<table>
<thead>
<tr>
<th><strong>Title</strong></th>
<th>A Geometric Framework for Investigating the Multiple Unicast Network Coding Conjecture</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Author(s)</strong></td>
<td>Xiahou, T; Wu, C; Huang, J; Li, Z</td>
</tr>
<tr>
<td><strong>Citation</strong></td>
<td>The IEEE International Symposium on Network Coding (NetCod 2012), Cambridge, Massachusetts, USA, 29-30 June 2012. In IEEE International Symposium on Network Coding Proceedings, 2012, p. 37-42</td>
</tr>
<tr>
<td><strong>Issued Date</strong></td>
<td>2012</td>
</tr>
<tr>
<td><strong>URL</strong></td>
<td><a href="http://hdl.handle.net/10722/160083">http://hdl.handle.net/10722/160083</a></td>
</tr>
<tr>
<td><strong>Rights</strong></td>
<td>IEEE International Symposium on Network Coding Proceedings. Copyright © I E E E.; ©2012 IEEE. Personal use of this material is permitted. However, permission to reprint/republish this material for advertising or promotional purposes or for creating new collective works for resale or redistribution to servers or lists, or to reuse any copyrighted component of this work in other works must be obtained from the IEEE.; This work is licensed under a Creative Commons Attribution-NonCommercial-NoDerivatives 4.0 International License.</td>
</tr>
</tbody>
</table>
A Geometric Framework for Investigating the Multiple Unicast Network Coding Conjecture

Tang Xiaohou¹, Chuan Wu¹, Jiaqing Huang⁵, Zongpeng Li¹†

¹ Department of Computer Science, University of Calgary, Canada
‡ Department of Computer Science, The University of Hong Kong
⁻ Dept. of Electronics & Information Engineering, Huazhong University of Science and Technology, China
† Institute of Network Coding, The Chinese University of Hong Kong

Abstract— The multiple unicast network coding conjecture states that for multiple unicast sessions in an undirected network, network coding is equivalent to routing. Simple and intuitive as it appears, the conjecture has remained open since its proposal in 2004 [1], [2], and is now a well-known unsolved problem in the field of network coding. Based on a recently proposed tool of space information flow [3]–[5], we present a geometric framework for analyzing the multiple unicast conjecture. The framework consists of four major steps, in which the conjecture is transformed from its throughput version to cost version, from the graph domain to the space domain, and then from high dimension to 1-D, where it is to be eventually proved. We apply the geometric framework to derive unified proofs to known results of the conjecture, as well as new results previously unknown. A possible proof to the conjecture based on this framework is outlined.

I. INTRODUCTION

Network coding encourages information flows to be “mixed” in the middle of a network, via means of coding [6], [7]. While network coding for a single communication session (unicast, broadcast or multicast) is well understood by now, the case of multiple sessions (multi-source, multi-sink) is much harder, with fewer results known [8]. The case of multiple independent one-to-one unicast sessions is probably the most basic scenario of the multi-source multi-sink setting. With routing, multiple unicast is equivalent to the combinatorial problem of multicommodity flows (MCF) [9], which is polynomial time computable (assuming fractional flows are allowed). With network coding, the structure and the computational complexity of the optimal solution are largely unknown.

If the network is directed, network coding can outperform routing for multiple unicast sessions. Fig. 1(A) shows a network coding solution for two unicast sessions, each with an end-to-end throughput of 1. If each link direction is fixed, then we can verify that achieving a throughput of 1 and 1 concurrently is infeasible without network coding. The potential of throughput improvement due to network coding is unbounded, for multiple unicast in a directed network [1].

Interestingly, the picture is drastically different in undirected networks, where the capacity of a link is flexibly sharable in two opposite directions. No example is known where network coding makes a difference from routing. Fig. 1(B) shows a MCF with end-to-end flow rate of 1 and 1, which is feasible if the underlying network in Fig. 1(A) is undirected. Harvey et al. [2] and Li and Li [1] conjectured that network coding is equivalent to routing for multiple unicast in undirected networks.

