ON THE ERROR TERM IN AN ASYMPTOTIC FORMULA
FOR THE SYMMETRIC SQUARE L-FUNCTION

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(Communicated by Wen-Ching Winnie Li)

Abstract. Recently Wu proved that for all primes \( q \),
\[
\sum_f L(1, \text{sym}^2 f) = \frac{\pi^4}{432} q + O(q^{27/28} \log^6 q)
\]
where \( f \) runs over all normalized newforms of weight 2 and level \( q \). Here we show that
\( 27/28 \) can be replaced by \( 9/10 \).

1. INTRODUCTION

Let \( q \) be a prime and
\[
\Gamma_0(q) = \left\{ \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \text{SL}_2(\mathbb{Z}) : q | c \right\}.
\]
We denote by \( S_2(q) \) the space of all holomorphic cusp forms for \( \Gamma_0(q) \) of weight 2. With respect to the inner product
\[
\langle f, g \rangle = \int_{\Gamma_0(q) \setminus \mathbb{H}} f(z) \overline{g(z)} \, dx \, dy,
\]
\( S_2(q) \) is a finite-dimensional Hilbert space, and there is an orthogonal basis \( B_2(q) \)
(which is the set of all normalized newforms in \( S_2(q) \)) such that
\begin{enumerate}
  \item each \( f \in B_2(q) \) is a common eigenvector of all Hecke operators \( T_n \) with \( (n, q) = 1 \), i.e. when \( f \in B_2(q) \) and \( (n, q) = 1 \),
  \[
  T_n f = \lambda_f(n) f;
  \]
  \item the Fourier expansion of \( f \in B_2(q) \) is
  \[
  f(z) = \sum_{n=1}^{\infty} \lambda_f(n) \sqrt{n} e(nz)
  \]
  where \( e(\alpha) = e^{2\pi i \alpha} \), \( \lambda_f(n) \) is the eigenvalue in (i) if \( (n, q) = 1 \) and \( \lambda_f(n)^2 = l^{-1} \lambda_f(m)^2 \) if \( n = lm \) where \( l \) is a power of \( q \) and \( (m, q) = 1 \) (see \( \text{[3]} \) (2.19) and (2.24)).
\end{enumerate}

Received by the editors September 17, 2002.
2000 Mathematics Subject Classification. Primary 11F67.
For the properties of $\lambda_f(n)$, it is known that they are all real and satisfy the Deligne bound $|\lambda_f(n)| \leq \tau(n)$. (Here and in the sequel $\tau(n) = \sum_{d|n} 1$ is the divisor function.) Moreover we have

\begin{equation}
\lambda_f(m)\lambda_f(n) = \sum_{d|(m,n)} \epsilon_q(d)\lambda_f\left(\frac{mn}{d^2}\right)
\end{equation}

where $\epsilon_q$ is the principal character mod $q$. In particular, we see that $\lambda_f(1) = 1$.

Associated to each $f \in B_2(q)$, we define the symmetric square $L$-function by

\begin{equation}
L(s, \text{sym}^2 f) = \sum_{n=1}^\infty \frac{\lambda_f(n^2)}{n^s} \text{ for } \Re s > 1,
\end{equation}

where $\zeta_q(s) = \prod_{p|q}(1 - p^{-s})^{-1}$. This $L$-function extends to an entire function over $\mathbb{C}$ and it satisfies a functional equation; more precisely, let us write

\begin{equation}
\Lambda(s, \text{sym}^2 f) = \left(\frac{q}{\pi^2}\right)^s \Gamma\left(\frac{s+1}{2}\right)^2 \Gamma\left(\frac{s+2}{2}\right)L(s, \text{sym}^2 f).
\end{equation}

Then we have $\Lambda(s, \text{sym}^2 f) = \Lambda(1-s, \text{sym}^2 f)$. Analogous to the Riemann zeta function, the values attained by $L(s, \text{sym}^2 f)$ in the critical strip are interesting. Particularly for $s = 1$ and all large prime $q$, we have the asymptotic formula

\begin{equation}
\sum_{f \in B_2(q)} L(1, \text{sym}^2 f) = \frac{\pi^4}{432} q + O(q^{9/10} \log^\beta q)
\end{equation}

for some constants $0 < \alpha < 1$ and $\beta > 0$. Here, we are concerned with the size of the error term. In [1], Akbary proved that $\alpha = 45/46$ is admissible and recently Wu gave an improvement to $\alpha = 27/28$ (see [3]). Our purpose is to show the refinement below.

