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Meromorphic solutions of higher order Briot–Bouquet differential equations

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(Received )

Abstract
For differential equations $P(y^{(k)}, y) = 0$, where $P$ is a polynomial, we prove that all meromorphic solutions having at least one pole are elliptic functions, possibly degenerate.

1. Introduction

According to a theorem of Weierstrass, meromorphic functions $y$ in the complex plane $\mathbb{C}$ that satisfy an algebraic addition theorem

$$Q(y(z + \zeta), y(z), y(\zeta)) \equiv 0, \quad \text{where } Q \neq 0 \text{ is a polynomial},$$

are elliptic functions, possibly degenerate [17, 1].

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More precisely, let us denote by \( W \) the class of meromorphic functions in \( \mathbb{C} \) that consists of doubly periodic functions, rational functions and functions of the form \( R(e^{az}) \) where \( R \) is rational and \( a \in \mathbb{C} \). Then each function \( y \in W \) satisfies an identity of the form (1.1), and conversely, every meromorphic function that satisfies such an identity belongs to \( W \).

One way to prove this result is to differentiate (1.1) with respect to \( \zeta \) and then set \( \zeta = 0 \). Then we obtain a Briot–Bouquet differential equation

\[ P(y', y) = 0. \]

The fact that every meromorphic solution of such an equation belongs to \( W \) was known to Abel and Liouville, but probably it was stated for the first time in the work of Briot and Bouquet [5, 6].

Here we consider meromorphic solutions of higher order Briot–Bouquet equations

\[ P(y^{(k)}, y) = 0, \quad \text{where } P \text{ is a polynomial.} \quad (1.2) \]

Picard [18] proved that for \( k = 2 \), all meromorphic solutions belong to the class \( W \). This work was one of the first applications of the famous Picard’s theorems on omitted values.

In the end of 1970-s Hille [12, 13, 14, 15] considered meromorphic solutions of (1.2) for arbitrary \( k \). The result of Picard was already forgotten, and Hille stated it as a conjecture. Then Bank and Kaufman [4] gave another proof of Picard’s theorem.

These investigations were continued in [8]. To state the main results from [8] we assume without loss of generality that the polynomial \( P \) in (1.2) is irreducible. Let \( F \) denote the compact Riemann surface defined by the equation

\[ P(p, q) = 0. \quad (1.3) \]

Then every meromorphic solution \( y \) of (1.2) defines a holomorphic map \( f : \mathbb{C} \to F \). According to another theorem of Picard, a Riemann surface which admits a non-constant holomorphic map from \( \mathbb{C} \) has to be of genus 0 or 1, (19, see also [2]). The following theorems were proved in [8]:

**Theorem A.** If \( F \) is of genus 1, then every meromorphic solution of (1.2) is an elliptic function.

**Theorem B.** If \( k \) is odd, then every meromorphic solution of (1.2) having at least one pole, belongs to the class \( W \).

The main result of the present paper is the extension of Theorem B to the case of even \( k \).

**Theorem 1.** If \( y \) is a meromorphic solution of an equation (1.2) and \( y \) has at least one pole, then \( y \in W \).

This can be restated in the following way. Let \( y \) be a meromorphic function in the plane which is not entire and does not belong to \( W \). Then \( y \) and \( y^{(k)} \) are algebraically independent.

It is easy to see that for every function \( y \) of class \( W \) and every natural integer \( k \) there exists an equation of the form (1.2) which \( y \) satisfies.

---

1 A “meromorphic function” in this paper means a function meromorphic in the complex plane, unless some other domain is specified. See [17, 20] for discussion of the equation (1.1) in more general classes of functions.
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It is not true that all meromorphic solutions of higher order Briot–Bouquet equations belong to $W$, a simple counterexample is $y''' = y$. We don’t know whether non-linear irreducible counterexamples exist.

