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<th>Automorphic orbit problem for polynomial algebras</th>
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<td><strong>Author(s)</strong></td>
<td>Yu, JT</td>
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<tr>
<td><strong>Citation</strong></td>
<td>Journal Of Algebra, 2008, v. 319 n. 3, p. 966-970</td>
</tr>
<tr>
<td><strong>Issued Date</strong></td>
<td>2008</td>
</tr>
<tr>
<td><strong>URL</strong></td>
<td><a href="http://hdl.handle.net/10722/156202">http://hdl.handle.net/10722/156202</a></td>
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AUTOMORPHIC ORBIT PROBLEM FOR POLYNOMIAL ALGEBRAS

JIE-TAI YU

Abstract. It is proved that every endomorphism preserving the automorphic orbit of a nontrivial element of the rank two polynomial algebra over the complex number field is an automorphism.

1. Introduction and the main results

In [13], Shpilrain raised the following

Problem 1.1. (Automorphic orbit problem for free groups) Let $F_n$ be the free group of rank $n$, $u \in F_n - \{e\}$, $\phi$ an endomorphism of $F_n$ preserving the automorphic orbit of $u$ in $F_n$, i.e. for each automorphism $\alpha$ of $F_n$, there exists an automorphism $\beta$ of $F_n$, such that $\phi(\alpha(u)) = \beta(u)$. Is $\phi$ an automorphism of $F_n$?


In the sequel all automorphisms (endomorphisms) of a polynomial algebra over a field $K$ are always $K$-automorphisms ($K$-endomorphisms). In view of Problem 1.1, it is natural and interesting to raise

Problem 1.2. (Automorphic orbit problem for polynomial algebras) Let $P_n$ be the polynomial algebra of rank $n$ over a field $K$, $p \in P_n - K$, $\phi$ an endomorphism of $P_n$ preserving the automorphic orbit of $p$ in $P_n$. Is $\phi$ an automorphism of $P_n$?

Recall that a polynomial $p \in P_n$ is a coordinate if there exists an automorphism $\psi$ of $P_n$ taking $x_1$ to $p$. A special case of Problem 1.1 when $u$ is a coordinate of $P_n$ is the following


Key words and phrases. Automorphic orbits, test polynomials, retracts, retrac-
tion, coordinates, polynomial algebras, outer rank.

The research of Jie-Tai Yu was partially supported by an RGC-CERG Grant.
Problem 1.3 (Coordinate preserving problem). Let $P_n$ be the polynomial algebra of rank $n$ over a field $K$. Is every endomorphism $\phi$ of $P_n$ taking all coordinates of $P_n$ to coordinates an automorphism?

Problem 1.3 is solved affirmatively for $n = 2$ when $K$ is an arbitrary field by van den Essen and Shpilrain [3], and is solved affirmatively for arbitrary $n$ when $K$ is an algebraically closed field of zero characteristic by Jelonek [8]. In this paper we solve Problem 1.2 for $n = 2$ when $K$ is the complex number field:

Theorem 1.4. Let $p \in \mathbb{C}[x, y] - \mathbb{C}$, $\phi$ an endomorphism of $\mathbb{C}[x, y]$ preserving the automorphic orbit of $p$. Then $\phi$ is an automorphism of $\mathbb{C}[x, y]$.

Recall the outer rank $k$ of a polynomial $p \in P_n$ is the minimal number $k$ such that under an automorphism $\phi$ of $P_n$, $\phi(p) \in P_k$. See Shpilrain and J.-T. Yu [15]. In our proof of Theorem 1.4, it is crucial to use the result below based on a theorem of Shpilrain and J.-T. Yu [17], which has its own interest.

Theorem 1.5. Let $p \in \mathbb{C}[x, y]$ has outer rank 2. Then $p$ is a test polynomial recognizing automorphisms among injective endomorphisms of $\mathbb{C}[x, y]$. Or, more precisely, if $\phi$ is an injective endomorphism of $\mathbb{C}[x, y]$ such that $\phi(p) = p$, then $\phi$ is an automorphism.

The above theorem can be viewed as an analogue of a result of Turner [18] for free groups.