**Theorem.** Let $q$ be a prime. There is an absolute constant $c > 0$ such that

\begin{equation}
\sum_{f \in B_2(q)} L(1, \text{sym}^2 f) = \frac{\zeta(2)^3}{2\pi^2} q + O(q^{9/10} \log^\epsilon q).
\end{equation}

(Note that $\zeta(2)^3/(2\pi^2) = \pi^4/432$.)

**Remark.** In decimal form we have $\frac{45}{46} \approx 0.978$, $\frac{27}{28} \approx 0.964$ and $\frac{9}{10} = 0.9$.

2. Some preparation

**Lemma 1.** Let $A > 1$ be any fixed constant and $y << y^{A}$ but $y \notin \mathbb{Z}$. We have

\begin{equation}
L(1, \text{sym}^2 f) = \zeta_q(2) \sum_{n \leq y} \frac{\lambda_f(n^2)}{n} + O(q^{9/10} \log^\epsilon y).
\end{equation}

where $\epsilon > 0$ is an arbitrarily small constant and the implied constant in the $O$-term depends on $\epsilon$.

**Proof.** This follows from the truncated Perron’s formula. Using the estimate

\begin{equation}
\Gamma\left(\frac{s+1}{2}\right)^2 \Gamma\left(\frac{s+2}{2}\right) \approx |t|^{(3\sigma+1)/2} e^{-3\pi|t|/4}
\end{equation}

for $s = \sigma + it$ where $\sigma << 1$ and $|t| \gg 1$, we can derive from the functional equation the convexity bound: for $0 \leq \sigma \leq 1$,

\begin{equation}
L(\sigma + it, \text{sym}^2 f) \ll (q|t|^{3/2})^{1-\sigma+\epsilon}.
\end{equation}
By Lemma 12.1, we see that for any $T > 1$,
\[
\zeta_q(2) \sum_{n \leq y} \frac{\lambda_f(n^2)}{n}
\]
\[= \zeta_q(2) \frac{\lambda_f(n^2)}{2\pi i} \int_{c-iT}^{c+iT} \sum_{n=1}^{\infty} \frac{\lambda_f(n^2) y^n}{n^{1+\epsilon}} ds + O(y^{\infty} \sum_{n=1}^{\infty} \frac{\tau(n)^2}{n^{1+\epsilon}} \min(1, (T \log \frac{y}{n})^{-1})).
\]

To evaluate the $O$-term, we split the summation over $n$ into three pieces: $n \leq y/2$, $n \geq 3y/2$ and $y/2 < n < 3y/2$. As $|\log(y/n)| \gg 1$ in the first two pieces, these two sums are $O(T^{-1} y^\epsilon)$. The third one is
\[\ll y(T^{-1} \sum_{y/2 < n < 3y/2} |y-n|^{-1}) + y^{-1+\epsilon} \ll y(T^{-1} + y^{-1}).\]

Thus the overall contribution is absorbed in the $O$-term in our lemma.

From Lemma 2, we can replace $\sum_{n=1}^{\infty} \lambda_f(n^2) n^{-(1+\epsilon)}$ in (5) by
\[\zeta_q(2 + 2s)^{-1} L(1 + s, \sym^2 f).
\]

Then we apply the residue theorem to the rectangular contour with vertices at $\epsilon \pm iT$ and $-1/2 + \epsilon \pm iT$. The integral in (5) equals a sum of two terms: the main term $L(1, \sym^2 f)$ from the pole at $s = 0$, and the remainder term which is
\[\ll \int_{1/2+\epsilon}^{1} \frac{L(1 + \alpha + iT, \sym^2 f)}{\zeta_q(2 + 2\alpha + iT)} \left| \frac{y^{\alpha}}{T} \right| \, d\alpha + y^{-1/2+\epsilon} \cdot \int_{-T}^{T} \frac{L(1/2 + \epsilon + it, \sym^2 f)}{\zeta_q(1 + 2\epsilon + it)} \left| \frac{dt}{1+|t|} \right|\]

Using the bound $\zeta(\sigma + it)^{-1} \ll \log(1+|t|)$ for $\sigma \geq 1$ and $|t| \gg 1$, the two $O$-terms are $\ll (qT)^{\epsilon}(y^{-1/2}q^{1/2}T^{3/4} + T^{-1})$. The proof is complete after setting $T = (y/q)^{2/7}$.