In the process of proving of Theorem 1 we will establish an estimate of the degrees of possible meromorphic solutions in terms of the polynomials $P$. Here by degree of a function of class $W$ we mean the degree of a rational function $y$, or the degree of $R$ in $y(z) = R(e^{az})$, or the number of poles in the fundamental parallelogram of an elliptic function $y$. Thus our result permits in principle the determination of all meromorphic solutions having at least one pole of a given equation (1·2).

Our method of proof is based on the so-called “finiteness property” of certain autonomous differential equations: there are only finitely many formal Laurent series with a pole at zero that satisfy these equations. The idea seems to occur for the first time in [12, p. 274] but the argument given there contains a mistake. This mistake was corrected in [8]. Later the same method was applied in [7] and [10] to study meromorphic solutions of other differential equations.

2. Preliminaries

We will use the following refined version of Wiman–Valiron theory which is due to Bergweiler, Rippon and Stallard.

Let $y$ be a meromorphic function and $G$ a component of the set $\{z : |y(z)| > M\}$ which contains no poles (so $G$ is unbounded). Set

$$M(r) = M(r, G, y) = \max\{|y(z)| : |z| = r, z \in G\},$$

and

$$a(r) = \frac{d \log M(r)}{d \log r} = r \frac{M'(r)}{M(r)}. \quad (2·1)$$

This derivative exists for all $r$ except possibly a discrete set. According to a theorem of Fuchs [11],

$$a(r) \to \infty, \quad r \to \infty,$$

unless the singularity of $y$ at $\infty$ is a pole. For every $r > r_0 = \inf \{|z| : z \in G\}$ we choose a point $z_r$ with the properties $|z| = r$, $|y(z_r)| = M(r)$.

**Theorem C.** For every $\tau > 1/2$, there exists a set $E \subset [r_0, +\infty)$ of finite logarithmic measure, such that for $r \in [r_0, \infty) \setminus E$, the disk

$$D_r = \{z : |z - z_r| < ra^{-\tau}(r)\}$$

is contained in $G$ and we have

$$y^{(k)}(z) = \left(\frac{a(r)}{z}\right)^k \left(\frac{z}{z_r}\right)^{a(r)} y(z) (1 + o(1)), \quad r \to \infty, \quad z \in D_r. \quad (2·2)$$

When $y$ is entire, this is a classical theorem of Wiman. Wiman’s proof used power series, so it cannot be extended to the situation when $y$ is not entire. A more flexible proof, not using power series is due to Macintyre [16]; it applies, for example to functions analytic and unbounded in $|z| > r_0$. The final result stated above was recently established in [3].
In what follows, we always assume that the polynomial $P$ in (1.2) is irreducible.

To state a result of [8] which we will need, we introduce the following notation. Let $A$ be the field of meromorphic functions on $F$. The elements of $A$ can be represented as rational functions $R(p, q)$ whose denominators are co-prime with $P$. In particular, $p$ and $q$ in (1.3) are elements of $A$. For $\alpha \in A$ and a point $x \in F$, we denote by $\text{ord}_x \alpha$ the order of $\alpha$ at the point $x$. Thus if $\alpha(x) = 0$ then $\text{ord}_x \alpha$ is the multiplicity of the zero $x$ of $\alpha$, if $\alpha(x) = \infty$ then $-\text{ord}_x \alpha$ is the multiplicity of the pole, and $\text{ord}_x \alpha = 0$ at all other points $x \in F$.

Let $I \subset F$ be the set of poles of $q$. For $x \in I$ we set $\kappa(x) = \text{ord}_x p / \text{ord}_x q$.

**Theorem D.** Suppose that an irreducible equation (1.2) has a transcendental meromorphic solution $y$. Let $f : \mathbb{C} \to F$ be the holomorphic map defined by $z \mapsto (y^{(k)}(z), y(z))$. Then:

a) Every pole of $p$ belongs to $I$.

b) For every $x \in I$, the number $\kappa(x)$ is either $1$ or $1 + k/n$, where $n$ is a positive integer.

c) If $\kappa(x) = 1 + k/n$ for some $x \in I$, then the equation $f(z) = x$ has infinitely many solutions, and all these solutions are poles of order $n$ of $y$.

d) If $\kappa(x) = 1$ for some $x \in I$, then the equation $f(z) = x$ has no solutions.