Despite a series of research effort devoted [10]–[12] to it, this fundamental problem in network coding has witnessed rather limited progresses towards its resolution. Besides “easy” cases where the cut set bounds can be achieved without network coding [1], [2], the conjecture has been verified only in small, fixed networks and their variations, such as the Okamura-Seymour network [10], [11]. It is worth noting that such verification already involves new tools such as information dominance [10], input-output equality and crypto equality [11].

In 2007, Mitzenmacher et al. compiled a list of seven open problems in network coding [13], where the multiple unicast conjecture appears as problem number 1. Chekuri commented that claiming an equivalence between network coding and routing for all undirected networks is a “bold conjecture”, and that the problem of fully understanding network coding for multiple unicast sessions is still “wild open” ([14], p51-55). A growing agreement is that new tools beyond a “simple blend” of graph theory and information theory are required for eventually settling the conjecture.

In this work, we apply a recently proposed tool, space information flow [3]–[5], to develop a geometric framework for studying the multiple unicast network coding conjecture.
The framework consists of four main steps. In Step 1, LP duality is applied for translating the conjecture from its throughput version to an equivalent cost version. In Step 2, graph embedding is performed, for translating the cost version from the network domain to the space domain. Step 3 aims at dimension reduction that brings the problem from a high dimension space to 1-D. Step 4 contains a direct proof in 1-D, where the cut condition on information flow transmission is readily applicable. Based on the geometric framework, we derive unified proofs to a number of known results on the conjecture, as well as new results unknown before.

Step 1 of the framework borrows an existing result from previous work [1]. Step 2 builds upon recent work on space information flow, where the optimal transmission of information flows, in a geometric space instead of in a fixed network topology, is studied. Step 3 exploits recent results developed in the space information flow paradigm and new results developed in this work. Step 4 is relatively simple, where the proof is done by taking an integration over the 1-D space on both sides of the cut condition inequality [4].

We hope that this framework will shed light on the original multiple unicast conjecture in network coding, and possibly other problems in network information flow.

II. MODEL AND PRELIMINARIES

We use \( G = (V, E) \) to represent an undirected network, with \( |V| = n \) nodes. Let \( c \in Q^E_+ \) be a link capacity vector, and \( w \in Q^E_+ \) be a link cost vector. Here \( Q_+ \) is the set of positive rational numbers. For the multiple unicast problem, the set \( V \) contains in particular \( k \) sender-receiver pairs, \( s_i \) and \( t_i \), \( 1 \leq i \leq k \). The \( k \) unicast sessions are independent, and have a desired throughput vector \( r = (r_1, \ldots, r_k) \).

In the max-throughput version of the multiple unicast problem, we are given a capacitated network \((G, c)\), and wish to maximize a ratio \( \alpha \geq 0 \), such that the throughput vector \( \alpha r \) or can be achieved. Let \( \alpha_{NC} \) and \( \alpha_{MCF} \) be the maximum values of \( \alpha \) possible, under network coding and routing (MCF), respectively, then the coding advantage is defined as the ratio \( \alpha_{NC}/\alpha_{MCF} \).

In the min-cost version of the multiple unicast problem, we are given a link-weighted network \((G, w)\), with each link having unlimited capacity. Under routing (MCF), the minimum cost for achieving a throughput vector \( r \) is \( \sum_i (d_i r_i) \), where \( d_i \) is the shortest path length between \( s_i \) and \( t_i \) in \( G \), under cost vector \( w \). Under network coding, we wish to minimize the total solution cost \( \sum_c (w(c) f^*(c)) \), such that vector \( f \) together with some code assignment forms a valid network coding solution for achieving throughput vector \( r \). Assume \( f^* \) is the underlying flow vector of an optimal network coding solution, we define the cost advantage of network coding as the ratio \( \sum_i (d_i r_i) / \sum_c (w(c) f^*(c)) \).