Our next task is to extend the admissible range in Lemma 2. To this end, we modify the mean square estimate result in Corollary 1. Suppose $M < n \leq q^9$, and $\{a_n\}_{1 \leq n \leq M}$ is a sequence of complex numbers. Then by taking $a_n = 0$ for $M < n \leq q^9$, Lemma 1 of [3] with $N = q^9$ gives
\[\sum_{f \in B_2(q)} \left| \sum_{n \leq M} a_n \rho_f(n) \right|^2 \ll q^9 \log q^{15} \sum_{n \leq M} |a_n|^2
\]
where $\rho_f(n) = \sum_{m^2 = n} \epsilon_q(m) \lambda_f(l^2)$. (Note that $B_2(q) = S_2(q)^*$ in [3] for prime $q$.)

Lemma 2. Let $M \gg 1$ and suppose that $\{a(n)\}_{M < n \leq 2M}$ satisfies
\[a(n) \ll \frac{(\tau(n) \log n)^A}{n}\]
for some constant $A > 0$. There exists a constant $B = B(A) \geq 0$ such that
\[\sum_{f \in B_2(q)} \left| \sum_{M < n \leq 2M} a(n) \lambda_f(n^2) \right|^2 \ll \max(1, q^9 M^{-1}) \log^B(qM).
\]
The implied constant depends on $A$. 

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Proof. When $M \geq q^3$, it follows immediately from \cite{4} Corollary 1 (by taking $N = M$). Consider the case $M < q^3$. From \cite{4} (16), we have

$$S := \sum_{f \in \mathbb{B}_2(q)} \left| \sum_{M < n \leq 2M} a(n) \lambda_f(n^2) \right|^2 = \sum_{f \in \mathbb{B}_2(q)} \left| \sum_{l < 2M} a_l \rho_f(l) \right|^2$$

where

$$a_l = \sum_{\sqrt{M/l} < m \leq \sqrt{2M/l}} \mu(m) \epsilon_q(m) a(lm^2) \leq \frac{(\tau(l) \log 2l)^A}{l} \sum_{\sqrt{M/l} < m \leq \sqrt{2M/l}} \frac{(\tau(m) \log 2m)^{2A}}{m^2} \leq (Ml)^{-1/2} (\log Ml)^B$$

(see the proof of \cite{4} Corollary 1 as well). $B$ denotes an unspecified positive constant depending on $A$ and its value may differ at each occurrence in the proof. By (6),

$$S \ll q^9 (\log q)^{15} \sum_{l < 2M} (Ml)^{-1} (\log Ml)^B \ll q^9 M^{-1} \log^B (qM).$$

Define for $1 \leq x < y$,

$$\omega_f(x, y) = \sum_{x \leq n < y} \frac{\lambda_f(n^2)}{n}.$$

Lemma 3. Let $x > 0$ and $x < y \ll q^A$ for some constant $A > 0$. Suppose $r \geq 1$ is a fixed integer satisfying $x^r \geq q^3$. Then there exists a constant $D = D(r) > 0$ such that

$$\sum_{f \in \mathbb{B}_2(q)} \omega_f(x, y)^{2r} \ll (\log q)^D$$

where the implied constant depends on $A$ and $r$.