Picard’s theorem on omitted values implies that $\kappa(x) = 1$ can happen for at most two points $x \in I$. For the convenience of the reader we include a proof of Theorem D in the Appendix.

The numbers $\kappa(x)$ can be easily determined from the Newton polygon of $P$. Thus Theorem D gives several effective necessary conditions for the equation (1.2) to have meromorphic or entire solutions.

**Remark.** The proof of Theorem D in [8] uses Theorem C which was stated in [8] but not proved. One can also give an alternative proof of Theorem D, using Nevanlinna theory instead of Theorem C, by the arguments similar to those in [9].

**Lemma 1.** Suppose that $y$ is a meromorphic solution of (1.2). If $\kappa(x) = 1$ for some $x \in I$ then $y$ has order one, normal type.

**Proof.** In view of Theorem A and Theorem D, d), we conclude that the genus of $F$ is zero. Therefore, we can find $t = R(p, q)$ in $A$ which has a single simple pole at $x$. Then $w = R(y^{(k)}, y)$ is an entire function by Theorem D, d). As $t$ has a simple pole at $x$, the element $1/t \in A$ is a local parameter at $x$, and in a neighborhood of $x$ we have

$$q = at^m + \ldots \quad \text{and} \quad p = bt^m + \ldots,$$

where $-m = \text{ord}_x p = \text{ord}_x q$ as $\kappa(x) = 1$, and the dots stand for the terms of degree smaller than $m$. Substituting $p = y^{(k)}$ and $q = y$ and differentiating the first equation $k$ times we obtain for $w$ a differential equation of the form

$$\frac{d^k}{dz^k} w^m + \ldots = \frac{b}{a} w^m,$$

(3.1)

where the dots stand for the terms of degree smaller than $m$. Now we use a standard argument of Wiman–Valiron theory. Applying Theorem C to the entire function $w^m$, with $G = \mathbb{C}$ and $z = z_r$, we compare the asymptotic relations (2.2) and (3.1) to conclude
that \( a(r) \sim cr \), where \( c \neq 0 \) is a constant. This implies \( \log M(r) \sim cr \), which means that \( w \) is of order 1, normal type. So \( y \) is also of order 1, normal type, because \( w \) and \( y \) satisfy a polynomial relation of the form \( P(y, w) = 0 \), where \( P \) is a polynomial with constant coefficients.

**Lemma 2.** Suppose that \( y \) is a meromorphic solution of (1.2). If \( \kappa(x_1) = \kappa(x_2) = 1 \) for two different points \( x_1 \) and \( x_2 \) in \( I \), then \( y \) is a rational function of \( e^{az} \), where \( a \in \mathbb{C} \).

**Proof.** As in the previous lemma, the genus of \( F \) is zero. Let \( t = R(p, q) \) be a function in \( A \) with a single simple pole at \( x_1 \) and a single simple zero at \( x_2 \). Then \( w = R(y^{(k)}, y) \) is an entire function of order 1, normal type (by Lemma 1) omitting 0 and \( \infty \) (by Theorem D, d). So \( w(z) = e^{az} \) for some \( a \in \mathbb{C} \). Since \( t \) is a generator of \( A \), by Lüroth’s theorem, both \( p \) and \( q \) are rational functions of \( t \) and the lemma follows.

**Lemma 3.** Suppose that \( k \) is even, the Riemann surface \( F \) is of genus zero, \( y \) is a non-constant meromorphic solution of (1.2), and \( \kappa(x) = 1 \) for at most one point \( x \in I \). Then the Abelian differential \( pdq \) is exact, that is \( pdq = ds \) for some \( s \in A \).

**Proof.** It is sufficient to show that under the assumptions of Lemma 3, the integral of \( pdq \) over every closed path in \( F \) is zero. As \( F \) is of genus zero, we only have to consider residues of \( pdq \). By Theorem D, a), all poles of our differential belong to the set \( I \).