A \( h \)-D space with \( p \)-norm distance is denoted as \( l^h_p \). For two nodes \( u \) and \( v \) in \( l^h_p \) with coordinates \((x_{u1}, \ldots, x_{uh})\) and \((x_{v1}, \ldots, x_{vh})\), respectively, the distance between \( u \) and \( v \) is:

\[
||u, v||_p = \left( \sum_{i=1}^{h} |x_{ui} - x_{vi}|^p \right)^{\frac{1}{p}}
\]

For the multiple unicast version of the space information flow problem, we are given \( k \) pairs of terminals, \((s_i, t_i)\), \( 1 \leq i \leq k \), in a space \( l^h_p \). We seek the min-cost solution that can achieve a throughput vector \( r \), under the rule that relay nodes can be inserted anywhere for free, and the cost of a one-hop transmission is proportional to both its flow rate and its geometric distance. Under routing (MCF), the optimal cost is \( \sum_i (||s_i t_i||_p r_i) \). Under network coding, let \( f^* \) be the underlying flow vector of the optimal solution. The cost is then \( \sum_{e \in f^*} (||e||_p f(e)) \). The cost advantage of network coding here is \( \sum_i (||s_i t_i||_p r_i) / \sum_{e \in f^*} (||e||_p f^*(e)) \).

III. THE GEOMETRIC FRAMEWORK

In this section, we describe the geometric framework for studying the multiple unicast conjecture, including its four major steps.

A. Step 1. From Throughput to Cost: LP Duality

In their original work where the multiple unicast conjecture was proposed [1], Li and Li first formulated the conjecture in the throughput domain, and then applied linear programming duality to translate it into the cost domain.

The Multiple Unicast Conjecture [1], [2]

**Throughput domain:** For \( k \) independent unicast sessions in a capacitated undirected network \((G, c)\), a throughput vector \( r \) is feasible with network coding if and only if it is feasible with routing.

**Cost domain:** Let \( f \) be the underlying flow vector of a network coding solution for \( k \) independent unicast sessions with throughput vector \( r \), in a cost-weighted undirected network \((G, w)\). Then \( \sum_c w(c) f(c) \geq \sum_i (d_i r_i) \).

Li and Li proved that the throughput version of the conjecture is equivalent to the cost version, by applying LP duality in the form of the Japanese Theorem. In particular, their proof leads to the following result that will be used in this work:

**Theorem 3.1.** (Li and Li, 2004 [1]) Given an undirected network \( G \) with \( k \) pairs of unicast terminals specified, and any desired throughput vector \( r \), the maximum coding advantage in \((G, c)\) over all \( c \in Q^E_+ \), equals the maximum cost advantage in \((G, w)\) over all \( w \in Q^E_+ \).

Intuitively, the throughput version of the conjecture claims that network coding cannot help improve throughput, while the cost version claims that network coding cannot help reduce transmission cost. In Step 1 of the framework, we apply Theorem 3.1 to translate the statement to be proven from its throughput version to cost version.
B. Step 2. From Network to Space: Graph Embedding

An embedding of a link-weighted graph \((G = (V, E), w)\) into a space \(l_p^m\) involves assigning a \(m\)-D coordinate to each node \(u \in V\). In the multiple unicast problem, we embed either the closure or the partial closure of \(G\). The closure network \(G'\) is a complete network defined over the same set of vertices \(V, E\), the shortest path length between \(u, v \in V\). The partial closure of \(G\) is \(G\) with direct links added between each pair of adjacent nodes, with cost \(d_i\).

A closure embedding has distortion \(\beta\) if \(\|uv\|_p \leq d(uv) \leq \beta \cdot \|uv\|_p, \forall u, v \in V\). A partial closure embedding has distortion \(\beta\) if \(\|si ti\|_p \leq d_i \leq \beta \|si ti\|_p\), and \(\|e\|_p \leq w(e), \forall e \in E\). In both cases, the embedding is isometric if \(\beta = 1\).

**Theorem 3.2.** For \(k\) pairs of unicast sessions in an undirected network \((G, w)\), with desired throughput vector \(r\), assume \(G\) has a \(\beta\)-distortion closure embedding in a space \(l_p^m\). If the cost advantage is 1 after the embedding, then it is upper-bounded by \(\beta\) before the embedding.