Proof. Following the argument in the proof of \cite{4} Lemma 4, one can show that

$$\omega_f(x, y)^r = \sum_{x^r \leq mn < y^r} \lambda_f(m^2) \frac{c(m, n)}{mn}$$

where $c(m, n)$ is independent of $f$ and $c(m, n) = 0$ if $n$ is not of the form $n = dn_1$ where $d|m$ and $n_1$ is squarefull. Moreover, $|c(m, n)| \leq \tau(mn)^\gamma$ for some integer $\gamma = \gamma(r) > 0$ depending on $r$. Then we write

$$\omega_f(x, y)^r = \sum_{H = 2^k} \sum_{\frac{H}{2^k} \leq n < 2H} \lambda_f(m^2) \frac{c(m, n)}{mn}$$

where the first summation runs over all nonnegative integers $k$. Define

$$c_H(m) = \sum_{\frac{H}{2^k} \leq n < 2H} \frac{c(m, n)}{n}.$$
Then, using \( \sum_{n \leq z} \tau(n) \gg z^{1/2}(\log z)^2 \), we have

\[
\tau(m) \ll \tau(m)^\gamma \sum_{d|m} \frac{1}{d} \sum_{H \leq dn < 2H} \frac{\tau(n)^\gamma}{n}
\]

\[
\ll \tau(m)^\gamma \left( \sum_{d|m} d^{-1} + \sum_{d|m} d^{-1} \sum_{H/d \leq n < 2H/d} \frac{\tau(n)^\gamma}{n} \right)
\]

(7) \( \ll H^{-1/2}(\tau(m)(\log m)(\log H))^D \).

Here we use \( D \) to denote a positive constant (depending on \( r \)) which may assume different values at other places. Making use of (4) for \( H \geq q \),

\[
\omega_f(x, y)^r = \sum_{H=2^k \leq q} \sum_{x'/2H < m \leq y'/H} \lambda_f(m^2) \frac{\epsilon_H(m) \sqrt{H}}{m} + O(q^{-1/2} \log D q).
\]

Squaring both sides and averaging over all \( f \in B_2(q) \) yields

\[
\sum_{f \in B_2(q)} \omega_f(x, y)^{2r} \ll \left( \sum_{H=2^k \leq q} H^{-1} \sum_f \left| \sum_{x'/2H < m \leq y'/H} \lambda_f(m^2) \frac{\epsilon_H(m) \sqrt{H}}{m} \right|^2 \right) \log D q
\]

as \( (\sum_{i \in I} a_i)^2 \ll |I| \sum_{i \in I} a_i^2 \) and \( |B_2(q)| \ll q \). For each \( H \), we split the range of the summation over \( m \) into dyadic intervals \( M < m \leq 2M \) where \( M \geq x'/2H \). It follows from Lemma 2 and (7) that

\[
\sum_f \left| \sum_{x'/2H < m \leq y'/H} \lambda_f(m^2) \frac{\epsilon_H(m) \sqrt{H}}{m} \right|^2 \ll \max(1, q^9 x^{-r} H) \log D q.
\]

Inserting it into (8), we conclude that

\[
\sum_{f \in B_2(q)} \omega_f(x, y)^{2r} \ll \log D q \sum_{H=2^k \leq q} \max(H^{-1}, q^9 x^{-r}),
\]

and our result follows in view of the condition \( x^r \geq q^9 \). \( \square \)

3. Proof of the Theorem

Define for \( f \in B_2(q), \omega_f = 4\pi(f, f) \), which is a positive real number. We have from [3] Lemma 2.5 that \( \omega_f = (2\pi^2)^{-1}qL(1, \text{sym}^2 f) \) and from [3] Corollary 2.2 (with \( \tau_3((m, n)) \leq \tau((m, n))^2 \leq \tau(m)\tau(n) \)) that

\[
\sum_{f \in B_2(q)} \omega_f^{-1} \lambda_f(m^2) \lambda_f(n^2) = \delta(m, n) + O(q^{-1}(mn)^{1/2} \tau(m)\tau(n))^2 \log 2mn)
\]

for \( \min(m, n) < q \), where \( \delta(\cdot, \cdot) \) is the Kronecker delta. (Note that \( \omega_f^{-1} = \omega_f \) in [3].) In particular, \( \sum_f \omega_f^{-1} \ll 1 \) as \( \lambda_f(1) = 1 \).