Consider first a point \( x \in I \) with \( \kappa(x) = 1 + k/n \). By Theorem D, c), we have a meromorphic solution \( y \) with a pole of order \( n \) at zero, such that the corresponding function \( f \) has the property \( f(0) = x \). In a neighborhood of \( x \) we have a Puiseaux expansion

\[
\sum_{j=J}^{\infty} c_j y^{-j/m} dq
\]

with some positive integer \( m \). We substitute \( p = y^{(k)} \), \( q = y \) and obtain

\[
y^{(k)} y' = \sum_{j \neq -m} c_j y^{-j/m} y' + ry^{-1} y', \quad (3.2)
\]

where \( r = c_m \) is the residue of \( pdq \) at \( x \). Now we notice that for even \( k \),

\[
y^{(k)} y' = \frac{d}{dz} \left\{ y^{(k-1)} y' - y^{(k-2)} y'' + \ldots + \frac{1}{2} (y^{(k/2)})^2 \right\}. \quad (3.3)
\]

Using this, we integrate (3.2) over a small circle around 0 in the \( z \)-plane, described \( m \) times anticlockwise. We obtain that \( 2\pi imr = 0 \), so \( r = 0 \).

Now we consider a point \( x \in I \) with \( \kappa(x) = 1 \). By the assumptions of the lemma, there is at most one such point. Then the residue of \( pdq \) at \( x \) is zero because the sum of all residues of a differential on a compact Riemann surface is zero. This proves the lemma.

Using (3.3) and Lemma 3, if the assumptions of Lemma 3 are satisfied, we can rewrite our differential equation

\[
y^{(k)} = p(y) \quad (3.4)
\]

as

\[
y^{(k-1)} y' - y^{(k-2)} y'' + \ldots + \frac{1}{2} (y^{(k/2)})^2 = s(y) + c, \quad (3.5)
\]
where $s \in A$ is an integral of the exact differential $pdq$, and $c$ is a constant that depends on the particular solution $y$. We have the relation $p(y) = ds/dy$.

**Lemma 4.** For a given differential equation of the form $(3 \cdot 5)$, there are only finitely many formal Laurent series with a pole at zero that satisfy the equation.

**Proof.** By making a linear change of the independent variable, we may assume that

$$s(y) = y^{2+k/n} + \ldots.$$

Then

$$p(y) = (2 + k/n)y^{1+k/n} + \ldots.$$

Now we substitute a Laurent series with undetermined coefficients

$$y(z) = \sum_{j=0}^{\infty} c_j z^{-n+j} \quad (3\cdot6)$$

to the equation $(3\cdot4)$, which is a consequence of $(3\cdot5)$. With even $k$ we have:

$$y^{(k)}(z) = \frac{(k + n - 1)!}{(n-1)!} c_0 z^{-n-k} + \frac{(k + n - 2)!}{(n-2)!} c_1 z^{-n-k-1} + \ldots + k! c_{n-k} z^{-k} + \frac{(k + 1)!}{1!} c_{n+k+1} z + \frac{(k + 1)!}{2!} c_{n+k+2} z^2 + \ldots;$$

and

$$y^{1+k/n}(z) = z^{-k-n} \left[ c_0^{1+k/n} + \left( 1 + k/n \right) c_0^{k/n} c_1 + \ldots \right] z^{n+k+1} + \left( 1 + k/n \right) c_0^{k/n} c_1 + \ldots + \left( 1 + k/n \right) c_0^{k/n} c_j + \ldots \right] z^j + \ldots. \quad (3\cdot7)$$