2. Preliminaries

First let us recall test polynomials and retracts of polynomial algebras. See [4, 5, 9, 10, 16, 17]. A polynomial $p \in P_n$ is called a test polynomial, if, for any endomorphism $\phi$ of $P_n$, $\phi(p) = p$ implies that $\phi$ is an automorphism. A subalgebra $R$ of $P_n$ is called a retract if there is a idempotent homomorphism ($\pi$ is called the retraction from $P_n$ to $R$) $\pi$ of $P_n$ such that $\pi(P_n) = R$. By a theorem of Costa [1], every proper retract of $K[x, y]$ (a retract of $K[x, y]$ different from $K$ and $K[x, y]$) is of the form $K[p]$ for some $p \in K[x, y]$ for arbitrary field $K$. Recently Shpilrain and J.-T. Yu [16, 17] have shown the close connection among test polynomials, retracts, and the Jacobian conjecture. See also [2, 10].

Lemma 2.1 (Shpilrain and J.-T.Yu [16]). Let $K$ be a field of zero characteristic. A polynomial $r \in K[x, y]$ generates a proper retract
of $K[x, y]$ if and only if there is an automorphism $\alpha$ of $K[x, y]$ such that $\alpha(r) = x + yq$ for some $q \in K[x, y]$. Moreover, under the above condition the retraction from $C[x, y]$ to $C[r]$ is $\alpha^{-1}\pi\alpha$, where $\pi$ is the retraction of $C[x, y]$ to $C[x + yq]$ defined by $\pi(x) = x + yq$ and $\pi(y) = 0$.

The next lemma is based on the main theorem and its proof in Drensky and J.-T. Yu [4].

**Lemma 2.2.** A polynomial $p \in C[x, y]$ belongs to a proper retract $C[r]$ if and only if $p$ is fixed by a non-injective endomorphism $\phi$ of $C[x, y]$. Moreover, under the above condition, if $p = f(r)$, $f(t) \in C[t] - C$, $\deg(f) = m$, then $\pi = \phi^m$ is the retraction from $C[x, y]$ to $C[r]$.

**Proof.** The first sentence is just the Theorem in [4]. Moreover, in the proof of the Theorem in [4], it is actually proved that $\pi = \phi^m$ is the retraction from $C[x, y]$ to $C[r]$ with $m = \lceil C(r) : C(p) \rceil$. By elementary algebra, $m = \deg(f)$, where $f \in K[t]$, and $p = f(r)$. $\square$

**Lemma 2.3.** Let $K$ be an arbitrary field, $u \in K[x, y]$ with outer rank 1, $\phi$ an endomorphism preserving the automorphic orbit of $u$. Then $\phi$ is an automorphism.

**Proof.** Write $u = f(p)$, where $f \in K[t]$, $p$ is a coordinate of $K[x, y]$. We may assume $p = x$. For any automorphism $\alpha$, $\phi\alpha(f(x)) = \beta(f(x))$ for some automorphism $\beta$. Hence $\beta^{-1}\phi\alpha(f(x)) = f(x)$, therefore $f(\beta^{-1}\phi\alpha(x)) = f(x)$. Let $\beta^{-1}\phi\alpha(x) = g(x, y)$. Compare the degrees of $y$ in both sides of $f(g(x, y)) = f(x)$, $g(x, y) = g(x, 0) = h(x) \in K[x]$. Compare the degrees in both sides of $f(h(x)) = f(x)$, $\deg(h(x)) = 1$, that forces $h(x) = \beta^{-1}\phi\alpha(x) = cx$, hence $\phi\alpha(x) = \beta(cx)$ for some $c \in K^*$ (in fact $c$ can only be some $m$-th root of unity, $m = \deg(f)$, but we do not need that). Therefore $\phi$ preserves coordinates of $K[x, y]$. By a result of Shpilrain and van den Essen [3], $\phi$ is an automorphism. $\square$

**Lemma 2.4.** Let $K$ be an arbitrary field, $p \in P_n = K[x_1, \ldots, x_n]$ a test polynomial. Then every endomorphism $\phi$ of $P_n$ preserving the automorphic orbit of $p$ is an automorphism.

**Proof.** Since $\phi(p) = \alpha(p)$ for some automorphism $\alpha$ of $P_n$, $\alpha^{-1}\phi(p) = p$, as $p$ is a test polynomial, $\alpha^{-1}\phi$, hence $\phi$, is an automorphism. $\square$

The following lemma is the main result of Shpilrain and J.-T. Yu [17].

**Lemma 2.5.** A polynomial $p \in C[x, y]$ is a test polynomial if and only if $p$ does not belong to any proper retract of $C[x, y]$. 
3. Proof of the main results

Proof of Theorem 1.5. Let \( p \in \mathbb{C}[x, y] \) has outer rank 2, \( \phi \) an injective endomorphism such that \( \phi(p) = p \). Suppose on the contrary, \( \phi \) is not an automorphism, then by Theorem 2 in [17], \( p \) has outer rank 1. This contradiction completes the proof.