We split the sum over \( n \) in Lemma 4 into two subsums \( \sum_{n \leq x} + \sum_{x < n \leq y} \) where \( 1 < x < q < y \). (Our choice will be \( x = q^{9/10} \) and \( y = q^{173/110} \).) Squaring the
formula in Lemma 3 together with the bound $L(1, \text{sym}^2 f) \ll \log^3 q$ (from [4, (18)]), we deduce that

$$\sum_{f \in B_2(q)} L(1, \text{sym}^2 f) = \frac{q}{2\pi^2} \sum_{f \in B_2(q)} w_f^{-1} L(1, \text{sym}^2 f)^2$$

(10)

$$= \frac{q}{2\pi^2} \zeta_q(2)(S_1 + 2S_2 + S_3) + O(q^{-r + (\frac{q}{y})^{2/7}})$$

where

$$S_1 = \sum_{f} w_f^{-1} \left( \sum_{n \leq x} \frac{\lambda_f(n^2)}{n} \right)^2,$$

$$S_2 = \sum_{f} w_f^{-1} \left( \sum_{n \leq x} \frac{\lambda_f(n^2)}{n} \right) \left( \sum_{x < n \leq y} \frac{\lambda_f(n^2)}{n} \right),$$

$$S_3 = \sum_{f} w_f^{-1} \left( \sum_{x < n \leq y} \frac{\lambda_f(n^2)}{n} \right)^2.$$

It follows from the bound $w_f^{-1} \ll q^{-1} \log q$ (see [4, (20)]) and Lemma 3 that if $x^r \geq q^9$,

$$S_3 \ll \frac{\log q}{q} \sum_{f} \omega_f(x, y)^2 \ll \left( \sum_{f} \omega_f(x, y)^2 \right)^{1/r} |B_2(q)|^{1 - 1/r} q^{-1} \log q$$

(11)

$$\ll q^{-1/r} \log^{c_{11}} q.$$

Throughout $c_i, i = 1, 2, \cdots$, denote unspecified positive constants. Using (10), we obtain that for $x < q$,

$$S_1 = \sum_{n \leq x} n^{-2} + O(q^{-1} \sum_{m,n \leq x} (mn)^{-1/2} \tau(m)^2 \tau(n)^2 \log 2mn)$$

(12)

$$= \zeta(2) + O(x^{-1} + q^{-1} x \log^{c_{11}} q).$$

To treat $S_2$, we split it into two parts: let $z = qx^{-1}$,

(13)

$$S_2 = \sum_{f} w_f^{-1} \sum_{n \leq x} \frac{\lambda_f(n^2)}{n} \sum_{x < n \leq y} \frac{\lambda_f(n^2)}{n} + \sum_{f} w_f^{-1} \sum_{z < n \leq x} \frac{\lambda_f(n^2)}{n} \sum_{x < n \leq y} \frac{\lambda_f(n^2)}{n}$$

$$= S_{21} + S_{22}, \text{ say.}$$

By (9), we have, provided that $z \leq x$ (or equivalently $x \geq q^{1/2}$),

$$S_{21} \ll q^{-1} (\log^{c_{11}} q) \sum_{m,z \leq n \leq y} \tau(m)^2 \tau(n)^2 (mn)^{-1/2} \ll \sqrt{\frac{y}{qx}} \log^{c_{11}} q.$$

Applying the argument in (12), we get that

$$\sum_{f} w_f^{-1} \left( \sum_{z < n \leq x} \frac{\lambda_f(n^2)}{n} \right)^2 \ll z^{-1} + q^{-1} x \log^{c_{11}} q \ll q^{-1} x \log^{c_{11}} q.$$
By $ab \ll |a|^2 + |b|^2$ and (11), we have $S_{22} \ll (q^{-1/r} + q^{-1}x) \log^{c_1} q$. Hence, by (13),

$$S_2 \ll (q^{-1/r} + q^{-1}x + \left(\frac{y}{qx}\right)^{1/2}) \log^{c_1} q.$$ 

Putting this estimate, (11) and (12) into (10), we infer that as $q(2) = \frac{(2)^3}{2\pi^2} + qO(q^{-1/r} + q^{-1}x) \log^{c_1} q$

$$+ q^r \left(x^{-1} + \left(\frac{y}{qx}\right)^{1/2} + \left(\frac{y}{q^r}\right)^{2/7}\right).$$

Subject to the condition $x^r \geq q^9$, we take $x = q^{9/r}$ and select $r = 10$, $x = q^{9/10}$ and $y = q^{173/110}$ by equating $q^{-1/r} = q^{-1}x$. This ends the proof.

References


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