In the last formula, the symbol $(\ldots)_j$ stands for a finite sum of products of the coefficients of the series $(3\cdot6)$ which contain no coefficients $c_i$ with $i \geq j$. Substituting to $(3\cdot4)$ and comparing the coefficients at $z^{-k-n}$ we obtain

$$\frac{(k + n - 1)!}{(n-1)!} c_0 = (2 + k/n)c_0^{1+k/n}.$$

This equation has finitely many non-zero roots $c_0$. We have

$$(2 + k/n)c_0^{k/n} = \frac{(k + n - 1)!}{(n-1)!}. \quad (3\cdot7)$$

Further we obtain

$$\frac{(k + n - 2)!}{(n-1)!} c_1 = (2 + k/n)c_0^{k/n}(1 + k/n)c_1 + \ldots \quad (3\cdot8)$$

Substituting here the value of $(2 + k/n)c_0^{k/n}$ from $(3\cdot7)$, we see that the coefficient at $c_1$ is different from zero, because

$$\frac{(k + n - 2)!}{(n-2)!} \neq \frac{(k + n - 1)!}{(n-1)!} \frac{k + n}{n}.$$

Thus $c_1$ is uniquely determined from $(3\cdot8)$. The situation is analogous for all coefficients.
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\(c_j\) with \(j < n + k\). These coefficients are uniquely determined from the equation (3-4) once \(c_0\) is chosen.

Now we consider the coefficients \(c_{n+k+j}\) with \(j \geq 0\). We have

\[
\frac{(k+j)!}{j!} c_{n+k+j} = (2+k/n)c_0^{k/n} c_{n+k+j} + \ldots + \frac{1}{n} c_{n+k+j}.
\]

Again we substitute the value of \((2+k/n)c_0^{k/n}\) from (3-7) and conclude that the coefficient at \(c_{n+k+j}\) equals

\[
\frac{(k+j)!}{j!} = \frac{(k+n)!}{n!}.
\]

This coefficient is zero for a single value of \(j\), namely \(j = n\). Thus \(c_{2n+k}\) cannot be determined from the equation (3-4), but once \(c_0\) and \(c_{2n+k}\) are chosen, the rest of the coefficients of the series (3-6) are determined uniquely.

To determine \(c_{2n+k}\) we invoke the equation (3-5):

\[
y^{(k-1)}y' - y^{(k-2)}y'' + \ldots + \frac{1}{2} (y^{(k/2)})^2 = y^{2+k/n} + \ldots,
\]

where the dots stand for the terms of lower degrees. We have

\[
y'(z) = -nc_0 z^{-n-1} + \ldots + c_{2n+k}(n+k) z^{n+k-1} + \ldots,
\]

\[
y''(z) = n(n+1)c_0 z^{-n-2} + \ldots + c_{2n+k}(n+k)(n+k-1) z^{n+k-2} + \ldots,
\]

\[
\ldots \ldots
\]

\[
y^{(k-1)}(z) = -n(n+1)\ldots(n+k-2)c_0 z^{-n-k+1} + \ldots + c_{2n+k}(n+k)(n+k-1)\ldots(n+2) z^{n+1} + \ldots.
\]

Substituting this to our equation (3-9) we write the condition that the constant terms in both sides of (3-9) are equal. This condition is a polynomial equation in \(c, c_0, \ldots, c_{2n+k}\) (it is linear with respect to \(c_{2n+k}\)) and the coefficient at \(c_{2n+k}\) in this equation equals

\[
c_0 \sum_{m=0}^{k-1} \frac{(n+m)!(n+k)!}{(n+m+1)!(n-1)!}.
\]

This expression is not zero because each term of the sum is positive. Thus \(c_{2n+k}\) is determined uniquely, and this completes the proof of the lemma.

Remark. It follows from this proof that the only meromorphic solutions of the differential equations

\[
y^{(k)} = y^m
\]

are exponential polynomials when \(m = 1\) and functions \(c(z - z_0)^{-n}\) where \(m = 1 + k/n, z_0 \in \mathbb{C}\) and \(c\) is an appropriate constant.

The rest of the proof of Theorem 1 is a repetition of the argument from [8].

By Theorems A and B, we may assume that \(F\) is of genus zero, and \(k\) is even. In view of Lemmas 2 and 3, it is enough to consider the case that the differential \(pdq\) is exact. Then every solution of (1-2) also satisfies (3-5) with some constant \(c\).