**Proof:** If there is a network coding solution in \(G\), with an underlying flow vector \(f\) satisfying \(\sum_w(w(e)f(e)) < \sum_i(d_i)\), then there is such a \(f'\) in \(G'\), by the definition of a closure network. The embedding of \(f'\) leads to a solution in \(l_p^m\), where \(\sum_w(||e||_p f'(e)) < \beta \cdot \sum_i(||si ti||_p d_i)\) due to the \(\beta\)-distortion property of the embedding. \(\square\)

A similar result holds for partial embedding as well.

**Theorem 3.3.** For \(k\) pairs of unicast sessions in an undirected network \((G, w)\), assume there is a \(\beta\)-distortion partial closure embedding of \(G\) in a space \(l_p^m\). If the cost advantage is 1 after the embedding, then it is upper-bounded by \(\beta\) before the embedding.

The proof of Theorem 3.3 is similar to that of Theorem 3.2, and is omitted. Informally, when the original link cost vector is ‘nice’, e.g., satisfying the triangular inequality, partial closure embedding may be preferred. Otherwise, closure embedding is likely to be more helpful. A special case of Theorem 3.2 and Theorem 3.3 is when \(\beta = 1\), then cost advantage is 1 after the embedding only if it is 1 before the embedding.

C. Step 3. From High Dimension to 1-D: Projection

Step 3 of the framework aims to simplify the statement to be proven from high dimension to 1-D. We introduce a few results useful for such dimension reduction.

**Theorem 3.4.** If there exists a configuration of \(k\) unicast sessions in \(l_{\infty}^n\), \(n > k\), where \(\sum_i(||e||_\infty f_e) < \sum_i(||si ti||_\infty r_i)\), then there exists a configuration of \(k\) unicast sessions in \(l_{\infty}^1\), where the same inequality holds.

**Proof:** For each session \(i\) of the \(k\) unicast sessions in the \(l_{\infty}^n\) space, let’s define the primary coordinate of \(i\) as \(\arg \max_j |x_{i,j} - x_{i,j}|\). We project the original multiple unicast instance from \(l_{\infty}^n\) to \(l_{\infty}^1\) by truncating the coordinate of each node in the following way: keep \(k\) coordinates including all the primary coordinates, dropping other coordinates.

After the projection from \(l_{\infty}^n\) to \(l_{\infty}^1\) above, the distance \(||si ti||_\infty\) remains unchanged, for each session \(i\). The distance between any two nodes \(u\) and \(v\) cannot increase. Therefore, \(\sum_i(||e||_\infty f_e)\) does not increase due to the projection, while \(\sum_i(d_ir_i)\) remains unchanged due to the projection, and hence the theorem is true.

By definition, the normed spaces are all equivalent in 1-D. In particular, there is no difference between \(l_2^1\) and \(l_{\infty}^1\). Therefore we drop the norm \(p\) from \(l_p^1\), and simply write \(l^1\).

**Theorem 3.5.** If there exists a configuration of \(k\) unicast sessions in \(l_{\infty}^2\), where \(\sum_i(||e||_\infty f_e) < \sum_i(||si ti||_\infty r_i)\), then there exists a configuration of \(k\) unicast sessions in \(l^1\), where the same inequality holds.

**Proof:** Let \(\vec{x}\) and \(\vec{y}\) be two vectors in a space \(l^h\). We define the projection of \(\vec{x}\) onto \(\vec{y}\) as \(\text{proj}(\vec{x}, \vec{y}) = \frac{\vec{x} \cdot \vec{y}}{||\vec{y}||^2} \vec{y}\), where \(\cdot\) is the inner product operation.