Proof of Theorem 1.4. We may assume \( \phi(p) = p \). By Lemma 2.4, we may assume \( p \) is not a test polynomial. By Lemma 2.5, we may assume \( p \) belongs to a proper retract \( \mathbb{C}[r] \) of \( \mathbb{C}[x, y] \). By Lemma 2.3, we may assume \( p \) has outer rank 2. By Theorem 1.5, we may assume \( \phi \) is non-injective. Suppose \( p = f(r) \), where \( f \in \mathbb{C}[t] - \mathbb{C}, \deg(f) = m \).

By Lemma 2.2, \( \pi = \phi^m \) is the retraction from \( \mathbb{C}[x, y] \) to \( \mathbb{C}[r] \). As \( \phi \) preserves the automorphic orbit of \( p \), so does \( \pi = \phi^m \). Applying Lemma 2.1 (suppose \( \alpha(r) = x + yq(x, y) \), where \( q(x, y) \notin K[y] \), \( \alpha \) is some automorphism of \( \mathbb{C}[x, y] \), replace \( r \) by \( \alpha(r) \), and \( \pi \) by \( \alpha\pi\alpha^{-1} \)), we have reduced our proof to the proof of the following

Lemma 3.1. Let \( r = x + yq(x, y) \), where \( q(x, y) \in \mathbb{C}[x, y] \), \( q(x, y) \notin \mathbb{C}[y] \), \( \pi \) the retraction of \( \mathbb{C}[x, y] \) to \( \mathbb{C}[r] \) defined by \( \pi(x) = x + yq(x, y) \), \( \phi(y) = 0 \), \( f \in \mathbb{C}[t] - \mathbb{C} \). Then \( \pi \) does not preserve the automorphic orbit of \( f(r) \).

Proof. Suppose on the contrary, \( \pi \) preserves the automorphic orbit of \( f(r) \). Then for any automorphism \( \alpha \) of \( \mathbb{C}[x, y] \), \( \pi\alpha(f(r)) = \beta(f(r)) \in \mathbb{C}[r] \) for some automorphism \( \beta \) of \( \mathbb{C}[x, y] \). Note that \( \pi\beta(f(r)) = \beta(f(r)) \).

By Lemma 2.2, \( \pi_{\deg(f)} = \pi \) is the retraction from \( \mathbb{C}[x, y] \) to the retract \( \mathbb{C}[\beta(r)] \) taking \( \beta(r) \) to \( \beta(r) \). By hypothesis, \( \pi \) is also the retraction of \( \mathbb{C}[x, y] \) to the retract \( \mathbb{C}[r] \) taking \( r \) to \( r \). This forces that \( \beta(r) = r \). Therefore \( \beta(x + yq(x, y)) = x + yq(x, y) \). Substituting \( y = 0 \), \( \beta(x) = x \). Hence \( \beta(yq(x, y)) = yq(x, y) \). But \( \beta \) is an automorphism, so \( \beta(y) = cy + h(x) \) where \( c \in \mathbb{C}^* \), \( h(x) \in \mathbb{C}[x] \). It follows easily that \( \beta(y) = y \), \( \beta \) is the identity automorphism. We have conclude that for all automorphisms \( \alpha \) of \( \mathbb{C}[x, y] \), \( \pi\alpha(f(r)) = f(r) \). Let \( M \) be a positive integer greater than \( \deg(q(x, y)) \), it is easy to see that \( x^M - y \) does not divide \( q(x, y) \) in \( \mathbb{C}[x, y] \). Let \( \alpha \) be the automorphism of \( \mathbb{C}[x, y] \) defined by \( \alpha(x) = x, \alpha(y) = y + x^M \). Then easy calculation shows that \( \pi\alpha(f(r)) = f(r + r^M q(r, r^M)) \). As \( x^M - y \) does not divide \( q(x, y) \), \( q(r, r^M) \neq 0 \). Therefore \( \pi\alpha(f(r)) = f(r + r^M q(r, r^M)) \neq f(r) \). This contradiction completes the proof. \( \Box \)
4. Acknowledgements

The author is grateful to the Beijing International Center for Mathematical Research and the Institut des Hautes Études Scientifiques for warm hospitality during his visit when this work was carried out. He also thanks V. Drensky, L. Makar-Limanov and V. Shpilrain for helpful discussions.

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