Assume that \(y\) is a transcendental meromorphic solution of (3-5), having at least one pole. By Theorem D, d), c), \(y\) has infinitely many poles \(z_j, j = 1, 2, 3, \ldots\). The functions \(y(z - z_j)\) satisfy the assumptions of Lemma 4, therefore some of them are equal. We
conclude that \( y \) is a periodic function. By making a linear change of the independent variable we may assume that the smallest period is \( 2\pi i \).

Consider the strip \( D = \{ z : 0 \leq \Im z < 2\pi \} \).

Case 1. \( y \) has infinitely many poles in \( D \). Applying Lemma 4 again, we conclude that \( y \) has a period in \( D \), so \( y \) is doubly periodic.

Case 2. \( y \) is bounded in \( D \cap \{ z : |\Re z| > C \} \) for some \( C > 0 \). Since \( y \) is \( 2\pi i \)-periodic, we have \( y(z) = R(e^z) \) where \( R \) is meromorphic in \( \mathbb{C}^* \). As \( R \) is bounded in some neighborhoods of 0 and \( \infty \), we conclude that \( R \) is rational.

Case 3. \( y \) has finitely many poles in \( D \) and is unbounded in \( D \cap \{ z : |\Re z| > C \} \) for every \( C > 0 \). As \( y \) is \( 2\pi i \)-periodic, we write \( y(z) = R(e^z) \) where \( R \) is meromorphic in \( \mathbb{C}^* \). Now \( R \) has finitely many poles and is unbounded either in a neighborhood of 0 or in a neighborhood of \( \infty \). Suppose that it is unbounded in a neighborhood of \( \infty \). Then the set \( \{ z : |R(z)| > M \} \), where \( M \) is large enough has an unbounded component \( G \) containing no poles of \( R \). On this component \( G \), the function \( R \) satisfies a differential equation

\[
\sum_{m=1}^{k} \left( \begin{array}{c} \frac{k}{m} \\
 \end{array} \right) w^m \frac{d^m R}{dw^m} = (c + o(1))R^\kappa,
\]

where \( c \) is some constant and \( \kappa = 1 \) or \( \kappa \) is one of the numbers \( 1 + k/n \) from Theorem D. Applying Theorem C in \( G \) as we did in the proof of Lemma 1, we obtain that \( \kappa = 1 \) and that \( R \) has a pole at infinity. Similar argument works for the singularity at 0, so \( R \) is rational, and this completes the proof.

4. Appendix

Proof of Theorem D. Statement a) is a special case of [9, Th. 10], but we give a simple independent proof using Theorem C. Proving it by contradiction, suppose that \( p \) has a pole at a point \( x \in F \) such that \( q(x) = b \in \mathbb{C} \). Let \( D_\epsilon \subset \mathbb{C} \) be a disk of radius \( \epsilon \) centered at \( b \), and \( V_\epsilon \subset F \) a component of \( q^{-1}(D_\epsilon) \) containing \( x \). We assume that the disk \( D_\epsilon \) is so small that \( V_\epsilon \) contains no other poles of \( p \), except the pole at \( x \). Let \( y \) be the meromorphic solution of our equation (1-2) and consider the map \( f : \mathbb{C} \to F \) given by \( f(z) = (y^{(k)}(z), y(z)) \). The image of this map is dense in \( F \) and the point \( x \) is evidently omitted by \( f \). Let \( G_\epsilon \subset \mathbb{C} \) be a component of the preimage \( f^{-1}(D_\epsilon) \). Consider the meromorphic function \( w = 1/(y - a) \). It is holomorphic and unbounded in \( G_\epsilon \), and \( |w(z)| = 1/\epsilon \) for \( z \in \partial G_\epsilon \). We conclude that \( G_\epsilon \) is unbounded. Now we apply Theorem C to \( w \) in \( G_\epsilon \).