![Fig. 2. Projecting a unit vector in \(l_{\infty}^2\) to the two diagonal lines. While unit vectors form a circle in \(l_{\infty}^2\), they form a square in \(l_{\infty}^1\). The total Euclidean length of the two projected vectors is constant, and is \(\sqrt{2}\), since \(||OD||_2 = ||OM||_2\). As shown in Fig. 2, given a unit length vector \((\vec{O}C)\) in \(l_{\infty}^2\), the total Euclidean length of the two projected line segments onto the two diagonal lines \((\vec{OD} + \vec{ON})\) is constant \(\sqrt{2}\). Since \(\sum_i(||e||_\infty f_e) < \sum_i(||si ti||_\infty r_i)\) by assumption, we have:

\[
\sum_i f_e (\text{proj}(e, \vec{OM}) + \text{proj}(e, \vec{ON})) < \sum_i r_i (\text{proj}(si ti, \vec{OM}) + \text{proj}(si ti, \vec{ON}))
\]

From the inequality above, we can conclude that for at least one of \(\vec{OM}\) and \(\vec{ON}\), the projected network coding solution still has a smaller total cost than the cost of the projected MCF solution. \(\square\)

**Theorem 3.6.** (Li and Wu, 2012 [4].) If there exists a configuration of \(k\) unicast sessions in \(l_{\infty}^2\), for any \(h \geq 2\), where \(\sum_i(||e||_2 f_e) < \sum_i(||si ti||_2 r_i)\), then there exists a configuration of \(k\) unicast sessions in \(l^1\), where the same inequality holds.

Below we formulate a conjecture that generalizes Theorem 3.5 from \(l_{\infty}^2\) to \(l_{\infty}^h\) for \(h \geq 2\). It can also be viewed as the transformation of Theorem 3.6 from \(l_{\infty}^2\) to \(l_{\infty}^h\). Later we show that this conjecture implies the original multiple unicast network coding conjecture.

**Conjecture 3.1.** If there exists a configuration of \(k\) unicast sessions in \(l_{\infty}^h\) for some \(h \geq 2\), where \(\sum_i(||e||_\infty f_e) < \sum_i(||si ti||_\infty r_i)\), then there exists a configuration of \(k\) unicast sessions in \(l^1\).
\[ \sum_i (||s_i t_i||_\infty r_i), \text{ then there exists a configuration of } k \text{ unicast sessions in } l^1, \text{ where the same inequality holds.} \]

\[ \text{D. Step 4. Prove Conjecture in 1-D: Integrating Cut Inequality} \]

In a recent work, Li and Wu prove the following equivalence between network coding and routing in 1-D spaces. Their approach is to write an inequality for the cut condition at a point in \( l^1 \), and then integrate both sides of the inequality over all points in \( l_1 \).

**Theorem 3.7.** (Li and Wu, 2012 [4].) For any configuration of \( k \) unicast sessions in \( l^1 \), we always have \( \sum_e (||e||_1 f_e) \geq \sum_i (||s_i t_i||_1 r_i) \).

IV. UNIFIED PROOFS TO PREVIOUS RESULTS

In this section, we demonstrate the application of the geometric framework designed in Sec. III, by providing unified proofs to three known results of the multiple unicast conjecture.

**A. The Case of Two Unicast Sessions**

**Theorem 4.1.** For two unicast sessions in an undirected network \((G, c)\), network coding is equivalent to routing (MCF), i.e., a throughput vector \((r_1, r_2)\) is feasible with network coding if and only if it is feasible with routing.

**Proof:**

**Step 1. Transformation:** Apply Theorem 3.1 to all network configurations with \( k = 2 \), to translate the statement from its throughput version to cost version.

**Step 2. Embedding:** Apply Theorem 3.2, to translate the statement to be proven from the network information flow domain to the space information flow domain, from \( G \) to \( l^\infty_2 \). A network \((G, w)\) with \( n \) nodes has an isometric closure embedding into \( l^\infty_2 \), as reviewed below.

