is the static stiffness in (1). Consider the mixed performance

$$H_{C1:2, 3} = (1 - \alpha)m_s^2H_{C1, 2} + \alpha H_{C1, 3} = f_5c + f_6c^{-1},$$

where

$$f_5 = ((1 - \alpha)m_s^2e_1 + \alpha a_1)k_t^2 + \alpha a_2(k_1 + k_2)^{-1} + \alpha a_3k_t^2(k_1 + k_2)^{-2},$$

$$f_6 = ((1 - \alpha)m_s^2e_2 + \alpha a_4)(k_1 + k_2)^{-2}k_t^{-2}.$$  

Denote $K_3 = \frac{-2(m_s^2(1 + \alpha a_1))}{\alpha a_2}$. For $K \geq K_0$, $H_{2, 3}$ has a unique minimum given by

$$k_2^{-1} = 0 \text{ and } c = \left(1 - \alpha\right)m_s^2e_2 + \alpha a_4 \right)^{1/2}.$$  

For $K < K_0$, $H_{2, 3}$ has a unique minimum given by

$$k_2 > K_3, \quad c = \frac{a_2a_3k_t}{2} \left(\frac{c_{num}}{c_{den}}\right)^{1/2},$$

$$k_1 = -\frac{k_2(a_2a_3k + 2m_s^2e_1(1 - \alpha) + 2a_1\alpha)}{2m_s^2e_1(1 - \alpha) + a_1\alpha},$$

where

$$c_{num} = \alpha^2(m_s^2e_2(1 - \alpha) + a_4),$$

$$c_{den} = (m_s^2e_1(1 - \alpha) + a_1\alpha^3(4a_3e_1m_s^2(1 - \alpha) + 4a_3a_1a - \alpha a_2^2)).$$

2) Mixed performance of $J_2$ and $J_3$ for $C_2$: Let $m_s, m_u, k_t$ be fixed and positive. Consider the mixed performance

$$H_{C2:2, 3} = (1 - \alpha)m_s^2H_{C2, 2} + \alpha H_{C2, 3} = f_7c + f_8c^{-1},$$

where

$$f_7 = ((1 - \alpha)m_s^2e_2b^{-2} + \alpha(a_3 + a_5b^{-1} + a_6b^{-2})),$$

$$f_8 = ((1 - \alpha)m_s^2e_5b^{-2} + e_5b^{-1} + a(5 + a_5b^{-1} + a_6b^{-2}))k^2 - 2(1 - \alpha)m_s^2e_2b^{-1} + \alpha(a_5 + 2a_6b^{-1}))k + (1 - \alpha)m_s^2e_2 + a_4.$$  

Denote $K_4 = \frac{m_u m_s k_t}{(m_s + m_u)^3}$ and $b_2 = \frac{-2(m_s^2e_3(1 + \alpha a_1))}{a a_5}$. $K$ is the static stiffness. If $K \leq K_4$, the network reduces to $S_4$ in [10]. For $K > K_4$, the unique minimum is obtained by $b = b_2$,

$$k = \frac{k_{num}}{k_{den}}, \quad c = \left(\frac{f_8}{f_7}\right)^{1/2}$$

where

$$k_{num} = (2m_s^2e_4(1 - \alpha) + a_5b + 2a_6b)b,$$

$$k_{den} = 2(a_2a_3^2 + (m_s^2e_5(1 - \alpha) + a_5)b + m_s^2e_4(1 - \alpha) + a_5).$$

Let $Q$ be the set of real, positive solutions $b$ of the equation after substituting (27) into (26) and differentiating with respect to $b^{-1}$. $b$ is equal to $b_2$ or $b \in Q \cap (0, b_2)$.

4) Numerical results: We simulate cases with static stiffness $K = 20, 55, 75 \text{ kN}^{-1}$, respectively. Fig. 12 and Fig. 13 are given by $m_u = 35 \text{ kg}$.

Fig. 11 shows that $C_1$ perform better than $C_2$ and $C_3$ do at low static stiffness and with the increasing of static stiffness, $C_1$ and $C_2$ tend to coincide, which is consistent in Fig. 12 and Fig. 13. It is also consistent with the analytical expression of optimal $H_{C1:2, 3}$. There is a critical point of static stiffness $K_0$ for $C_1$, beyond which $C_1$ reduces to $C_2$. For $J_2$ performance,
as shown in Fig. 12, \( C_1 \) performs better than \( C_3 \) for all range \( K \), but for \( J_3 \) performance, it only performs better at low static stiffness shown in Fig. 13. In summary, if one is more concerned about suspension deflection and tyre grip performance, for a low static stiffness, \( C_1 \) performs better than \( C_2 \) and \( C_3 \) do, even though an inerter is used in \( C_3 \).

V. Conclusion

This paper has presented the analytical solutions for the multi-objective performance optimization including ride comfort, suspension deflection and tyre grip for a quarter-car model. The results show that the considered structure has an advantage in suspension deflection and tyre grip performance at low static stiffness compared with the one with a similar complexity employing the inerter. When considering ride comfort and suspension deflection together the considered structure performs better for suspension deflection performance; for mixed ride comfort and tyre grip performance, it performs better for tyre grip performance at low static stiffness; for mixed suspension deflection and tyre grip performance, it performs better for suspension deflection in all range static stiffness and for tyre grip performance at low static stiffness. In other words, the contributions of inerter for tyre grip performance are mainly for the high static stiffness range. The considered structure may be a good alternative of the one with inerter at low static stiffness where suspension deflection and tyre grip are more important than ride comfort.

References