Set \( M(r) = \max \{|w(z)| : |z| = r, z \in G_\epsilon \} \) and let \( a(r) \) be defined as in (2-1). For any \( r > r_0 = \inf \{|z| : z \in G_\epsilon \} \), we choose a point \( z_r \) with \( |z| = r \) and \( |w(z_r)| = M(r) \). By Theorem C, we have

\[
|w^{(j)}(z_r)| = \left( \frac{a(r)}{r} \right)^j |w(z_r)|(1 + o(1)) = \frac{a(r)^j}{r^j} M(r)(1 + o(1)) \tag{4.1}
\]

where \( r \to \infty \) outside a set of finite logarithmic measure.

From Lemma 6.10 of [3], we have for every \( \beta > 0 \),

\[
(a(r))^\beta = o(M(r)), \tag{4.2}
\]

as \( r \to \infty \) outside a set of finite logarithmic measure.
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Differentiating the equation $y = 1/w + a$ we obtain

$$y^{(k)} = \frac{1}{w} Q \left( \frac{w'}{w}, \frac{w''}{w}, \ldots, \frac{w^{(k)}}{w} \right),$$

(4.3)

where $Q$ is a polynomial. On the other hand, from the Puiseaux expansion at the point $x$ we obtain

$$y^{(k)} = (c + o(1))w^\alpha, \quad w \to \infty,$$

(4.4)

where $c \neq 0$ is a constant and $\alpha > 0$. Combining (4.3) and (4.4) we obtain

$$Q \left( \frac{w'}{w}, \ldots, \frac{w^{(k)}}{w} \right) = (c + o(1))w^{1+\alpha}.$$

Inserting to this asymptotic relation $z = z_r$ and using (4.1) and (4.2) we obtain a contradiction which proves a).

Consider now a point $x \in I$. From the Puiseaux expansion we obtain

$$y^{(k)} = (c + o(1))y^{\kappa(x)}, \quad y \to \infty.$$  

(4.5)

If $x$ has a preimage under the map $f$, then this preimage is a pole $z_0$ of $y$. If this pole is of order $n$ we have $y(z) \sim c_1(z-z_0)^{-n}$ and $y^{(k)}(z) \sim c_2(z-z_0)^{-n-k}$ as $z \to z_0$. Substituting to (4.5) we conclude that $\kappa(x) = 1 + k/n$. Thus if $x$ has at least one preimage under $f$ then $\kappa(x) = 1 + k/n$ with a positive integer $n$, and every preimage of $x$ is a pole of order $n$ of $y$. This implies d).

Now suppose that a point $x \in I$ has only finitely many preimages. Let $U_\epsilon = \{z \in \mathbb{C} : |z| > 1/\epsilon\}$ be a neighborhood of infinity, and $V_r \subset F$ a component of the preimage $q^{-1}(U_r)$. We may assume that $\epsilon > 0$ is so small that $V_r$ does not contain other poles of $q$ except $x$. Let $G_r$ be a component of the preimage $f^{-1}(V_r)$. If $G_r$ is bounded then $f : G_r \to U_\epsilon$ is a ramified covering of a finite degree, and $f$ takes the value $x$ somewhere in $G$. As we assume that $f$ is transcendental but $x$ has only finitely many preimages, there should exist an unbounded component $G_r$. Choosing a smaller $\epsilon$ if necessary, we achieve that this unbounded component $G_r$ contains no $f$-preimages of $x$. Then $y$ is a holomorphic function in $G_r$, $|y(z)| = 1/\epsilon$, $z \in \partial G_r$, and $y$ is unbounded in $G_r$. Applying Theorem C to the function $y$ in $G_r$, we obtain the asymptotic relation (2.2). Putting $z = z_r$ in this relation, taking (4.2) into account, and comparing with (4.5) we conclude that $\kappa = 1$ in (4.5). This implies c). Thus in any case $\kappa = 1 + k/n$ or $\kappa = 1$, which proves b).

REFERENCES

A. Eremenko, L.W. Liao and T.W. Ng