Let \( u \) and \( v \) be two nodes in \( l^\infty_2 \), at location \((x_{u1}, \ldots, x_{un})\) and \((x_{v1}, \ldots, x_{vn})\), respectively. The \( \infty \)-norm distance, or Chebyshev distance, between \( u \) and \( v \) is:

\[ ||u, v||_\infty = \lim_{p \to \infty} \left( \sum_{i=1}^n |x_{ui} - x_{vi}|^p \right)^{1/p} = \max_i |x_{ui} - x_{vi}| \]

We number the nodes in \( G \) and hence \( G' \) as \( u_1, u_2, \ldots, u_n \). We can embed each node \( u_i, 1 \leq i \leq n \) by assigning it the coordinates \((x_{i1}, x_{i2}, \ldots, x_{in}) = (d_{i1}, d_{i2}, \ldots, d_{in})\), where \( d_{ij} \) is the shortest path length between \( u_i \) and \( u_j \) in \( G \). After such an embedding, we can verify that for any \( 1 \leq i \leq k \leq n \), \( |d_{ik} - d_{jk}| \leq |d_{ij}| \) due to the triangular inequality satisfied by cost metric \( d \) in \( G' \), and hence \( ||u_i, u_j||_\infty = d_{ij} \) by the definition of \( \infty \)-norm distance above.

**Step 3. Projection:** Apply Theorem 3.4 to reduce the space from \( l^\infty_2 \) to \( l^2_2 \) \((k = 2)\), and then apply Theorem 3.5 to further reduce to \( l^1 \).

**Step 4. 1-D Proof:** Apply Theorem 3.7 to prove the statement in \( l^1 \), concluding the proof to Theorem 4.1.

---

**2012 International Symposium on Network Coding (NetCod)**

**B. The \( O(\log k) \) Upper-Bound in The General Case**

**Theorem 4.2.** For \( k \) unicast sessions in a undirected capacitated network \((G, c)\) with \( n \) vertices, the coding advantage is upper-bounded by \( O(\log k) \).

**Proof:** **Step 1. Transformation:** Apply Theorem 3.1, to translate the statement from throughput version to its cost version.

**Step 2. Embedding:** We apply Theorem 3.3 to translate the problem from \( G \) to a Euclidean space. It is known that a partial closure embedding of \( G \) with distortion \( O(\log k) \) is always possible (e.g., see Section 4 of [15]).

**Step 3. Projection:** Apply Theorem 3.6 to reduce the space from \( l^2_2 \) to \( l^1 \). Here \( h \) is the dimension required for the embedding in Step 2 to be feasible.

**Step 4. 1-D Proof:** Apply Theorem 3.7 to prove the statement in \( l^1 \), concluding the proof to Theorem 4.2. \( \square \)

**C. Multiple Unicast in Star Networks**

A network \( G \) is a star network, if there is a (center) node \( u \) in \( G \), such that every other node is directly connected to \( u \) only. It has been previously studied in the literature of network coding for multiple unicast sessions [16].

**Theorem 4.3.** For \( k \) unicast sessions in an undirected network \((G, c)\) with a star topology that satisfies the following property, network coding is equivalent to routing: for each session \( i \), at least one of \( s_i \) or \( t_i \) locates at a node that is a source or destination of at most three sessions.

**Proof:**

**Step 1. Transformation:** Apply Theorem 3.1 to undirected star networks, to translate the statement from throughput version to its cost version.

**Step 2. Embedding:** We apply Theorem 3.3 to transform the problem from \( G \) to \( l^2_\infty \). We show a partial closure embedding of the star network \((G, w)\) into \( l^2_\infty \), with \( \beta = 1 \), guaranteeing the distance between every pair of \( s_i \) and \( t_i \) remains unchanged during the embedding.

![Fig. 3. Embedding a star network with heterogeneous cost into \( l^2_\infty \). (a) Original network \( G \). (b) Embedding in \( l^2_\infty \).](image-url)

As shown in Fig. 3, we first embed the center node to the origin \( O \) in \( l^2_\infty \). The other nodes are distributed on the four quadrants. For each pair of \( s_i \) and \( t_i \), if neither is at center \( O \), then embed the pair to different quadrants. The distance between this pair is \((d_{s_iO} + d_{t_iO})\), thus the pairwise weights
remain unchanged. Given the condition in the theorem, such an isometric embedding into $l_\infty^2$ is always possible.

**Step 3. Projection:** Apply Theorem 3.5 to reduce the space from $l_\infty^2$ to $l^1$.

**Step 4. 1-D Proof:** Apply Theorem 3.7 to prove the statement in $l^1$, concluding the proof to Theorem 4.3. \hfill \Box

V. NEW RESULTS IN COST DOMAIN

In this section, we further apply the geometric framework from Sec. III to prove a number of new results.

A. Complete Networks

We prove that in a complete network with uniform cost, network coding can not outperform routing, for multiple unicast sessions.

**Theorem 5.1.** For $k$ unicast sessions in a network $(G,w)$, if $G$ is a complete graph and $w$ is a uniform cost vector, then the cost advantage is 1.

**Proof:**

**Step 1. Transformation:** In this case, we are proving network coding is equivalent to coding in the cost domain only. Step 1 in the framework does not apply.

**Step 2. Embedding:** We describe an isometric closure embedding of the uniform complete network $G$ into $l_1^n$. For each vertex $i$, $i = 1,2,\ldots,n$, let all the coordinates of $i$ be zero, except that the $ith$ coordinate is $\sqrt{2}/2$. Consequently, the distance between any two points is 1 in the target space, and we obtain an isometric embedding of $G$. We can then apply Theorem 3.2 to transform the problem from $G$ to $l_1^n$.

**Step 3. Projection:** Apply Theorem 3.6 to reduce the space from $l_1^n$ to $l^1$.

**Step 4. 1-D Proof:** Apply Theorem 3.7 to prove the statement in $l^1$, concluding the proof to Theorem 5.1. \hfill \Box

If the complete network $(G,w)$ does not have a uniform cost in $w$, then we can similarly prove that the cost advantage is upper-bounded by the maximum heterogeneity in list costs $\max_{e_1,e_2 \in G} w(e_1)/w(e_2)$. The difference in the proof lies in Step 3, where we resort to an embedding with a distortion, instead of an isometric one.

B. Grid Networks

**Theorem 5.2.** For $k$ pairs of unicast sessions in a 2-D square grid network $(G,w)$ with uniform cost in $w$, the cost advantage is at most $\sqrt{2}$ (Fig. 4). If each pair of $s_i$ and $t_i$ is further on the same row or column, then cost advantage is 1.

**Proof:** **Step 1. Transformation:** Not applicable.

**Step 2. Embedding:** We perform a partial closure embedding of the grid network into $l_2^n$ in the straightforward way. The distortion is upper-bounded by $\sqrt{2}$. If each pair of $s_i$ and $t_i$ is on the same horizontal or vertical line, then we obtain an isometric partial closure embedding. Then we can apply Theorem 3.3 to transform the problem from $G$ to $l_2^n$.

**Step 3. Projection:** Apply Theorem 3.6 to reduce the space from $l_2^n$ to $l^1$.

**Step 4. 1-D Proof:** Apply Theorem 3.7 to prove the statement in $l^1$, concluding the proof to Theorem 5.2. \hfill \Box

The result in Theorem 5.2 can be enhanced and generalized in a number of directions. For instance, if the original network $G$ is a uniform $h$-D grid instead of a 2-D grid, for some $h \geq 2$, then we can embed $G$ into $l_2^h$ with distortion $\sqrt{h}$, leading to an upper-bound of $\sqrt{h}$ on the cost advantage of network coding for multiple unicast sessions.

![Fig. 5. Embedding a grid network with diagonal links into $l_\infty^2$.](image)

Furthermore, consider a 2-D uniform grid network $G$ that further includes diagonal lines within all minimal squares, also with unit cost. We can embed the partial closure of $G$ into $l_\infty^2$ in an isometric fashion, as shown in Fig. 5. Here the isometric embedding is obtained by applying the most straightforward way of embedding $G$ into a plane. Applying this as Step 2 in the framework, we can prove that network coding is equivalent to routing in $G$.

C. Layered Networks

A layered network is a generalization of a bipartite network into multi-partite, such that edges exist between neighboring partite/layers only, as shown in Fig. 6. We prove that, if links from each layer have uniform cost, then the cost advantage for multiple unicast is 1. If links from each layer have heterogeneous costs, the cost advantage can still be bounded by the degree of intra-layer cost heterogeneity.

In a layered network $(G,w)$, let $L$ be a layer of links. Define cost heterogeneity of layer $L$ as $\max_{e_1,e_2 \in L} w(e_1)/w(e_2)$.

**Theorem 5.3.** For $k$ pairs of unicast sessions in an undirected layered network $(G,w)$, with each pair of $s_i$ and $t_i$ lying at different layers, the cost advantage of network coding is 1.
if each layer has uniform link costs. Otherwise, the cost advantage is upper-bounded by the maximum cost heterogeneity over all layers.

**Proof:** Step 1. Transformation: Not applicable.

Step 2. Embedding: We embed the partial closure of the layered network \( G \) into \( l_{\infty}^2 \), as shown in Fig. 6. If each layer has uniform link cost, then in the embedding, the distance between any pair of \( s_i \) and \( t_i \) remains unchanged. The distance between any other pair of nodes can only decrease. If each layer has heterogeneous link costs, then the distortion of the embedding can be upper-bounded by the maximum cost heterogeneity over all layers. We can then apply Theorem 3.3 to transform the problem from \( G \) to \( l_{\infty}^2 \).

Step 3. Projection: Apply Theorem 3.5 to reduce the space from \( l_{\infty}^2 \) to \( l^1 \).

Step 4. 1-D Proof: Apply Theorem 3.7 to prove the statement in \( l^1 \), concluding the proof to Theorem 5.2.

![Fig. 6. Embedding a layered network (can be viewed as generalization of both a bipartite network and a combination network \( C_{n,k} \) into \( l_{\infty}^2 \), \( d_i \) is the uniform cost of links in layer \( i \). If link costs in layer \( i \) is not uniform, we scale them to uniform before embedding, losing a factor equivalent to the cost heterogeneity.](image1)

![Fig. 7. A specific layered network and its embedding.](image2)

A special case of a layered network, as shown in Fig. 7(a), was used to demonstrate that network coding can have an arbitrarily large coding advantage for multiple unicast sessions [1]. There are \( k \) pairs of unicast sessions. Each \( s_i \) is connected to node \( A \) and all the receivers except \( t_i \), \( A \) is connected to \( B \). Each receiver \( t_i \) is connected to \( B \) as well as all the senders except \( s_i \). If we assume each link has a unit cost (instead of a unit capacity [1]), Fig. 7(b) depicts the embedding of this network into \( l_{\infty}^2 \). From Theorem 5.3, we know that network coding does not make a difference here, contrasting the arbitrarily large coding advantage under uniform link capacities.

VI. CONCLUSION

We applied a recently proposed tool, space information flow, to design a geometric framework for analyzing the multiple unicast conjecture, a well-known open problem in network coding. Based on the framework, we obtain unified proofs to a number of new results as well as existing results on the multiple unicast conjecture. We conclude by suggesting the following direction for proving the conjecture itself, based on the framework:

**A possible proof to the multiple unicast conjecture**

**Step 1. Transformation:** Apply Theorem 3.1 to translate the conjecture from its throughput version to cost version.

**Step 2. Embedding:** Based on the isometric closure embedding of \( G \) into \( l_{\infty}^n \), apply Theorem 3.3 to transform the problem from \( G \) to \( l_{\infty}^n \).

**Step 3. Projection:** Prove and then apply Conjecture 3.1, to reduce the problem from \( l_{\infty}^n \) to \( l^1 \).

**Step 4. 1-D Proof:** Apply Theorem 3.7 to prove the statement in \( l^1 \), concluding the proof to the conjecture.

ACKNOWLEDGEMENT

The last author would like to thank Michael Langberg, Sidharth Jaggi and Lap Chi Lau for helpful discussions.

REFERENCES


